Identity of indiscernibles in Quine's "New Foundations" and related theories

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June 10, 2020

We demonstrate that equality is dispensable as a logical primitive in set theory with stratified comprehension for open sentences and without any extensionality axiom, and so in all nontrivial fragments of Quine's system "New Foundations" of [4].

We work in a first order language in which the only predicate is $\in$. The results apply to languages with additional predicates (and in fact to languages with function symbols) if the notion of stratification is extended appropriately.

Definition: A function σ from variables to integers is called a *stratification of* ϕ iff for each atomic subformula '$x \in y$' of ϕ we have $\sigma(`x') + 1 = \sigma(`y')$. (If equality were present as a primitive, we would also require for each atomic subformula '$x = y$' that $\sigma(`x') = \sigma(`y')$.) A formula ϕ is said to be *stratified* iff there is a stratification of ϕ.

The point of this definition is that a formula is stratified iff there is a way to assign types to the variables in the formula which makes sense in the simple theory of types of Russell (as simplified by Ramsey). The following axiom scheme (introduced by Quine in his paper [4]) is known to be consistent; it has the same paradox-averting powers as the simple theory of types while avoiding the syntactical encumbrances of type theory:

Axiom Scheme of Stratified Comprehension: If ϕ is a stratified formula in which the variable A is not free, $(\exists A.(\forall x.x \in A \leftrightarrow \phi))$. Informally, $\{x \mid \phi\}$ exists when ϕ is stratified (in the absence of extensionality assumptions, there may be more than one such "set"; any use of the notation $\{x \mid \phi\}$ in this paper refers to some witness to the appropriate instance of the comprehension axiom, and does not implicitly assume the existence of a unique or even a canonical witness).

Definition: A stratification of the abstract $\{x \mid \phi\}$ is defined as a stratification of the formula $(\forall x.x \in A \leftrightarrow \phi)$.

Quine proposed this axiom as part of the theory *NF* ("New Foundations") introduced in [4]. The only other axiom of *NF* is the axiom of extensionality.

He suggested the same definition of equality which we will give below, but he did not address the question as to whether the defined notion could be shown to have the correct logical properties; in his later [5] he gives a different definition of equality (from which the logical properties of equality could not be derived) then postulates an axiom from which substitutivity of identity can be derived.

The consistency of *NF* remains an open question. The axiom scheme of stratified comprehension is known to be consistent, because Jensen showed in [3] that *NFU*, the theory in which extensionality is restricted to nonempty sets (or, equivalently, in which extensionality is restricted to sets and non-sets with no elements are permitted) is consistent. Marcel Crabbé showed in [1] that *SF*, the theory whose axioms consist entirely of the axiom scheme of stratified comprehension, interprets *NFU*.

The axiom scheme of stratified comprehension can be liberalized. There is no need for variables free in an instance of the stratified comprehension scheme to satisfy stratification restrictions, since any instance of naive comprehension which fails to be stratified for this reason is actually a substitution instance of an instance of stratified comprehension. This motivates the following:

Definition: A function σ from variables to integers is called a *weak stratification of* ϕ iff for each atomic subformula '$x \in y$' of ϕ in which 'x' and 'y' are both bound we have $\sigma(\text{'}x\text{'}) + 1 = \sigma(\text{'}y\text{'})$. (If equality were present as a primitive, we would also require for each atomic subformula '$x = y$' in which 'x' and 'y' were bound that $\sigma(\text{'}x\text{'}) = \sigma(\text{'}y\text{'})$.) A formula ϕ is said to be *weakly stratified* iff there is a weak stratification of ϕ.

Observation: It is a consequence of the Axiom Scheme of Stratified Comprehension that $(\exists A.(\forall x.x \in A \leftrightarrow \phi))$ holds whenever ϕ is a formula in which A is not free and $(\forall x.x \in A \leftrightarrow \phi)$ is weakly stratified (notice that it is not sufficient for ϕ to be weakly stratified; free occurrences of 'x' in ϕ must be subject to stratification restrictions as well as the bound variables of ϕ).

Definition: We say that a set abstract $\{x \mid \phi\}$ is weakly stratified exactly when the formula $(\forall x.x \in A \leftrightarrow \phi)$ is weakly stratified.

As we stated above, we will not work in the full theory *NF* or even in *SF*. We adopt the version of the axiom scheme of stratified comprehension restricted to quantifier-free formulas ϕ. This axiom scheme has been considered by Forster in [2]; the theory with extensionality and this scheme (and with equality as a logical primitive) is there called *NF0*, so our working theory may be called *NF0U*. We remind the reader that we do not adopt equality as a logical primitive.

We define equality as follows:

Definition: We define $x = y$ as an abbreviation for the formula $(\forall z.x \in z \leftrightarrow y \in z)$.

Definition: For any formula ϕ and variables x and y, we define $\phi[y/x]$ as the result of substituting y for all free occurrences of x in ϕ. We define

$\phi[u,v/x,y]$ as the result of simultaneous substitution of u and v for free occurrences of x and y, respectively.

The definition can be seen as motivated by Leibniz's principle of the identity of indiscernibles. The essential property of equality is that $x = y$ should imply $\phi[x/a] \leftrightarrow \phi[y/a]$. Our definition guarantees this when there is a set $\{a \mid \phi\}$, but we are only provided with such a set when the abstract $\{a \mid \phi\}$ is weakly stratified. It is not immediately obvious that objects which are distinguishable by their unstratified properties must be separated by sets. It may not be obvious – but it does turn out to be true!

Lemma: If ϕ is a quantifier-free formula, then $x = y \rightarrow \phi[x/a] \leftrightarrow \phi[y/a]$.

Proof of Lemma: If $\{a \mid \phi\}$ is weakly stratified, this is obvious. The only way in which this formula can fail to be weakly stratified is if ϕ contains occurrences of $a \in a$. We define a modified formula ϕ', obtained by replacing each occurrence of a with one of the new variables (not occurring in ϕ) a_0 or a_1, in such a way that each subformula $a \in a$ becomes $a_0 \in a_1$. It is clear that the abstracts $\{a_0 \mid \phi'\}$ and $\{a_1 \mid \phi'\}$ are weakly stratified. Now $\phi[x/a] = \phi'[x,x/a_0,a_1] \leftrightarrow x \in \{a_1 \mid \phi'[x/a_0]\} \leftrightarrow y \in \{a_1 \mid \phi'[x/a_0]\} \leftrightarrow \phi[x,y/a_0,a_1] \leftrightarrow x \in \{a_0 \mid \phi'[y/a_1]\} \leftrightarrow y \in \{a_0 \mid \phi'[y/a_1]\} \leftrightarrow \phi'[y,y/a_0,a_1] = \phi[y/a]$ follows by repeated applications of the hypothesis $x = y$ (as defined above) and the axiom scheme of stratified comprehension for open formulas. The proof of the Lemma is complete.

Theorem: For any formula ϕ, $x = y \rightarrow \phi[x/a] \leftrightarrow \phi[y/a]$.

Suppose without loss of generality that ϕ is given in prenex normal form, and proceed by induction on the number of leading quantifiers. The basis case of the induction is handled by the Lemma. We now present the induction step. Suppose that $\phi = (\exists z.\psi)$. We prove $\phi[x/a] \rightarrow \phi[y/a]$. Suppose that $\phi[x/a] = (\exists z.\psi[x/a])$ is true. Select a witness w which makes this existential statement true. Apply the inductive hypothesis to $\psi[x,w/a,z]$ to conclude that $\psi[y,w/a,z]$ holds, and so that $\phi[y/a]$ holds. The same argument with the roles of x and y interchanged establishes the converse implication. Since universal sentences are negations of existential sentences, this is all that is needed. The proof of the Theorem is complete.

It should be clear that our claim of the logical adequacy of the defined notion of equality has been established.

This result applies to almost all of the known consistent fragments of *NF*. It is clear that the result applies to *NF* itself and to *NFU*. We claim that it also applies to the theory NF_3 of Grishin (extensionality plus stratified comprehension for formulas ϕ with stratifications with range having only three elements) and the theory *NFP* of Crabbé (predicative *NF*: extensionality plus stratified comprehension for those abstracts $\{x \mid \phi\}$ with stratifications σ whose range includes nothing larger than $\sigma(‘x’) + 1$ and which map only free variables to

$\sigma(`x') + 1)$. Both of these theories are known to be consistent and admit interesting known-to-be-consistent extensions. This claim can be established by a careful consideration of weakly stratified abstracts $\{x \mid \phi\}$ with ϕ quantifier-free. Any such abstract is a substitution instance of an abstract $\{x \mid \phi'\}$ in which no free variable appears more than once. Such an abstract admits a stratification with range $\{0, 1, 2\}$ obtained by assigning x type 1, then assigning each free variable which occurs in an atomic subformula with x type 0 or 2 as the syntax requires, and assigning all remaining free variables type 0. The resulting stratification satisfies the conditions placed on stratifications by both NF_3 and *NFP*.

The result appears to apply as well to the intuitionistic version of *NF* (studied recently by Dzierzgowski; not known to be consistent nor known to be as strong as classical *NF*). Prenex normal form is not available, so a more complex structural induction on formulas is needed, with clauses for propositional connectives as well as quantifiers, which appears to succeed.

References

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[4] W. V. O. Quine, "New Foundations for Mathematical Logic," *American Mathematical Monthly*, 44 (1937), pp. 70–80.

[5] W. V. O. Quine, *Mathematical Logic*, second edition, Harvard, 1951.