On hereditarily small sets in ZF

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Abstract

We show in ZF (the usual set theory without Choice) that for any set X, the collection of sets Y such that each element of the transitive closure of $\{Y\}$ is strictly smaller in size than X (the collection of sets hereditarily smaller than X) is a set. This result has been shown by Jech in the case $X = \omega_1$ (where the collection under consideration is the set of hereditarily countable sets).

In [2], Thomas Jech showed that the collection of hereditarily countable sets exists in ZF by showing that it has ordinal rank $\leq \omega_2$. In [1], Thomas Forster observes in effect that this result can be extended to show the existence of $H(\kappa)$ for any aleph κ: the result he actually proves is in the context of NF and stated differently, but the reasoning is transferable. In this note, we demonstrate the existence for any set X of the set $H(|X|)$ of all sets which are hereditarily of size less than $|X|$, by generalizing Jech's technique of establishing a bound on the ranks of the elements of this collection. The generalization is not altogether trivial. We give two different proofs, one being the way we first established the result, and one emulating Jech's original argument for the existence of $H(\aleph_1)$ more closely.

We work in ZF, in which we have all the usual axioms of set theory except Choice.

Definition (transitive closure): We define $\mathrm{TC}(A)$ for any set A as the intersection of all transitive sets which include A as a subset (so $A \notin TC(A)$; when we want to include A we consider $\mathrm{TC}(\{A\})$).

Definition (ordinals, order types): Ordinals for us are the usual von Neumann ordinals. We will take well-orderings to be non-strict and define the order type of a well-ordering $\leq$ to be the unique ordinal α such that the natural non-strict order on α is isomorphic to $\leq$. We define the α'th element of a set of ordinals A as the image of α under the isomorphism from the natural order on the order type of A to the natural order on A.

Definition: For any set x, let $\rho(x)$ denote the rank of x, an ordinal defined by transfinite recursion as the supremum of ordinals $\rho(y)+1$ for $y \in x$.

Definition (cardinality): We define the relation $X \sim Y$ of equinumerousness as usual as obtaining iff there is a bijection from X to Y. We cannot define $|X|$ as the initial ordinal equinumerous with X as we do not have choice, so we define it as the collection of sets of minimal rank equinumerous with X, following [3]. Relations of order on cardinals are defined as usual.

Definition (the collection of sets hereditarily smaller than X): For any set X, we define $H(|X|)$ as $\{A \mid (\forall B \in \mathrm{TC}(\{A\})).|B| < |X|)\}$.

Theorem: For any set X, $H(|X|)$ is a set.

Proof: Fix a set X.

Let α be the collection of all order types of well-orderings of partitions of sets Y with $|Y| < |X|$. We can without losing any order types restrict Y to be a subset of X, in which case the relevant collection of well-orderings of partitions becomes a set, and it is clear that the collection of order types of such well-orderings is a set, and indeed an initial ordinal.

Observe that for any set S with $|S| < |X|$, the collection $\rho``S$ of ranks of elements of $S$ is the same size as the partition of $S$ into sets $\{s \in S \mid \rho(s) = \beta\}$ and supports a well-ordering, and so $|\rho``S| < |\alpha|$.

Lemma (and implicit definition): For any initial ordinal β, we observe that there is a least initial ordinal $\chi(\beta)$ such that the order type of any union of $< |X|$ sets of ordinals each with order type $< \beta$ is less than $\chi(\beta)$.

Proof of Lemma: Let C be an injection from a set Y with $|Y| < |X|$ to sets of ordinals each with order type (in the natural order) $< \beta$.

We say that an element a of $\bigcup \mathtt{rng}(C)$ has (y, γ) as a code iff a is the γ'th element of $C(y)$: let $\mathtt{codes}(a)$ be the collection of all codes for a. Now observe that the natural order on $\bigcup \mathtt{rng}(C)$ is isomorphic to an order on the collection of sets $\mathtt{codes}(a)$, which is a partition of a subset of $Y \times \beta$, so all such order types are bounded above by the collection of order types of well-orderings of partitions of subsets of sets $Y \times \beta$ with $|Y| < |X|$ (where we can without losing order types assume that Y is a subset of X, and so see clearly that this is a set and an ordinal).

Define a sequence of sets of ordinals F_i^S for each set $S \in H(|X|)$. The set F_0^S is ρ“S, the set of ranks of elements of S. The set F_{n+1}^S is defined as $\bigcup_{T \in S} F_n^T$. It should be clear that F_n^S is the set of all ranks of elements of $\bigcup^n S$. Clearly this recursive definition succeeds.

We prove by induction on n that the order type of the set F_n^S for any $S \in H(|X|)$ is less than $\chi^n(\alpha)$.

For $n = 0$ this has already been observed: the order type of $F_0^S = \rho$“S is less than $\chi^0(\alpha) = \alpha$.

Suppose that the result is true for F_k^S for all $S \in H(|X|)$. Fix an S. F_{k+1}^S is the union for $T \in S$ of F_k^T, and so the union of $< |X|$ sets of ordinals of order type $< \chi^k(\alpha)$ by inductive hypothesis, and so of order type $< \chi^{k+1}(\alpha)$.

The union of all F_n^S's for a fixed S is the collection of all ranks of elements of TC(S). The collection of all ranks of elements of TC(S) is an ordinal and so exactly the rank of S (to see this, suppose that $\gamma < \beta \in \rho$“TC(S) and $\gamma \notin \rho$“TC(S); for any δ with $\gamma < \delta \leq \beta$, define $f^i(\delta)$ as the least rank $> \gamma$ of an element of a TC(T) where $T \in$ TC(S) is of rank δ; clearly $f(\delta) < \delta$; the sequence of $f^i(\beta)$'s is strictly decreasing and can only terminate at $\gamma + 1$, and the presence of an element of rank $\gamma + 1$ in $TC(S)$ implies the presence of an element of rank γ in $TC(S)$). We define $\chi^\omega(\alpha)$ as the supremum of the ordinals $\chi^i(\alpha)$, and observe that the union of the F_n^S's is the union of a countable set of sets of ordinals each of order type less than $\chi^\omega(\alpha)$, and so bounded above in size by the set of order types of well-orderings of partitions of subsets of $\omega \times \chi^\omega(\alpha)$, which does not depend on the choice of S, whence we can see that we have established a uniform bound on the rank of elements of $H(|X|)$, so $H(|X|)$ is a subcollection of a rank of the cumulative hierarchy and thus a set.

We give a second proof, emulating Jech's argument more closely.

Alternative Proof: Fix a set X and define α as above. For each $S \in H(|X|)$ we define a partial function G_S sending finite sequences of ordinals less than α to ranks of elements of $\mathrm{TC}(S)$: $G_S((\beta_0))$ is defined as the β_0'th element of $\rho``S$, if there is one, and otherwise is undefined; $G_S((\beta_0, \ldots, \beta_{n-1}, \beta_n))$ is defined as the β_n'th member of $\{G_T((\beta_0, \ldots, \beta_{n-1})) \mid T \in S\}$, if there is one, and otherwise is undefined. This induction clearly works, as any value of a G_S function is determined by values of G_T's for T with lower rank at sequences of smaller length. We claim that any G_S maps a subset of the set of sequences of length $n \geq 1$ of ordinals less than α onto the set of ranks of elements of $\bigcup^{n-1} S$. We prove this by induction. For $n = 1$ this has already been observed: the order type of $\rho``S$ is less than α, for any S. Suppose that this is true for $n = k$ and any set S. $G_S((\beta_0, \ldots, \beta_{k-1}, \beta_k))$ is defined as the β_k'th element of $\{G_T((\beta_0, \ldots, \beta_{k-1})) \mid T \in S\}$. This set is a collection of ordinals of size less than α, because it is the same length as a well-ordering of a partition of S by value of $G_T((\beta_0, \ldots, \beta_{k-1}))$. Thus the range of G_S includes each $\{G_T((\beta_0, \ldots, \beta_{k-1})) \mid T \in S\}$, and by inductive hypothesis thus includes the ranks of every element of $\bigcup^k S$, because we know by inductive hypothesis that the sets $\{G_T((\beta_0, \ldots, \beta_{k-1})) \mid T \in S\}$ contain all ranks of elements of $\bigcup^{k-1} T$, and $\bigcup_{T \in S} \bigcup^{k-1} T = \bigcup^k S$. This shows that a subset of the collection of all finite sequences of elements of α can be mapped onto $\rho``\mathrm{TC}(S)$, and since $\rho``\mathrm{TC}(S)$ is the ordinal $\rho(S)$, the order type of a well-ordering of the finite sequences of elements of α is an upper bound of size not dependent on S on $\rho(S)$, so once again we have shown that the rank of the elements of $H(|X|)$ is bounded and so $H(|X|)$ is a set.

References

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[2] Jech, Thomas, "On Hereditarily Countable Sets", *Journal of Symbolic Logic*, Vol. 47, No. 1 (Mar., 1982), pp. 43-47

[3] Scott, Dana, "Definitions by abstraction in axiomatic set theory", *Bull. Amer. Math. Soc.*, vol. 61, p. 442, 1955.