A triangle for calculating the unsigned Genocchi numbers (of the first kind) A110501.

Peter Bala, April 24, 2017

The first few Genocchi numbers A110501 may be rapidly calculated by means of the following lower triangular array:

1										
↓										
1	— x1 —>	**1**								
↓		↓								
1	— x2 —>	**3**	— x1 —>	**3**						
↓		↓		↓						
1	— x4 —>	**7**	— x2 —>	**17**	— x1 —>	**17**				
↓		↓		↓		↓				
1	— x6 —>	**13**	— x4 —>	**69**	— x2 —>	**155**	— x1 —>	**155**		
↓		↓		↓		↓		↓		
1	— x9 —>	**22**	— x6 —>	**201**	— x4 —>	**959**	— x2 —>	**2073**	— x1 —>	**2073**
⋮										

The multiplication factors on the horizontal lines $[1, 2, 4, 6, 9, ...]$ come from the sequence $[1, 2, 4, ..., n(n-1), n^2, ...]$.

Background Theory.

Recall a 2-Motzkin path is a lattice path in the integer plane $\mathbb{Z} \times \mathbb{Z}$ from $(0,0)$ to (n,n) using horizontal steps $H = (2,0)$, up steps $\mathrm{U} = (1,1)$ and down steps $D = (1,-1)$, and which never passes below the $x-$axis. To every 2-Motzkin path, one can associate a "weight" keeping track of the height of the up, down and horizontal steps. A down step is of the form $(1,-1)$, so it goes from a lattice point (x,y) to the point $(x+1, y-1)$ for some integers x,y. We say that this down step occurs at height y. To each down step at height y, we assign the weight d_y. In the same way, we assign the weight h_y to a horizontal step between the points (x,y) and $(x+2,y)$. We assign the weight 1 to any up step.

Oste and van der Jeugt [1, Section 7] show that the continued fraction

$$\cfrac{1}{1 - zh_0 - \cfrac{zd_1}{1 - zh_1 - \cfrac{zd_2}{1 - zh_2 - \cfrac{zd_3}{1 - zh_3 - \cdots}}}} \tag{1}$$

is the generating function for 2-Motzkin paths weighted in this manner. This combinatorial interpretation allows one to rapidly calculate the terms of a sequence whose generating function can be expressed as a continued fraction of the form (1). The results are conveniently displayed in the form of a triangle.

In the particular case of the unsigned Genocchi numbers, the generating function can be expressed as the continued fraction

$$\cfrac{z}{1-\cfrac{z}{1-\cfrac{2z}{1-\cfrac{4z}{1-\cfrac{6z}{1-\cfrac{9z}{1-\cdots}}}}}}$$

where the coefficents of the partial numerators are given by the sequence $[1, 1, 2, 4,$ $..., n(n-1), n^2, ...]$. Therefore the Genocchi numbers have a combinatorial interpretation in terms of weighted 2-Motzkin paths with down steps having weights of the form $n(n-1)$ or n^2 and with no horizontal steps (i.e. weighted Dyck paths). Note that in order to have our calculation triangle read down the page, we have reflected the diagram of 2-Motzkin paths about the $x-$axis and then rotated the diagram clockwise, so that the horizontal weights h_i now appear along the diagonals of the lower triangular array and the down weights d_i now appear along the rows of the array.

References

[1] R. Oste and J. Van der Jeugt, Motzkin paths, Motzkin polynomials and recurrence relations, Electronic Journal of Combinatorics 22(2) (2015), #P2.8. Section 7