HILBERT–KUNZ MULTIPLICITY OF QUADRICS VIA EHRHART THEORY

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Abstract. Watanabe and Yoshida conjectured a lower bound on the Hilbert–Kunz multiplicity of quadrics. For large enough characteristic, the conjecture is now a theorem of Trivedi. We give a combinatorial approach to this result. Namely, we show that the Hilbert–Kunz multiplicity of a d-dimensional quadric of characteristic $p > 0$ is a rational function of p composed from Ehrhart polynomials of polytopes. This allows us to show that the Hilbert–Kunz multiplicity of quadrics is a decreasing function of dimension and allows us to restate the asymptotic form of the Watanabe–Yoshida conjecture as a lower bound on the second coefficient of the Ehrhart polynomial of the Fibonacci polytope.

1. Introduction

Hilbert–Kunz multiplicity is a multiplicity theory native to positive characteristic, one of the numerical invariants defined via the iterates of the Frobenius endomorphism.

Definition (Monsky)**.** Let $(R, \mathfrak{m})$ be a local ring of positive characteristic $p > 0$. Then the *Hilbert–Kunz multiplicity* of R is defined as the limit

$$\mathrm{e}_{\mathrm{HK}}(R) := \lim_{e\to\infty} \frac{\lambda(R/\mathfrak{m}^{[p^e]})}{p^{e \dim R}},$$

where $\mathfrak{m}^{[p^e]}$ is the ideal generated by all p^eth powers of elements in $\mathfrak{m}$.

The ideal $\mathfrak{m}^{[p^e]}$ is called the *Frobenius power* of $\mathfrak{m}$; this terminology stresses the analogy to Samuel's definition of the classical Hilbert–Samuel multiplicity. Hilbert–Kunz multiplicity originates in the work of Kunz [Kun69, Kun76]: [Kun69] shows that flatness of the Frobenius characterizes regular rings and [Kun76] extensively studies the sequence whose limit is the Hilbert–Kunz multiplicity as it measures the failure of flatness of the Frobenius. The invariant was defined by Monsky [Mon83], whose motivation came from the Iwasawa theory.

From its very origins, the development of the Hilbert–Kunz theory was motivated by its use as a tool of understanding singularities and its similarity with the Hilbert–Samuel theory. For example, the celebrated theorem of Nagata ([Nag62]) asserts that a local ring R is regular if and only if it is unmixed[1] and its multiplicity is 1. In a parallel result, Watanabe and Yoshida showed in [WY00] that Hilbert–Kunz multiplicity detects regular rings: a local ring R is regular if and only if it is unmixed and $\mathrm{e}_{\mathrm{HK}}(R) = 1$.

However, there is a major difference between two theories of multiplicity: Hilbert–Kunz multiplicity need not be an integer; in fact, it may be even irrational ([Bre]). Overall, we know little about the set of values of Hilbert–Kunz multiplicity. A very natural question was raised by Blickle and Enescu in [BE04]: what is $\inf\{R \mid \mathrm{e}_{\mathrm{HK}}(R) > 1, \operatorname{char} R = p, \dim R = d\}$? Equivalently, this question asks for the best lower bound on the Hilbert–Kunz multiplicity of an unmixed singular ring. Blickle and Enescu gave a lower bound $\mathrm{e}_{\mathrm{HK}}(R) \geq 1 + \frac{1}{d!p^d}$, showing that 1 is not a limit point in the space of values of Hilbert–Kunz multiplicity. Another bound was given in [CDHZ12].

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[1] The unmixed condition means that the completion $\hat{R}$ has no lower-dimensional components: lower-dimensional components do not contribute to the limit.

In a different direction, Watanabe and Yoshida [WY05] conjectured what singular rings minimize Hilbert–Kunz multiplicity.

Conjecture 1. *Let $p > 2$ be a prime number and define the simple (A_1)-singularity*[2] *by the equation*

$$A_{p,d} := \mathbb{F}_p[[x_0, \ldots, x_d]]/(x_0^2 + \cdots + x_d^2).$$

Let $(R, \mathfrak{m})$ be a local ring of dimension $d \geq 1$, characteristic p, and set $k = R/\mathfrak{m}$. Then

1. *if $\mathrm{e}_{\mathrm{HK}}(R) \neq 1$, then $\mathrm{e}_{\mathrm{HK}}(R) \geq \mathrm{e}_{\mathrm{HK}}(A_{p,d})$,*
2. *if R is formally unmixed and k is algebraically closed, then $\mathrm{e}_{\mathrm{HK}}(R) = \mathrm{e}_{\mathrm{HK}}(A_{p,d})$ if and only if*

$$\widehat{R} \cong k[[x_0, \ldots, x_d]]/(x_0^2 + \cdots + x_d^2) = A_{p,d} \mathbin{\widehat{\otimes_{\mathbb{F}_p}}} k.$$

Conjecture 1 is now known in dimensions at most 8 ([WY05, AE08, AE13, CSA24, CR]) and for complete intersections ([ES05, CR]).

An important subtlety is that Conjecture 1 does not immediately provide a numerical lower bound because a closed formula for $\mathrm{e}_{\mathrm{HK}}(A_{p,d})$ is not known. In her thesis [Han92], Han developed an algorithm computing the Hilbert–Kunz multiplicity of diagonal hypersurfaces, which was published in a simplified form in [HM93]. The algorithm gives a way to compute $\mathrm{e}_{\mathrm{HK}}(A_{p,d})$ for given p and d, but an actual formula was only presented for $d \leq 4$. However, Gessel and Monsky [GM] observed that the algorithm computes the limits of Hilbert–Kunz multiplicities

$$\lim_{p\to\infty} \mathrm{e}_{\mathrm{HK}}(A_{p,d}) = 1 + \frac{\mathcal{A}_d}{d!}$$

where $\mathcal{A}_d$ are the Euler (zigzag) numbers given by the Taylor–Maclaurin series

$$\sec x + \tan x = \sum \frac{\mathcal{A}_d}{d!} x^d.$$

This result motivated the following characteristic-free bound.

Conjecture 2 (Watanabe–Yoshida, [WY05])**.** *For any dimension $d \geq 1$ and characteristic $p > 2$, one has $\mathrm{e}_{\mathrm{HK}}(A_{p,d}) \geq 1 + \mathcal{A}_d/d!$.*

Recently, in [Tri23], Trivedi showed that this conjecture is true for any d and $p > d - 1$. Even more recently, Meng announced a complete[3] solution in [Men]. Combined with the result of Enescu and Shimomoto [ES05] there is now a good characteristic-free bound on Hilbert–Kunz multiplicity of complete intersections, but, in general, we only have a much weaker characteristic-free bound observed in [AE08]:

$$\mathrm{e}_{\mathrm{HK}}(R) \geq 1 + \frac{1}{d(d!(d-1)+1)^d}.$$

Note that it is known that $\mathcal{A}_d/d! \sim 2(2/\pi)^{d+1}$.

It is believed that the bound in Conjecture 2 is not sharp. In fact, the last named author conjectured in the unpublished note [Yos19] that $\mathrm{e}_{\mathrm{HK}}(A_{p,d})$ is a strictly decreasing sequence in p for a fixed d. This is known to be true in small dimensions, where an explicit formula for $\mathrm{e}_{\mathrm{HK}}(A_{p,d})$ can be computed, and for p sufficiently large (depending on d), as a byproduct of Trivedi's approach to Conjecture 2 in [Tri23].

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[2] As discussed in the unpublished notes of the last-named author [Yos19] the equation of $A_{2,d}$ should be different, see also [JNS$^+$23, CR].

[3] Meng assumes that $p > 3$, but the conjecture holds for $p = 2$ by a direct computation in [CR] if $A_{2,d}$ is appropriately defined.

Results. This note provides an approach to Conjecture 2 that is drastically different from [Tri23]. Trivedi's work is quite intricate, it builds on Achinger's computation of Frobenius pushforwards of vector bundles on quadrics in order to study the *Hilbert–Kunz density functions* (a theory developed in [Tri18]) and then treats the densities by analytic tools. Meng's improvement in [Men] results from even more sophisticated analytic tools. On the other hand, the approach of this paper is more elementary, based on the algorithm of Han and Monsky [HM93]. For quadrics the algorithm can be interpreted using linear algebra ([Yos19]): for the families of $(2n+1)\times(2n+1)$ square matrices

$$T_n = \begin{bmatrix} 2 & \cdots & 2 & 1 & 0 & \cdots & 0 \\ \vdots & & \iddots & \vdots & \ddots & & \vdots \\ 2 & \iddots & & \vdots & & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\ 0 & \ddots & & \vdots & & \iddots & 2 \\ \vdots & & \ddots & \vdots & \iddots & & \vdots \\ 0 & \cdots & 0 & 1 & 2 & \cdots & 2 \end{bmatrix} \text{ and } N_n = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \iddots & \vdots & \ddots & & \vdots \\ 0 & \iddots & & \vdots & & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\ 0 & \ddots & & \vdots & & \iddots & 0 \\ \vdots & & \ddots & \vdots & \iddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

one computes (Corollary 3.7) that

$$\mathrm{e}_{\mathrm{HK}}(A_{p,d}) = 1 + \frac{[T^{d+1}_{\lfloor p/2 \rfloor}]_{(1,1)} - p^d}{p^d - [N^{d+1}_{\lfloor p/2 \rfloor}]_{(1,1)}},$$

where $[M]_{(1,1)}$ denotes the $(1,1)$–entry of M. Using the special shape of the matrices T_n and N_n, we prove the following.

Theorem (Corollaries 3.10, 3.12, 3.14, 3.15)**.** *Let $F_d(n)$ and $E_d(n)$ be the Ehrhart polynomials of the d-dimensional* Fibonacci *and* extended Fibonacci *polytopes (Definition 2.1). Then for all $p > 2$,*

$$\mathrm{e}_{\mathrm{HK}}(A_{p,d}) = 1 + \frac{2^d F_d\left(\frac{p-3}{2}\right)}{p^d - E_{d-2}\left(\frac{p-1}{2}\right)}.$$

As a consequence, we have

(1) *there exist polynomials $f, g \in \mathbb{Q}[x]$ of degree d such that $\mathrm{e}_{\mathrm{HK}}(A_{p,d}) = \frac{f(p)}{g(p)}$ for all $p > 2$;*
(2) *(Gessel–Monsky,* [GM]*)* $\lim_{p\to\infty} \mathrm{e}_{\mathrm{HK}}(A_{p,d}) = 1 + \mathcal{A}_d/d!$*;*
(3) $\mathrm{e}_{\mathrm{HK}}(A_{p,d}) > \mathrm{e}_{\mathrm{HK}}(A_{p,d+1})$ *for any $p > 2$.*

The first corollary significantly strengthens [Tri23, Theorem (B)], which shows that $\mathrm{e}_{\mathrm{HK}}(A_{p,d})$ is a rational function for $p > 2^{\lfloor (d-1)/2 \rfloor (d-3)}$ with the degrees of the denominator and numerator bounded by $(d+2)^{d+4}$. As another corollary, we obtain (Remark 3.13) an easy algorithm for computing the rational function $\mathrm{e}_{\mathrm{HK}}(A_{p,d})$ (d fixed, $p > 2$ varies). In particular, computationally, we verify computationally that for all $d \leq 100$ the function $p \mapsto \mathrm{e}_{\mathrm{HK}}(A_{p,d})$ is nonincreasing and Conjecture 2 holds.

Furthermore, our result provides a direction for attacking Conjecture 2 combinatorially. While we cannot solve this problem – and believe that it should involve an intricate analysis of the Ehrhart polynomials – we restate it in terms of a certain statistic for alternating permutations introduced by Coons and Sullivant [CS23]. We show that the asymptotic version of the conjecture, established in [Tri23], amounts to finding a good bound on the second coefficient c_2 of the Ehrhart polynomial $\mathcal{A}_d/d!k^d + c_1k^{d-1} + c_2k^{d-2} + \cdots$ of the Fibonacci polytope. We are able to compute c_1, but by a "miraculous" cancellation this is not enough.

2. The Fibonacci polytope and its Ehrhart polynomial

We begin by fixing notation. If P is an integer polytope, we use $|P|$ to denote the number of integer points in P and kP for its k-dilation. It is known that the function $k \mapsto |kP|$ is a polynomial, which is called the Ehrhart polynomial. In this work we will need the following two families of polytopes.

Definition 2.1. For $d \geq 1$, define the d-dimensional *Fibonacci polytope* as a subset of $[0,1]^d$ given by inequalities

$$x_i + x_{i+1} \leq 1, \ \ i = 1, \ldots, d-1$$

and the *extended d-dimensional Fibonacci polytope* as a subset of $[-1,1]^d$ given by inequalities:

$$|x_i| + |x_{i+1}| \leq 1, \ \ i = 1, \ldots, d-1.$$

Remark 2.2. All integer points in the Fibonacci polytope are its vertices; by recursion one can show that the number of vertices is given by the Fibonacci numbers.

The numbers of integer points $p(d)$ in the extended Fibonacci polytope of dimension d form the Jacobsthal sequence $(2^{d+2} - (-1)^{d+2})/3$ ([Slo, A001045]). Namely, it satisfies the linear recurrence $p(d+1) = p(d) + 2p(d-1)$ with the standard initial conditions: if we increase the dimension by adding x_{d+1}, then any integer point can be extended by $x_{d+1} = 0$. However, if $x_d = 0$, then the point can be also extended by ±1.

The number of vertices $v(d)$ of the extended Fibonacci polytope satisfies linear recurrence $v(d) = 2v(d-2) + 2v(d-3)$ with the initial condition $v(1) = 2, v(2) = 4$ ([Slo, A107383]). First, note that vertices correspond to words on the alphabet $\{-1, 0, 1\}$ such that there are no consecutive nonzero entries and no '0' can be replaced by '1'. The last condition means that the sequence cannot start or end with '00' and cannot contain three consecutive '0's. Such a sequence must therefore end with '0 ± 1' or '0 ± 10'. In the first case, removing the ending gives a word in $v(d-2)$ and in the second a word from $v(d-3)$. The recursion follows.

The Fibonacci polytope is affinely equivalent to the order polytope of the zigzag poset ([Sta10, Section 3.8]), i.e., the set of points in $[0,1]^d$ such that $x_1 \geq x_2 \leq x_3 \geq \cdots$ by mapping $x_i \to y_i$ where $y_i = 1 - x_i$ if i is odd and $y_i = x_i$ if i is even. Thus, the two polytopes have equal Ehrhart polynomials and a result of Stanley gives a canonical triangulation of the Fibonacci polytope.

Theorem 2.3 (Stanley)**.** *The order polytope of a partial order P on $[n]$ has a canonical triangulation indexed by linear extensions $P \to [n]$, i.e., permutations of $[n]$ that adhere to the partial order.*

Hence the order polytope of the zigzag poset is triangulated by *alternating* permutations. The number of such permutations is the *Euler up-down number* $\mathcal{A}_d$. It is known that $1 + \sum_{d \geq 1} \mathcal{A}_d/d!\, x^d = \sec x + \tan x$ [And81].

Corollary 2.4. *The volume of the Fibonacci polytope of dimension d equals $\mathcal{A}_d/d!$.*

To the best of our knowledge, there is no closed-form description of the Ehrhart polynomial of this polytope, but a recent paper of Coons and Sullivant [CS23] gives a combinatorial formula for the h^* vector.

Definition 2.5. Let σ be an alternating permutation in S_n. Its *permutation statistic* $\operatorname{swap}(\sigma)$ is the number of $i < n$ such that $\sigma^{-1}(i) < \sigma^{-1}(i+1) - 1$. Equivalently, $\operatorname{swap}(\sigma)$ is the number of $i < n$ such that i is to the left of $i+1$ and swapping i and $i+1$ in σ yields another alternating permutation.

By [CS23, Theorem 1.9] the Ehrhart series of the order polytope Z_d of the zigzag poset is

$$\frac{\sum_\sigma t^{\mathrm{swap}(\sigma)}}{(1-t)^{d+1}}.$$

over alternating permutations. This result essentially follows from Theorem 2.3 after realizing that the adjacency of two simplices is given by a swap ([CS23, Proposition 2.1]). Note that the definition of swap is asymmetric to avoid double counting. Coons and Sullivant showed that the h^*-vector is symmetric and unimodal (see also [PZ25] for a generalization).

Corollary 2.6. *The swap statistic provides a formula for the Ehrhart polynomial: if* $s_d(m) = \#\{\sigma \mid \mathrm{swap}(\sigma) = m\}$, *then*

$$|kF_d| = \sum_{m=0}^{d-2} s_d(m)\binom{k+d-m}{d} = \sum_{i=0}^{d-2}(-1)^i\binom{k+d-i}{d-i}\left(\sum_{m=0}^{d-2}\binom{m}{i}s_d(m)\right).$$

Proof. The first equality follows from the formula for the Ehrhart series. The second formula can be obtained either by manipulating the binomial coefficients or by using the inclusion-exclusion formula on the Stanley triangulation based on [CS23, Proposition 2.1]: note that $\binom{k+d-i}{d-i}$ is the Ehrhart polynomial of a $(d-i)$-dimensional unit simplex. ☐

Remark 2.7. Overall, there seems to be little known about the numbers $s_d(m)$. It seems that their table coincides with [Slo, A205497]. The table of $\sum_{m=0}^{d-2}\binom{m}{i}s_d(m)$ is given in [Slo, A079502] based on a paper of Kreweras [Kre76, table on page 20]. Kreweras denotes by u_r^n the number of *alternating surjective* maps $f\colon [n] \to [r]$, i.e., surjections satisfying the up-down condition $f(1) > f(2) < f(3) > \cdots$. The equality $u_{n-r}^n = \sum_{m=0}^{n-2}\binom{m}{r}s_n(m)$ can be shown by constructing f from a permutation $\pi\colon [n] \to [n]$ with r chosen swaps by identifying the values $\pi(i)$ and $\pi(j)$ with $i < j$ if they form a swap. This reduces the image to $n - r$ numbers. The swap condition guarantees that the map is still alternating.

Lemma 2.8. *We have a relation*

$$\sum_{m=0}^{d-2} ms_d(m) = \mathcal{A}_d\left(\frac{d}{2}-1\right).$$

Proof. It was shown in [CS23, Theorem 4.1] that the sequence $s_d(0), \dots, s_d(d-2)$ is symmetric. Hence $is_d(i) + (d-2-i)s_d(d-2-i) = (d/2-1)(s_d(i) + s_d(d-2-i))$. Thus, the lemma easily follows from the relation $\sum_{m=0}^{d-2} s_d(m) = \mathcal{A}_d$. ☐

We are now able to expand the Ehrhart polynomial of the Fibonacci polytope to the next term.

Corollary 2.9. *The Ehrhart polynomial* $P_d(k) := |kF_d|$ *of the d-dimensional Fibonacci polytope is*

$$P_d(k) = \frac{\mathcal{A}_d}{d!}k^d + \frac{3}{2}\frac{\mathcal{A}_d}{(d-1)!}k^{d-1} + \frac{1}{(d-2)!}\left(\mathcal{A}_d\frac{-3d^2+17d+2}{24} + \sum_{m=0}^{d-2}\binom{m}{2}s_d(m)\right)k^{d-2} + O(k^{d-3}).$$

In particular,

$$P_d\left(x-\frac{3}{2}\right) = \frac{\mathcal{A}_d}{d!}x^d + \frac{1}{(d-2)!}\left(\mathcal{A}_d\frac{-3d^2+17d-25}{24} + \sum_{m=0}^{d-2}\binom{m}{2}s_d(m)\right)x^{d-2} + O(x^{d-3}).$$

Proof. We computed the coefficient at k^{d-1} in the preceding proof, so it remains to compute the coefficient at k^{d-2}. By Corollary 2.6 it can be written as

$$\frac{\sum_{1\le i<j\le d} ij}{d!} \times \sum_{m=0}^{d-2} s_d(m) - \frac{\binom{d}{2}}{(d-1)!} \times \sum_{m=0}^{d-2} ms_d(m) + \frac{1}{(d-2)!} \times \sum_{m=0}^{d-2}\binom{m}{2}s_d(m).$$

Since $\sum_{1\le i<j\le d} ij = \frac{(d-1)d(d+1)(3d+2)}{24}$, it now remains to use Lemma 2.8 to compute that

$$\frac{\sum_{1\le i<j\le d} ij}{d!}\sum_{m=0}^{d-2} s_d(m) - \frac{\binom{d}{2}}{(d-1)!}\sum_{m=0}^{d-2} m s_d(m) = \frac{\mathcal{A}_d}{(d-2)!}\frac{(d+1)(3d+2)}{24} - \frac{\mathcal{A}_d}{(d-2)!}\frac{d(d-2)}{4}.$$

The second assertion now can be verified straightforwardly. □

Question 2.10. It appears that the sequence $s_d(1)$ coincides with a shift of [Slo, A001924] ($s_3(1) = 1$). Can it be shown that $s_{d+1}(1) = s_d(1) + s_{d-1}(1) + d - 1$?

Remark 2.11. Recent work [FMP] implies that all coefficients of the Ehrhart polynomial of the Fibonacci polytope are positive.

3. THE HAN–MONSKY ALGORITHM AND ITS MATRIX INTERPRETATION

3.1. **Han–Monsky algorithm for computing the Hilbert–Kunz multiplicity.** In [HM93] Han and Monsky provided an approach to the calculation of Hilbert–Kunz multiplicity for a special class of local rings. Their key idea is based on a concept of *representation ring*, which is a certain Grothendieck ring.

Let k be a fixed field and consider the subcategory $\mathrm{Mod}_{\mathrm{T}}(k)$ of finitely generated $k[T]$-modules with a nilpotent action of the variable T. On pairs (M, N) of such modules there is a natural equivalence relation: $(M_1, N_1) \equiv (M_2, N_2)$ if and only $M_1 \oplus N_2 \cong M_2 \oplus N_1$. The equivalence class of a pair (M, N) will be denoted as $M - N$.

Definition 3.1. The representation ring Γ_k consists of the equivalence classes $M - N$ with the operations of direct sum and tensor product:

$$(M_1 - N_1) \times (M_2 - N_2) := [(M_1 \otimes_k M_2) \oplus (N_1 \otimes_k N_2)] - [(M_1 \otimes_k N_2) \oplus (N_1 \otimes_k M_2)].$$

The equivalence class $k - 0$ is the identity element of Γ_k.

The standard examples of elements in Γ_k are provided by $\delta_i = k[T]/(T^i) - 0$, $i \in \mathbb{Z}_{>0}$. Since $k[T]$ is a PID, any $M \in \mathrm{Mod}_{\mathrm{T}}(k)$ decomposes as $M = \oplus_{i=1}^{\infty} k[T]/(T^i)^{\oplus a_i}$ and the function D_k is defined by setting

$$D_k(M) := \dim_k M/TM = \sum_{i=1}^{\infty} a_i.$$

This observation shows that Γ_k is a free $\mathbb{Z}$-module with a basis $\{\delta_i\}$ and the function D_k extends by linearity, $D_k\colon \Gamma_k \to \mathbb{Z}$ is then the sum of the coordinates in this basis.

The following result is a restatement of [HM93, Lemma 5.6].

Theorem 3.2. *For a given field k of positive characteristic $p > 0$ define*

$$R := k[[x_0, \ldots, x_d]]/(x_0^{n_0} + \cdots + x_d^{n_d}).$$

For a fixed integer $e \ge 0$ and each $i = 0, \ldots, d$, define the remainder r_i determined by $p^e = n_i a_i + r_i$, $0 \le r_i < n_i$. In the above notation,

$$\dim_k R/(x_0^{p^e}, \ldots, x_d^{p^e}) = D_k\left(\prod_{i=0}^{d} (n_i - r_i)\delta_{a_i} + r_i \delta_{a_i+1}\right).$$

Proof. Let us sketch the idea of the proof. For a positive integer n, consider $k[x]/(x^{p^e})$ as a $k[T]$-module with the action $Tf := x^n f$. Denote this module as M_n. Consider the division with the remainder $p^e = an + r$. Then the submodule $k[T]x^i = k\langle x^i, x^{i+n}, \ldots\rangle \subset M$ is isomorphic to $k[T]/(T^{a+1})$ if $0 \le i < r$ and to $k[T]/(T^a)$ if $r \le i < n$. This gives a direct sum decomposition

$$M_n = \oplus_{0\le i<n} k[T]x^i = [k[T]/(T^{a+1})]^{\oplus r} \oplus [k[T]/(T^a)]^{\oplus n-r}$$

as a $k[T]$-module.

Furthermore, $M_{n_0} \otimes_k \cdots \otimes_k M_{n_d} \cong k[x_0, \dots, x_d]/(x_0^{p^e}, \dots, x_d^{p^e})$ with $T \mapsto x_0^{n_0} + \cdots + x_d^{n_d}$, so $\dim_k R/(x_0^{p^e}, \dots, x_d^{p^e}) = D_k(M_{n_0} \otimes_k \cdots \otimes_k M_{n_d})$. □

The difficulty of using the theorem is decomposing the product of basis elements $\delta_i\delta_j$ as a linear combination of the basis elements. Han and Monsky were able to provide the needed multiplication rules. In particular, their work gives the following recipe.

Corollary 3.3. *Let k be a field of characteristic $p > 2$ and $a = (p-1)/2$. Then*

$$\mathrm{e}_{\mathrm{HK}}(A_{p,d}) := \mathrm{e}_{\mathrm{HK}}(k[[x_0, \dots, x_d]]/(x_0^2 + \cdots + x_d^2)) = 1 + \frac{D_k((\delta_a + \delta_{a+1})^{d+1}) - p^d}{p^d - (-1)^{a(d+1)} D_k((\delta_{a+1} - \delta_a)^{d+1})}.$$

Proof. Essentially, this is explained in [HM93, Example 2, page 134]. Since p is odd, $\mu = 1$ can be used in [HM93, Theorem 5.3], so [HM93, Theorems 5.5, 5.7] will show that

$$\dim_k A_{p,d}/(x_0^p, \dots, x_d^p) = p^d\,\mathrm{e}_{\mathrm{HK}}(A_{p,d}) + (1 - \mathrm{e}_{\mathrm{HK}}(A_{p,d}))D_k((-1)^a(\delta_{a+1} - \delta_a)^{d+1})$$

and we solve the equation for the Hilbert–Kunz multiplicity. □

3.2. **A linear algebra approach.** In order to use Corollary 3.3 we need some of the multiplication rules in the representation ring Γ_k. Han and Monsky found it more convenient to work with another basis of Γ_k instead of $\{\delta_i\}$. We set $\lambda_0 = \delta_1$ and $\lambda_i = (-1)^i(\delta_{i+1} - \delta_i)$, so that $\delta_i = \lambda_0 - \lambda_1 + \cdots + (-1)^{i-1}\lambda_{i-1}$. Then, by definition, $D_k(M - N)$ is the coefficient at λ_0 of the decomposition of $M - N$ in the basis $\{\lambda_i\}$.

The following is an easy restatement of [HM93, Theorem 2.5].

Theorem 3.4 (Han–Monsky)**.** *Let k be a field of characteristic $p > 0$ and $0 \le i < j \le p - 1$. Then*

1. *if $i + j \le p - 1$ then $\lambda_i\lambda_j = \sum_{j-i}^{j+i} \lambda_k$.*
2. *if $i + j \ge p$ then $\lambda_i\lambda_j = \lambda_{p-1-i}\lambda_{p-1-j}$.*

Example 3.5. The multiplication matrices of λ_1 and λ_2 in the basis $\{\lambda_i\}_{i=0}^{p-1}$ are given by:

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \text{ and } \lambda_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix}.$$

Combining this with Theorem 3.2 we get the following.

Corollary 3.6. *Let k be a field of characteristic $p > 0$ and $R = k[[x_0, \dots, x_d]]/(x_0^{n_0} + \cdots + x_d^{n_d})$. For a fixed positive integer n, write $p = an + r$ with $0 \le r < n$ and set $b = n - r$. Consider the*

$p \times p$ matrix

$$M_n = \begin{bmatrix} n \cdots n & r & 0 \cdots\cdots 0 \\ \vdots \quad \iddots & b & \ddots \quad \vdots \\ n \quad \iddots & r & \ddots \quad \vdots \\ r \;\; b \;\; r & & \ddots \quad 0 \\ 0 \quad \ddots & & r \;\; b \;\; r \\ \vdots \quad \ddots & r & \iddots \quad n \\ \vdots \quad \ddots & b & \iddots \quad \vdots \\ 0 \cdots\cdots 0 & r & n \cdots n \end{bmatrix},$$

where the “triangular corners” of ‘n’ have length a, the “triangular corners” of ‘0’ have length $p-a-1$, the the middle rectangle has alternating ‘r’ and ‘b’.

Then $\dim_k R/(x_0^p, \dots, x_d^p) = \prod_{i=0}^d ((n_i - r_i)\delta_{a_i} + r_i \delta_{a_i+1})$ *is the* $(1,1)$*-entry of the matrix product* $M_{n_0} M_{n_1} \cdots M_{n_d}$.

Proof. We note that $(n-r)\delta_a + r\delta_{a+1} = n(\lambda_0 - \lambda_1 + \cdots + (-1)^{a-1}\lambda_{a-1}) + r(-1)^a \lambda_a$ and the matrix of the multiplication by this element in the basis of $\{\lambda_i\}_{i=0}^{p-1}$ is M_n with alternating signs:

$$\begin{bmatrix} n & -n \cdots\cdots & (-1)^{a-1}n & (-1)^a r & 0 \cdots\cdots\cdots\cdots 0 \\ -n & n \quad \cdots & (-1)^a r & (-1)^{a+1} b & (-1)^{a+2} r \quad 0 \quad \cdots \quad 0 \\ \vdots & \vdots \quad \vdots & \vdots & \vdots & \vdots \quad \vdots \quad \vdots \quad \vdots \end{bmatrix}.$$

Since the signs only depend on the parity and are identical in the matrices, the product of these matrices is again an alternating sign version of $M_{n_0} M_{n_1} \cdots M_{n_d}$. ☐

We now obtain the matrix form of Corollary 3.3.

Corollary 3.7. *For an integer $a \geq 1$, define the square matrices T_a and N_a of size $(2a+1)$*

$$T_a = \begin{bmatrix} 2 \cdots 2 & 1 & 0 \cdots 0 \\ \vdots \quad \iddots & \vdots & \ddots \quad \vdots \\ 2 \;\; \iddots & \vdots & \ddots \;\; 0 \\ 1 \cdots\cdots & 1 & \cdots\cdots 1 \\ 0 \;\; \ddots & \vdots & \iddots \;\; 2 \\ \vdots \quad \ddots & \vdots & \iddots \quad \vdots \\ 0 \cdots 0 & 1 & 2 \cdots 2 \end{bmatrix} \quad \text{and } N_a = \begin{bmatrix} 0 \cdots 0 & 1 & 0 \cdots 0 \\ \vdots \quad \iddots & \vdots & \ddots \quad \vdots \\ 0 \;\; \iddots & \vdots & \ddots \;\; 0 \\ 1 \cdots\cdots & 1 & \cdots\cdots 1 \\ 0 \;\; \ddots & \vdots & \iddots \;\; 0 \\ \vdots \quad \ddots & \vdots & \iddots \quad \vdots \\ 0 \cdots 0 & 1 & 0 \cdots 0 \end{bmatrix}.$$

Let k be a field of characteristic $p > 2$. If $a = (p-1)/2$, then

$$\mathrm{e_{HK}}(A_{p,d}) = \mathrm{e_{HK}}(k[[x_0, \dots, x_d]]/(x_0^2 + \cdots + x_d^2)) = 1 + \frac{[T_a^{d+1}]_{(1,1)} - p^d}{p^d - [N_a^{d+1}]_{(1,1)}}.$$

3.3. **Main results.** The following lemma is a partial case and a warm-up result for the main proof.

Lemma 3.8. *Fix a positive integer n and define a $(2n+1) \times (2n+1)$-matrix*

$$Z = \begin{bmatrix} 1 \cdots 1 & 0 & 1 \cdots 1 \\ \vdots \quad \iddots & \vdots & \ddots \quad \vdots \\ 1 \;\; \iddots & \vdots & \ddots \;\; 1 \\ 0 \cdots\cdots & 0 & \cdots\cdots 0 \\ 1 \;\; \ddots & \vdots & \iddots \;\; 1 \\ \vdots \quad \ddots & \vdots & \iddots \quad \vdots \\ 1 \cdots 1 & 0 & 1 \cdots 1 \end{bmatrix}.$$

Then $[Z^{d+1}]_{(1,1)} = 2^d |(n-1)F_d|$, where F_d denotes the d-dimensional Fibonacci polytope.

Proof. By the definition of the matrix product, we may write

$$[Z^{d+1}]_{(1,1)} = \sum_{1\le i_1,\dots,i_d\le 2n+1} Z_{(1,i_1)}Z_{(i_1,i_2)}\cdots Z_{(i_d,1)}.$$

Let $S = \{(1,i_1,\dots,i_d,1) \in [1,n]^d \mid i_k + i_{k+1} \le n+1 \text{ for all } k\}$ and observe that the map

$$S \ni (1,i_1,\dots,i_d,1) \mapsto (i_1-1, i_2-1,\dots,i_d-1)$$

is a bijection with the integer points in the $(n-1)$th dilation of the d-dimensional Fibonacci polytope F_d. Thus it suffices to verify that the number of tuples $(1,i_1,\dots,i_d,1)$ such that $Z_{(1,i_1)}\cdots Z_{(i_d,1)} \neq 0$ is $2^d|S|$.

Now, we note that S corresponds to the products $Z_{(1,i_1)}\cdots Z_{(i_d,1)}$ where each entry is in the top-left triangular block. On the other hand, if $Z_{i,i_1} \neq 0$ then it may belong to either the top-left (which S) or bottom-left triangular blocks. These choices are equivalent and, after making either choice, we will similarly have two possible corners for i_2 and so on. ☐

Theorem 3.9. *Let F_d denote the d-dimensional Fibonacci polytope. Consider the matrix T_n defined in Corollary 3.7. Then for any positive integer n, we have*

$$[T_n^{d+1}]_{(1,1)} = (2n+1)^d + 2^d|(n-1)F_d|.$$

Proof. First, we observe that $(2n+1)^d$ is the upper-left entry of $\mathbb{I}^{d+1}$, where $\mathbb{I}$ is a $(2n+1)\times(2n+1)$ square matrix such that $\mathbb{I}_{(i,j)} = 1$ for all i, j. We will work with nonzero products in the expression

$$[T_n^{d+1}]_{(1,1)} = \sum_{1\le i_1,\dots,i_d\le 2n+1} t_{1,i_1}t_{i_1,i_2}\cdots t_{i_d,1}.$$

For convenience, we label the product $t_{1,i_1}t_{i_1,i_2}\cdots t_{i_d,1}$ by the tuple of indices $(1,i_1,\dots,i_d,1)$. Similarly, we use a $(d+2)$-tuple $(j_0,\dots,j_{d+1})$ to label a product of elements of $\mathbb{I}$. In correspondence with the shapes of T_n and N_n, let us introduce 2 regions: the first is the middle rhombus $\mathcal{R}$ and the second, $\mathcal{C}$, consists of the four corners of length n. We will use these regions for the pairs of positive integers.

Suppose that $t_{1,i_1}t_{i_1,i_2}\cdots t_{i_d,1} \neq 0$. Due to the structure of T, the product must be equal to 2^l for some $0 \le l \le d+1$. The number of products (equivalently, tuples) which are equal to 2^{d+1} is counted easily: in this case $t_{i_k,i_{k+1}}$ must always belong to the upper left corner of T_n, so there are $|(n-1)F_d|$ such tuples by the proof of Lemma 3.8.

We now may assume that $l \neq d+1$. We give a procedure P that from a given tuple $(1,i_1,\dots,i_d,1)$ will produce 2^l tuples $(j_0,\dots,j_{d+1})$ (i.e., 2^l elements of $\mathbb{I}^{d+1}$).

Procedure 1. *Let k_0 be the smallest integer k such that $t_{i_k,i_{k+1}} = 1$. We now analyze all $i_0, i_1,\dots,i_{d+1}$ starting from the left:*

(1) *set $j_{d+1} = j_0 = 1$;*
(2) *if $m < k_0$ and $t_{i_m,i_{m+1}} = 2$, then we put two values for j_{m+1}: i_{m+1} and $2n+2-i_{m+1}$;*
(3) *if $m > k_0$ and $t_{i_m,i_{m+1}} = 2$, then we put two values for j_m: i_m and $2n+2-i_m$;*
(4) *if $m > k_0$ and $t_{i_m,i_{m+1}} = 1$ (i.e., $(i_m,i_{m+1}) \in \mathcal{C}$), then $j_m = i_m$.*

Claim 3.9.1. *The map P has the following properties:*

(a) *the image of P covers all tuples $(1,j_1,\dots,j_d,1)$ corresponding to $(d+1)$-fold products of elements in $\mathbb{I}$ with at least one entry in $\mathcal{R}$;*
(b) *if $(1,i_1,\dots,i_d,1) \neq (1,i'_1,\dots,i'_d,1)$ then $P(1,i_1,\dots,i_d,1) \cap P(1,i'_1,\dots,i'_d,1) = \emptyset$.*

Proof. First, observe that for a tuple $(1,j_1,\dots,j_d,1) \in P(1,i_1,\dots,i_d,1)$, we have $(j_m,j_{m+1}) \in \mathcal{R}$ if and only if $(i_m,i_{m+1}) \in \mathcal{R}$ (hence, it belongs to the rhombus of T_n). This is due to the symmetry:

the transformation

$$(i_0, i_1, \ldots, i_m, \ldots, i_{d+1}) \mapsto (i_0, i_1, \ldots, 2n+2-i_m, \ldots, i_{d+1})$$

reflects (i_{m-1}, i_m) horizontally and (i_m, i_{m+1}) vertically, but the two regions, $\mathcal{R}$ and $\mathcal{C}$, are stable under reflections. Note that the distinguished entry $t_{i_{k_0}, i_{k_0+1}} = 1$ is no different, since the values i_{k_0} and i_{k_0+1} are operated with while checking $t_{i_{k_0-1}, i_{k_0}}$ and $t_{i_{k_0+1}, i_{k_0+2}}$. In particular, the image of P contains only tuples with at least one entry in the rhombus $\mathcal{R}$.

Now, in order to prove the two claims it suffices to show that given a tuple $(1, j_1, \ldots, j_d, 1)$ there is a unique tuple $(1, i_1, \ldots, i_d, 1)$ such that $t_{1,i_1} \cdots t_{i_d,1} \neq 0$ and $(1, j_1, \ldots, j_d, 1) \in P(1, i_1, \ldots, i_d, 1)$. This is essentially due to the fact that T_n has two corners filled with 0s, so there is no ambiguity arising in steps (2), (3). Namely, we start by observing that k_0 is the smallest integer k such that $\mathbb{I}_{(j_k, j_{k+1})} \in \mathcal{R}$ due to the preservation of the rhombus $\mathcal{R}$. We know that $t_{i_m, i_{m+1}} = 2$ for all $k_0 > m$. Thus, starting with $m = 1$, we set $i_m = \min\{j_m, 2n+2-j_m\}$ so that the entry (i_{m-1}, i_m) belongs to the top-left corner – this is the inverse to the step (2) of the construction of P. This proceeds until i_{k_0}, but i_{k_0+1} cannot be recovered this way. Instead, we approach it starting from the tail. Set $i_{d+1} = 1$ and proceed inductively reversing steps (3) and (4) of the construction of P. Given i_{m+1}, (j_m, j_{m+1}) determines i_m: if $(j_m, j_{m+1}) \in \mathcal{R}$ then we set $i_m = j_m$; otherwise, we set $i_m = \min\{j_m, 2n+2-j_m\}$ when $i_{m+1} \leq n$ or $i_m = \max\{j_m, 2n+2-j_m\}$ if $i_{m+1} > n$ (so that (i_m, i_{m+1}) is either in the top-left or the bottom-right corners). We proceed this way until reaching $m = k+1$ in which case all values are determined. By the construction, we see now that $(1, j_1, \ldots, j_d, 1) \in P(1, i_1, \ldots, i_d, 1)$. ☐

The claim now shows that $[T_n^{d+1}]_{(1,1)} - 2^{d+1}|(n-1)F_d|$ is the number of all tuples $(1, j_1, \ldots, j_d, 1)$ that contain at least one entry in the rhombus $\mathcal{R}$. It remains to count the number of tuples that have no entry in $\mathcal{R}$. Lemma 3.8 shows that the number of such tuples is $2^d|(n-1)F_d|$. It now follows that

$$(2n+1)^d = [\mathbb{I}^{d+1}]_{1,1} = [T_n^{d+1}]_{(1,1)} - 2^d|(n-1)F_d|.$$

☐

Corollary 3.10. *Let F_d and E_d denote the Fibonacci and extended Fibonacci polytopes. If k is a field of characteristic $p > 2$ and $a = (p-1)/2$, then*

$$\mathrm{e}_{\mathrm{HK}}(k[[x_0, \ldots, x_d]]/(x_0^2 + \cdots + x_d^2)) = 1 + \frac{2^d|(a-1)F_d|}{(2a+1)^d - |aE_{d-2}|}.$$

Proof. By Corollary 3.7,

$$\mathrm{e}_{\mathrm{HK}}(k[[x_0, \ldots, x_d]]/(x_0^2 + \cdots + x_d^2)) = 1 + \frac{[T_a^{d+1}]_{(1,1)} - (2a+1)^d}{(2a+1)^d - [N_a^{d+1}]_{(1,1)}}.$$

By Theorem 3.9 the numerator has the required form. As for the denominator, we interpret

$$[N_a^{d+1}]_{(1,1)} = \sum_{-a \leq i_1, \ldots, i_d \leq a} n_{1,a+1+i_1} \cdots n_{a+1+i_d,1}.$$

Since we need all $n_{i,j} \neq 0$, we must have $i_1 = i_d = 0$ and the remaining indices clearly correspond to the integer points in an a-dilation of the extended Fibonacci polytope. ☐

Example 3.11. Suppose that $p = 3$, so $a = 1$. By Theorem 3.9 and Remark 2.2, we see that

$$\mathrm{e}_{\mathrm{HK}}(k[[x_0, \ldots, x_d]]/(x_0^2 + \cdots + x_d^2)) = 1 + \frac{2^d 3}{3^{d+1} - 2^d + (-1)^d}$$

giving the sequence $2, 3/2, 4/3, 23/19, \ldots$.

Corollary 3.12. *The function* $p \mapsto \mathrm{e}_{\mathrm{HK}}(\mathbb{F}_p[[x_0,\dots,x_d]]/(x_0^2+\cdots+x_d^2))$ *is a rational function of* p *and its numerator and denominator are polynomials of degree* d*.*

Remark 3.13. One might compute this rational function by polynomial interpolation of the first $d+1$ values for the numerator and denominator polynomials (computed by taking powers T_k^{d+1} and N_k^{d+1} for $k = 1,\dots,d+1$). One may also compute the initial terms of the Ehrhart polynomial of the Fibonacci polytope using a recursion[4] in [PZ25, Proposition 3.17] and replace the polynomial interpolation by [PZ25, Remark 3.18]. Alternatively, the Ehrhart polynomial of the Fibonacci polytope can be also computed via the recursive algorithm given by Kreweras ([Kre76], [Slo, A079502]).

Our experiments indicate that the rational function in Corollary 3.12 is actually a function of p^2, i.e., individually the numerator and denominator are degree d polynomials in p that only contain even or odd powers of p depending on the parity of d.

From Corollary 3.10 we recover an unpublished result of Gessel and Monsky ([GM], see also [BLM12, Theorem 4.1]) and improve its convergence estimate.

Corollary 3.14. *Let* k *be a field of characteristic* $p > 2$*. Then*

$$\mathrm{e}_{\mathrm{HK}}(A_{p,d}) := \mathrm{e}_{\mathrm{HK}}(k[[x_0,\dots,x_d]]/(x_0^2+\cdots+x_d^2)) = 1 + \frac{\mathcal{A}_d}{d!} + O(p^{-2}).$$

Proof. Using Corollary 2.9 we can rewrite

$$|(a-1)F_d| = |(\frac{p-3}{2})F_d| = \frac{\mathcal{A}_d}{d!}\frac{p^d}{2^d} + O(p^{d-2}).$$

Thus

$$\frac{2^d|(a-1)F_d|}{(2a+1)^d - |aE_{d-2}|} = \frac{\frac{\mathcal{A}_d}{d!} + O(p^{-2})}{1 + O(p^{-2})} = \frac{\mathcal{A}_d}{d!} + O(p^{-2}).$$

□

Corollary 3.15. *For any field* k *of characteristic* $p > 2$ *and any* $d \geq 0$*, we have*

$$\mathrm{e}_{\mathrm{HK}}(A_{p,d}) > \mathrm{e}_{\mathrm{HK}}(A_{p,d+1}).$$

Proof. We apply the formula in Corollary 3.10. First, observe that $|aF_{d+1}| < (a+1)|aF_d|$. This follows from the definition: we add x_{d+1} such that $x_d + x_{d+1} \leq a$, so there are at most $(a+1)$ options but some are not viable. Hence

$$\frac{2^{d+1}|(a-1)F_{d+1}|}{(2a+1)^{d+1}} < \frac{(2a)2^d|(a-1)F_d|}{(2a+1)^{d+1}} < \frac{2^d|(a-1)F_d|}{(2a+1)^d}.$$

Similarly, $|aE_{d+1}| < (2a+1)|aE_d|$ and, therefore, $\frac{|aE_{d-1}|}{(2a+1)^{d+1}} < \frac{(2a+1)|aE_{d-2}|}{(2a+1)^{d+1}}$. Thus

$$\begin{aligned}\mathrm{e}_{\mathrm{HK}}(A_{p,d+1}) &= 1 + \frac{2^{d+1}|(a-1)F_{d+1}|}{(2a+1)^{d+1} - |aE_{d-1}|} = 1 + \frac{2^{d+1}|(a-1)F_{d+1}|/(2a+1)^{d+1}}{1 - |aE_{d-1}|/(2a+1)^{d+1}} \\ &< 1 + \frac{2^d|(a-1)F_d|/(2a+1)^d}{1 - |aE_{d-2}|/(2a+1)^d} = \mathrm{e}_{\mathrm{HK}}(A_{p,d}).\end{aligned}$$

□

Remark 3.16. The corollary also holds for $p = 2$ by a direct computation performed in [CR]. Notably, this result demonstrates that $\mathrm{e}_{\mathrm{HK}}(A_{p,d-1}) = \mathrm{e}_{\mathrm{HK}}(A_{p,d}/(L)) > \mathrm{e}_{\mathrm{HK}}(A_{p,d})$ for *every* linear form L. Thus, there is no analogue of *superficial* elements for Hilbert–Kunz multiplicity (recall that $\mathrm{e}(R) = \mathrm{e}(R/(L))$ when L is superficial).

---

[4] Is there a similar recursion for the extended Fibonacci polytope?

As the last corollary we strengthen an easy reduction for the Watanabe–Yoshida conjecture observed in [ES05].

Corollary 3.17. *It suffices to prove Conjecture 1 for isolated singularities.*

Proof. Suppose that the Watanabe–Yoshida conjecture holds for isolated singularities. Let $(R, \mathfrak{m})$ be a formally unmixed local ring of positive characteristic $p > 0$. Since Hilbert–Kunz multiplicity does not change upon completion, we may assume that R is complete. Hence its regular locus is open. Let $\mathfrak{p} \neq \mathfrak{m}$ be a minimal prime of its singular locus. Since R is equidimensional, we have $\mathrm{e}_{\mathrm{HK}}(R) \geq \mathrm{e}_{\mathrm{HK}}(R_{\mathfrak{p}}) \geq \mathrm{e}_{\mathrm{HK}}(A_{p,\dim R_{\mathfrak{p}}}) > \mathrm{e}_{\mathrm{HK}}(A_{p,\dim R})$. ☐

3.4. **On the asymptotic Watanabe–Yoshida conjecture.** Trivedi [Tri23] showed that $\mathrm{e}_{\mathrm{HK}}(A_{p,d})$ is eventually decreasing. For this result, we may ignore the lower-dimensional term in the denominator. By Corollary 2.9 it thus would suffice to show that for large d

$$\sum_{m=0}^{d-2} \binom{m}{2} s_d(m) > \mathcal{A}_d \frac{3d^2 - 17d + 25}{24}$$

where $s_d(m)$ is the swap statistic. This can also be restated directly in terms of the alternating maps of Kreweras (Remark 2.7) as $u^n_{n-2} \geq \frac{3n^2-17n+25}{24} u^n_n$. We checked this conjecture for $n \leq 600$ and it seems that $24u^n_{n-2}/u^n_n - 3n^2 + 17n - 25$ is decreasing slowly, perhaps logarithmically, to 0.

An attempt can be made by utilizing that $s_d(m)$ is symmetric in m by [CS23]. For example, when d is even, we may pair the symmetric terms by writing them as $c \pm a$ for $c = d/2 - 1$. Then

$$\sum_{m=0}^{d-2} \binom{m}{2} s_d(m) = \binom{c}{2} s_d(c) + \sum_{a=1}^{c} \left(\binom{c-a}{2} + \binom{c+a}{2} \right) s_d(c+a) = \binom{c}{2} \mathcal{A}_d + \sum_{a=1}^{c} a^2 s_d(c+a).$$

Unfortunately, one still has to analyze the remaining terms since

$$\binom{c}{2} = \binom{d/2-1}{2} = \frac{1}{8}d^2 - \frac{3}{4}d + 1 < \frac{3d^2 - 17d + 25}{24}.$$

For d odd, we similarly extract $\frac{(d-3)^2}{8}\mathcal{A}_d$.

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Appendix A. Asymptotic behavior of a more general family of matrices

Our Theorem 3.9 was guided by experiments on a more general family of $(2k+1) \times (2k+1)$-matrices

$$Q(q,k) = \begin{bmatrix} q & \cdots & q & 1 & 0 & \cdots & 0 \\ \vdots & & \iddots & \vdots & \ddots & & \vdots \\ q & \iddots & & \vdots & & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & \cdots & \cdots & 1 \\ 0 & \ddots & & \vdots & & \iddots & q \\ \vdots & & \ddots & \vdots & \iddots & & \vdots \\ 0 & \cdots & 0 & 1 & q & \cdots & q \end{bmatrix}.$$

As in Theorem 3.9, $[Q(q,k)^d]_{1,1}$ is a polynomial in q, k and, motivated by Corollary 3.7, we would like to compute its leading coefficient $\lim_{k\to\infty} \frac{[Q(q,k)^{d+1}]_{(1,1)}}{(2k+1)^d}$.

First, we need some combinatorial preliminaries. Following Chebikin [Che08] we introduce the following definition.

Definition A.1. The index i is an alternating descent of a permutation π if either i is odd and $\pi(i) > \pi(i+1)$, or i is even and $\pi(i) < \pi(i+1)$.

We will use $A(n,k)$ to denote the number of permutations of $[n]$ having k alternating descents.

For convenience, we will also define alternating descents for a general chain of inequalities $x_1 < x_2 < \cdots$. We now give a key lemma.

Lemma A.2. *Consider a partition of the hypercube* $[0,1]^n$ *by the hyperplanes* $x_i + x_{i+1} = 1$ *for* $i = 1, \ldots, n-1$. *The total volume of all regions that involve exactly* k *'$>$' signs is equal to* $A(n,k)/n!$.

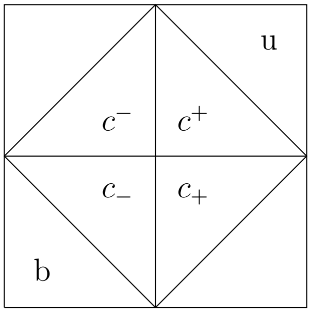
Proof. The affine transformation $y_i = x_i$ for i odd and $y_i = 1 - x_i$ will swap inequalities at the even place. Hence the number of the original '$>$' is the number of alternating descents on the new variables y_i. This transformation does not change the volume. Now for each region with the prescribed number of '$>$' corresponds a partial order on the variables y_i. We may now triangulate the region by extending the partial order on y_i to a total order. Each total order then naturally corresponds to a permutation, by writing $i_1, \ldots, i_n$ so that $y_{i_1} < y_{i_2} < \cdots$, with the same signature as the original region. The assertion now follows. □

We will now use that

$$[Q(q,k)^{d+1}]_{1,1} = \sum_{1 \le i_1, \ldots, i_d \le k+1} Q_{1,i_1} Q_{i_1,i_2} \cdots Q_{i_{d-2},i_{d-1}} Q_{i_{d-1},1}$$

and will plot the indices $(i_1, \ldots, i_{d-1})$ as integer points on a cube $[1, 2k+1]^{d-1}$. The value of this product is thus a function on the cube. After translating, this becomes a question about integer points in the k-dilations of certain regions of the cube $[-1,1]^{d-1}$.

The value of each $Q_{i_c,i_{c+1}}$ is given by simple inequalities depending on what region of the matrix Q it is in. Due to the matrix multiplication rules, two consecutive elements may only be in the same or in the adjacent regions, i.e., cannot move from the top-left to the bottom-right corner immediately. This motivated the following regions – for convenience, we will mirror the picture to have more natural signs.



Since we will be working with a dilation of $[-1,1]^n$, we will similarly define regions on each pair of consecutive coordinates (x_i, x_{i+1}).

Thus, we may use words in the alphabet $E = \{b, u, c^+, c^-, c_+, c_-\}$ to encode regions of the cube $[-1,1]^n$ corresponding to nonzero products. It is easy to say when a word defines a non-empty region:

1. u, c^+, c^- can be only followed by u, c^+, c_+
2. b, c_+ and c_- can be only followed by b, c^-, c_-,
3. the first letter must be either u, c^+, or c^- and the last letter must be either u, c^+, or c_+ (this rule corresponds to the fact that $Q_{1,i_1} \neq 0$ and $Q_{i_n,1} \neq 0$).

Let W_n denote the set of n-letter words on this alphabet. We define the signature of the word, $\sigma(w)$, to be the number of occurrences of letters of type $c^{\pm}$ or $c_{\pm}$.

Example A.3. The word uc^+ describes the following region in the cube $[-1,1]^3$: $x_1 + x_2 > 1$ (from 'u'), $x_2 + x_3 \leq 1$ (from 'c^+'), which, after simplifications, is given by $x_1 > 1 - x_2 \leq x_3$.

Theorem A.4. *Let $A(n,k)$ be the number of permutations of $[n]$ having k alternating descents. Then*

$$\sum_{w \in W_n, \sigma(w)=k} \operatorname{vol}(w) = \frac{1}{(n+1)!} 2^{\max\{0,k-1\}} A(n+1,k).$$

Proof. We want to express the volume of the region given by a word in W_n. The substitution $x_i' \mapsto -x_i$ does not change the volume of a region or the permutation defined by the word. We define a map ϕ from W_n to n-letter words on the alphabet $\{g,l\}$ by sending $b, u \mapsto g$ and $c^{\pm}, c_{\mp} \to l$. Of course, a word on $\{g,l\}$ can be considered as a region given by $[0,1]^n$ by u, c^+, in this way ϕ does not change the volume. We may identify to a word in $\{g,l\}^n$ a chain of inequalities given by $x_i + x_{i+1} > 1$ for g and $x_i + x_{i+1} < 1$ for l. Thus the volume of the regions with k letters g is $A(n,k)/n!$ by Lemma A.2.

Now, given a word $w \in \{g,l\}^n$ with k letters g it remains to show that $|\phi^{-1}(w)| = 2^{\max\{0,k-1\}}$. For $k=0$, this is trivial as the word $u \dots u$ is the only mapping to $g \dots g$. For $k=1$ and 'l' at the ith spot, we note that a preceding 'g' or the boundary condition, when $i=1$, requires that $x_i \geq 0$ and, similarly, the succeeding letter will force $x_{i+1} \geq 0$. Thus we must have c^+ in the unique lift.

Now, we may assume that $k \geq 2$ and use induction on the length of the word to describe the number of pieces. First, if we extend a word w to wg, then any option for $v \in \phi^{-1}(w)$ can be only extended to vu due to the boundary conditions. Note that this is a valid extension: if $|w| = n$, then $x_n \geq 0$ was a boundary condition which is now required by 'u'. There cannot be other elements in $\phi^{-1}(wg)$ for the same reason. Thus $|\phi^{-1}(w)| = |\phi^{-1}(wg)|$ does not change in such a case.

Second, suppose that we extend a word w of signature $k-1$ and length n to wl. Suppose w ends with 'g' and write it as $w = vg \dots g$. Then for any lift $\tilde{v}$ of v we have two lifts of w: $\tilde{v}u \dots uc^+$ and $\tilde{v}b \dots bc^-$. Note that all lifts of w have the form $\tilde{v}u \dots u$ due to the boundary conditions, so we must have $2|\phi^{-1}(w)| = |\phi^{-1}(wl)|$.

In the last case, when w is ending on 'l' itself, say, $w = vl$, then for any lift of w has the form $\tilde{v}c^+$ or $\tilde{v}c^-$ due to the boundary condition (the actual sign may depend on $\tilde{v}$). However, in wl it is now not affected by the boundary condition and, for example, the lift $\tilde{v}c^-$ will be extended by both $\tilde{v}c^-c^+$ and $\tilde{v}c_-c^-$. Thus we have $2|\phi^{-1}(w)| = |\phi^{-1}(wl)|$ again. ☐

Recall that the alternating Eulerian polynomials are defined by $A_n(x) = \sum_{k=0}^{n-1} A(n,k)x^k$. Chebikin [Che08] computed the exponential generating function

$$\sum_{n \geq 1} A_n(x) \frac{z^n}{n!} = \frac{\sec((1-x)z) + \tan((1-x)z) - 1}{1 - x\sec((1-x)z) - x\tan((1-x)z)}.$$

Corollary A.5. *The leading term of $k \mapsto [Q(q,k)^{n+1}]_{1,1}$ is $\frac{q^2}{2\,n!}\big(A(n,0) + A_n(2q)\big)\,k^n$.*

Proof.

$$\sum_{w \in W_n} \operatorname{vol}(w) = \frac{1}{(n+1)!} \sum_{k=0}^{n} 2^{\max\{0,k-1\}} A(n+1,k) = \frac{1}{2(n+1)!}(A_{n+1}(2) + A(n+1,0)).$$

☐

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