Old and New Problems From 55 Years of OEIS

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Experimental Math Seminar, Oct 10 2019

(Additional notes added Oct 12 2019 - see next slide)

Additional notes added after the talk

1. Concerning the first section of the talk, Bradley Klee observed (October 11 2019) that since the sides of an equilateral triangle of area 1 have length 1.51967…, which is greater than the diagonal of a square of area 1, all three edges of the triangle have to be cut, and so a two-piece dissection of a triangle to a square is impossible. (The existence of a 3-piece dissection is still open.)

2. In the last section, when I thanked some of the people who keep the OEIS running, I accidentally omitted some key names, which have now been added.

3. After the talk, two people asked what is involved in being an editor. Answer: look at the page on the wiki called https://oeis.org/wiki/Instructions_For_Associate_Editors

Outline

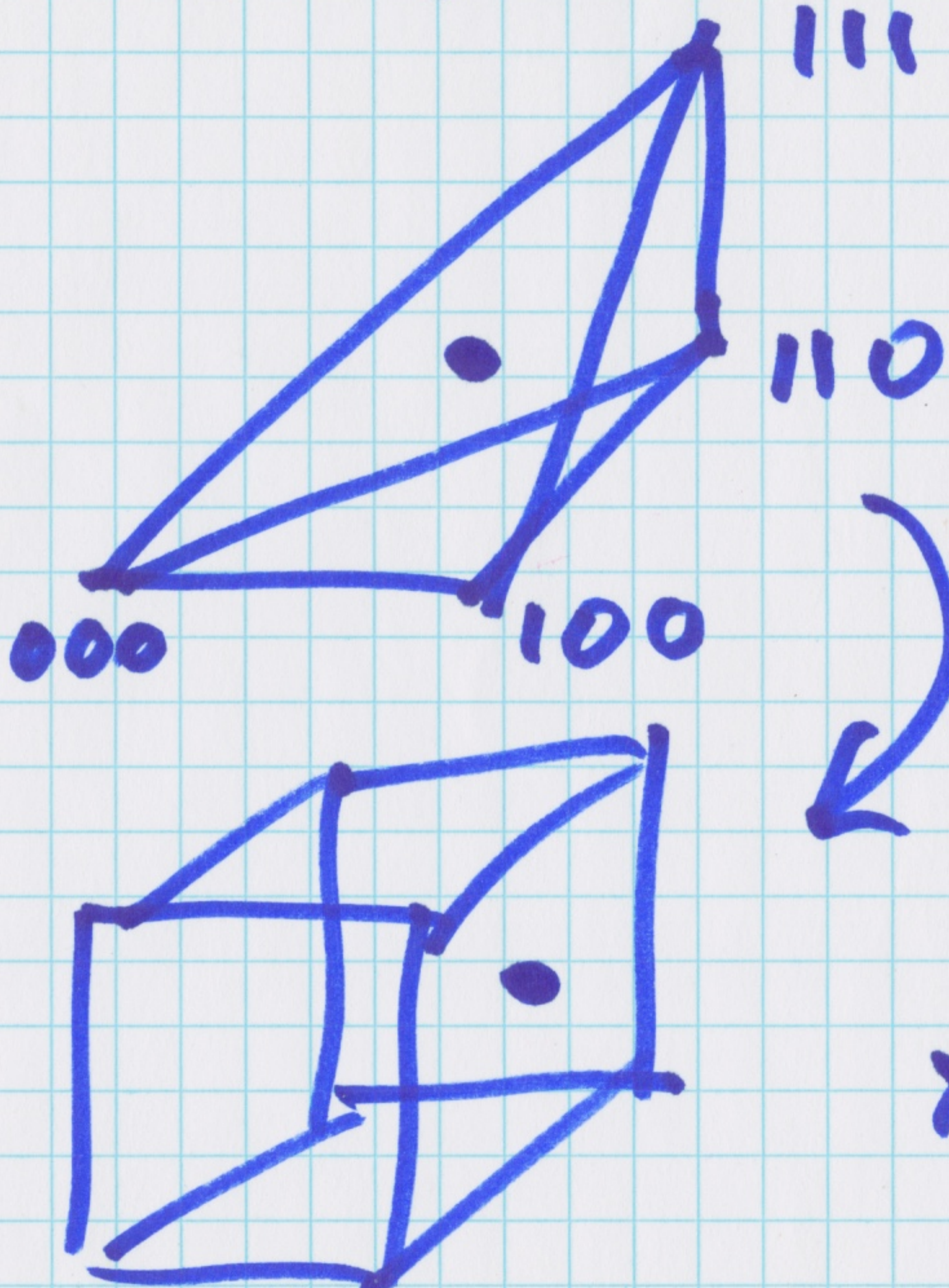
- Dissections
- Roots of theta series; kissing numbers
- The Recaman Hypothesis
- Forest Fire; covering [1..n] with GPs
- All differences are distinct
- A Curious Property of 909
- Éric Angelini
- The OEIS at 55 and the future

Dissections

(Geometrical)

$$\begin{array}{cccccccc} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ & & i & & j & k & & \end{array}$$

$$0 \le i < j < k \le n-1$$



DISSECT

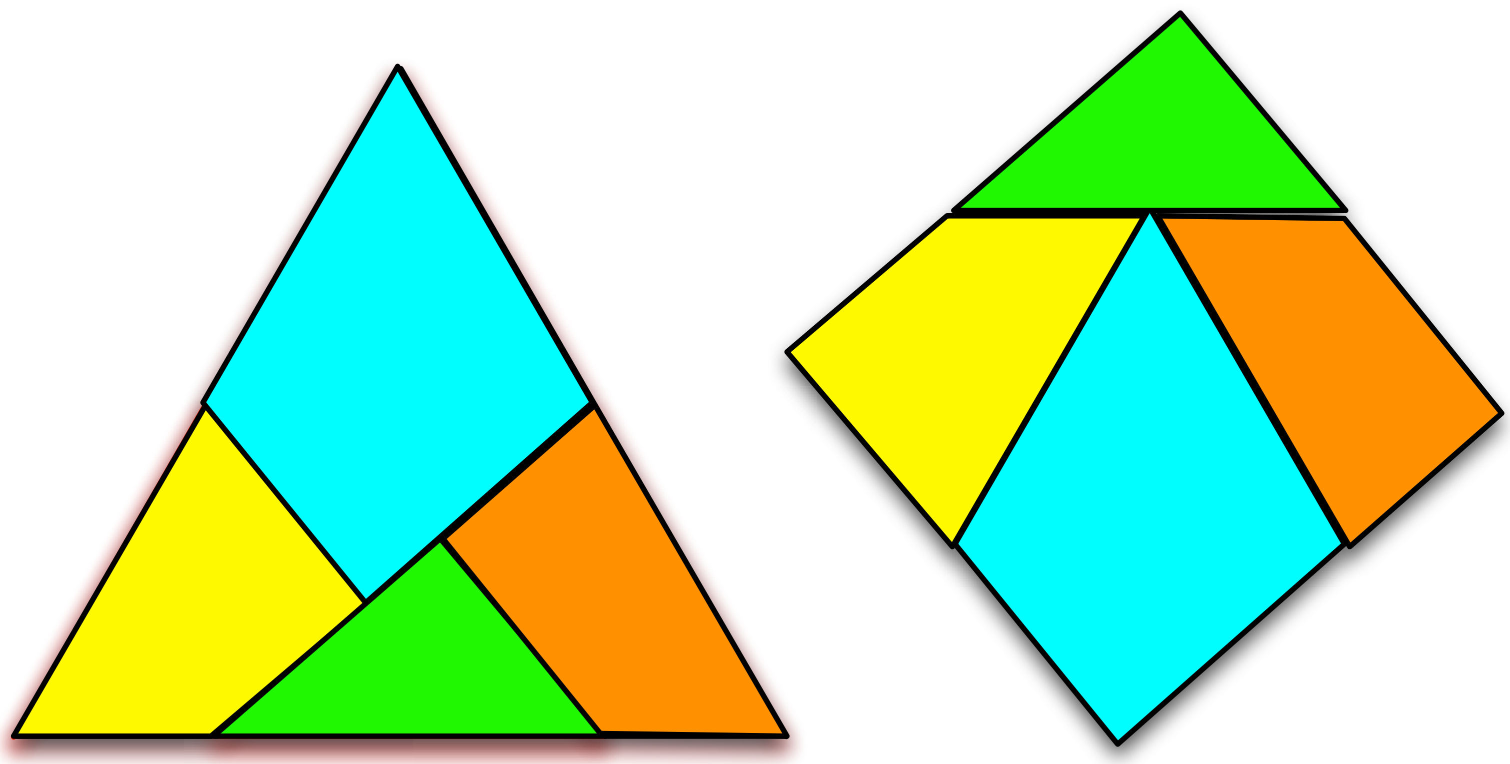
$$x, y, z \in [0,1]^3$$

NJAS, V. Vaishampayan, arXiv:0710.3857

Show there is no 3-piece dissection of an equilateral triangle into a square

Min. no. of pieces for dissecting n-gon into square?

a(3) = 4 ?



4?, 1, 6?, 5?, 7?, 5?, 9?, 7?

(A110312)

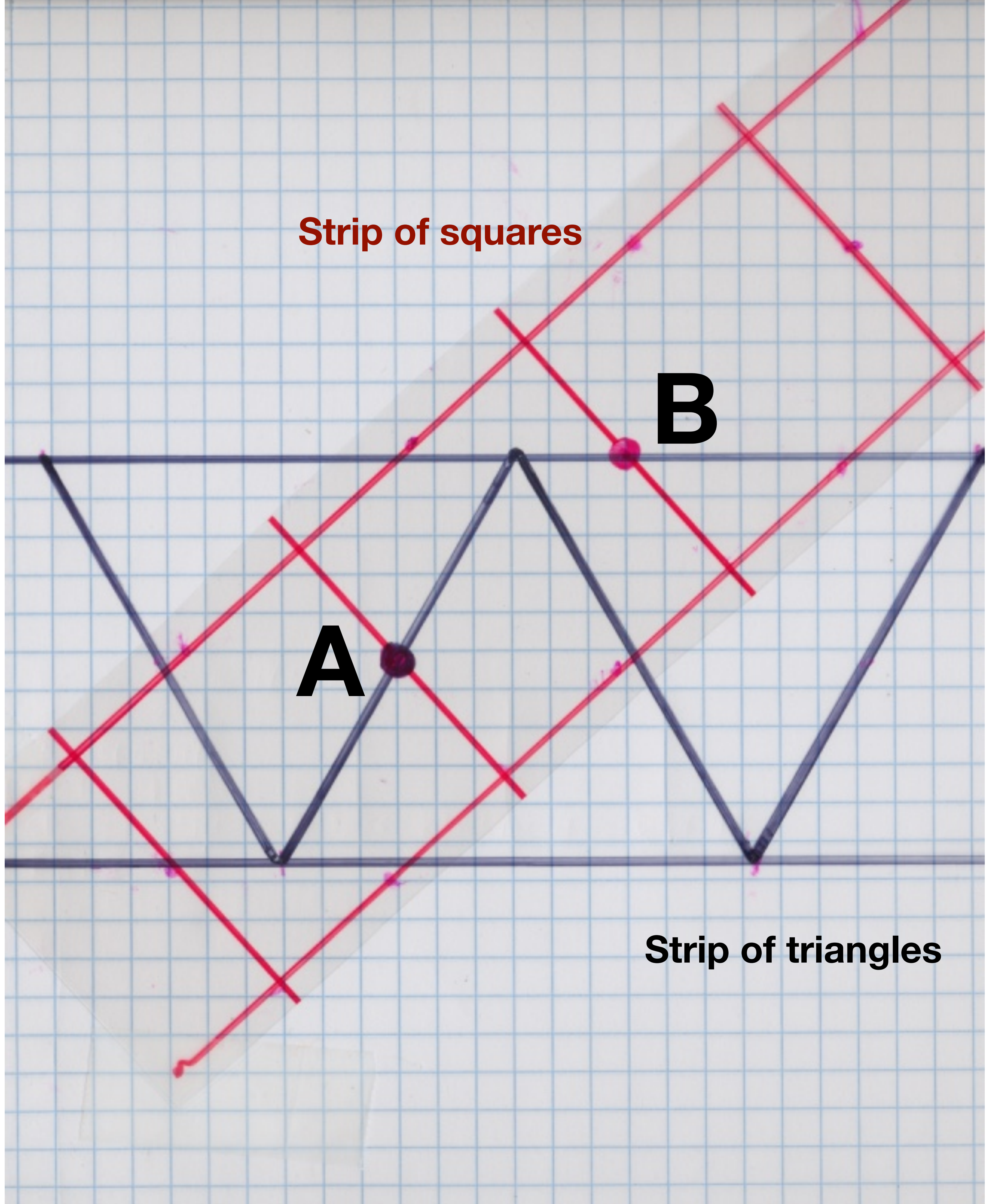
pieces must be polygons, can turn pieces over

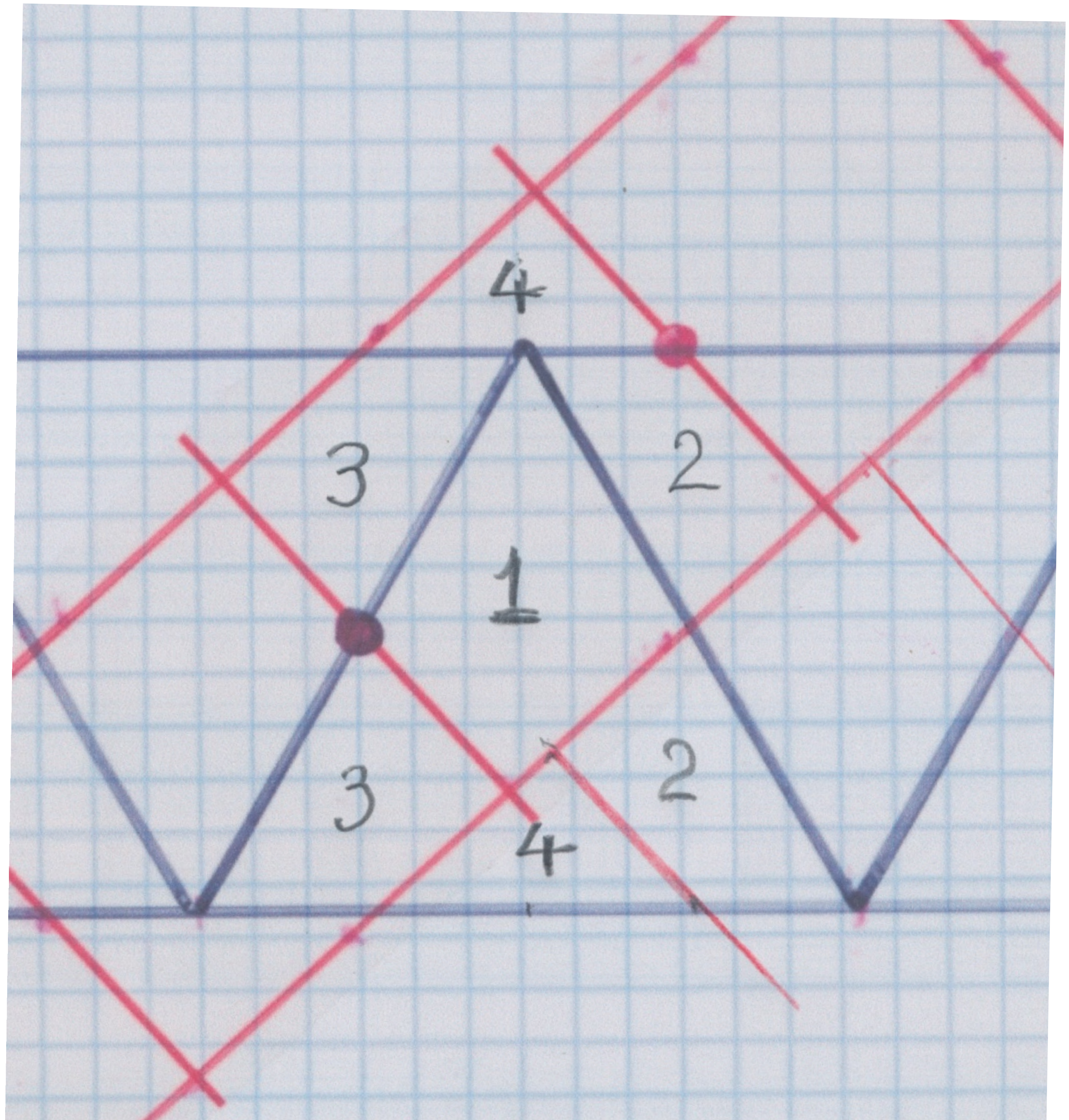
Triangle to square dissection

A match midpoints of sides

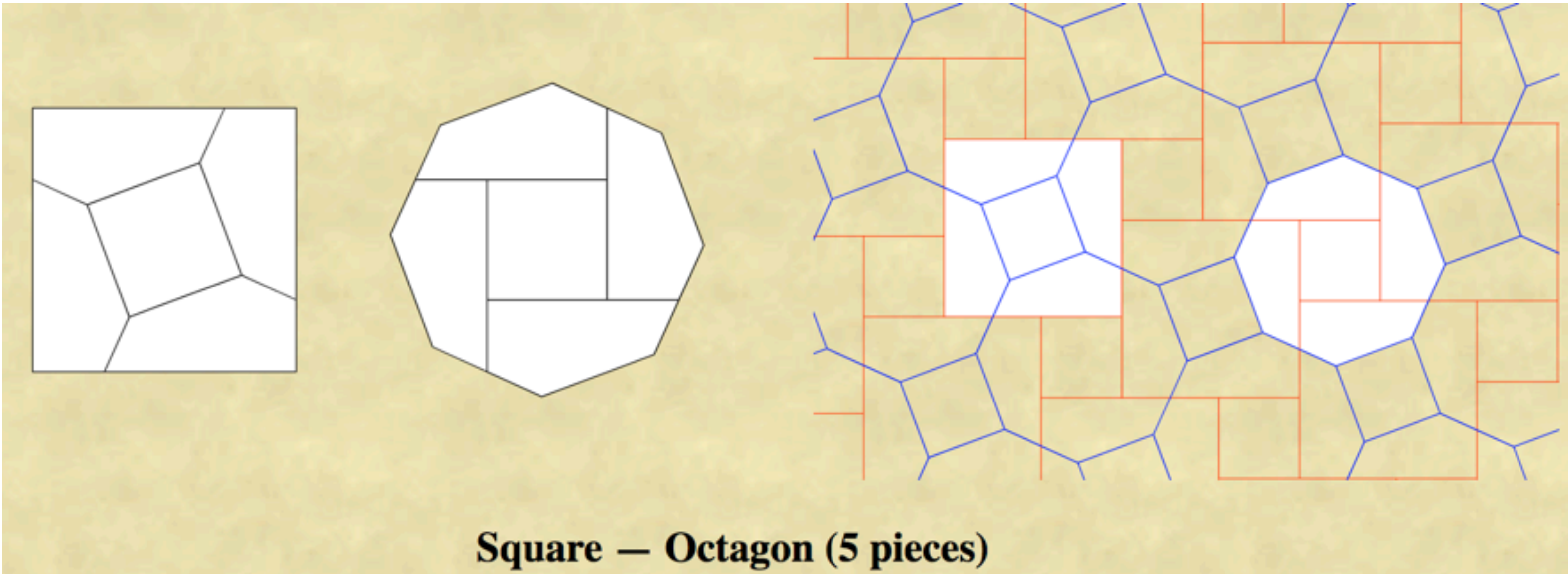
B rotate strip of squares until midpoint is on edge of triangle strip

Probably unique?
Probably minimal?




4
3
2
1
3
2
4

Octagon to Square



Geoffrey Bennett, 1926

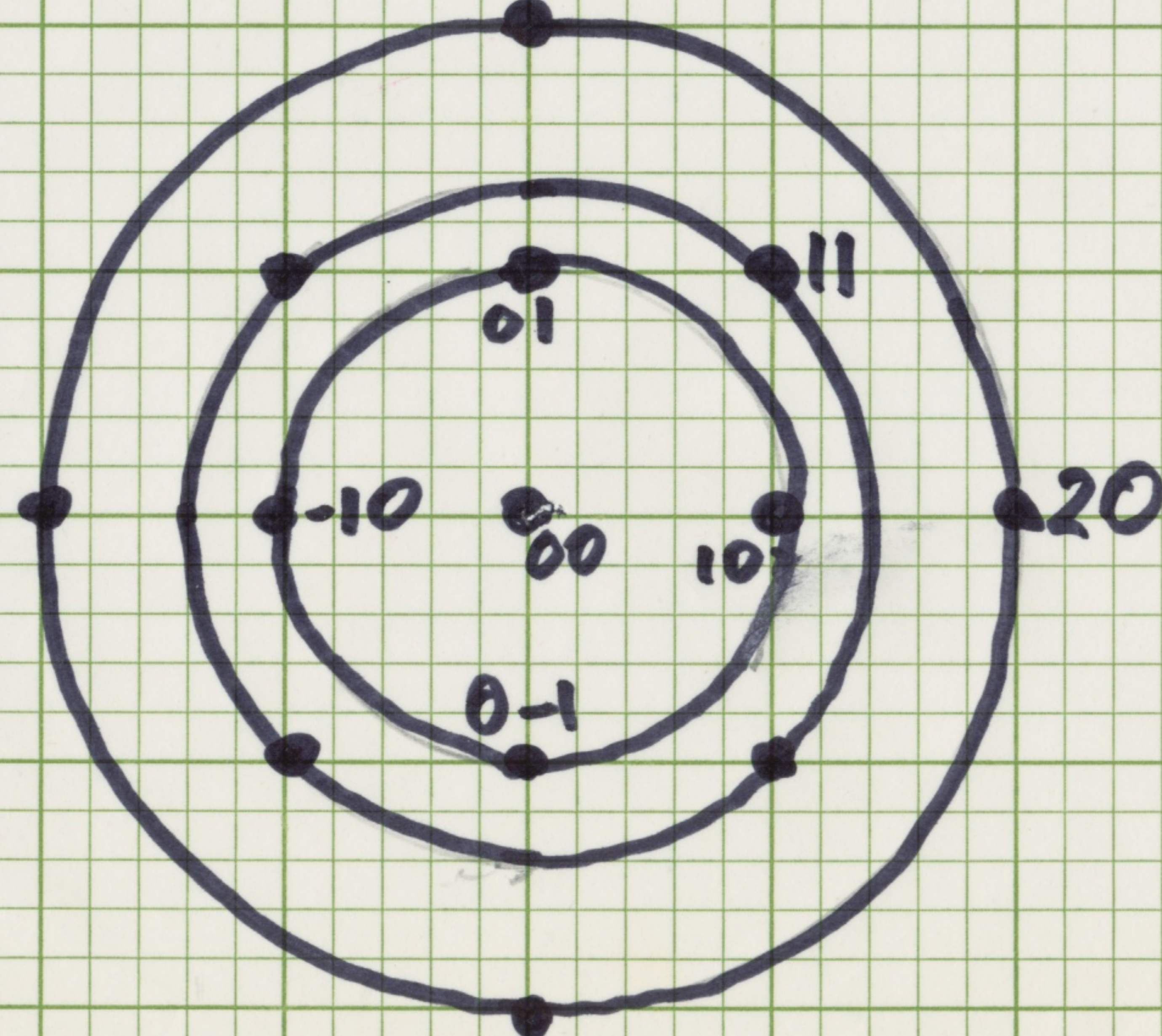
Show a(8) = 5

Roots of Theta Series

and Kissing Numbers

THETA SERIES OF $\mathbb{Z}^2$ LATTICE

$$1 + 4q^1 + 4q^2 + 4q^4 + 8q^5 + \cdots$$



THETA SERIES OF LATTICE Λ :

$$\theta_\Lambda(q) = \sum_{u \in \Lambda} q^{u \cdot u}$$

$$= \sum_{k \geq 0} a(k) q^k$$

$a(k)$ = no. of vectors u of squared length k

$$\theta_{E_8} = 1 + 240 q^2 + 2160 q^4 + 6720 q^6 + \cdots$$

$$= 1 + 240 \sum_{k=1} \sigma_3(k) q^{2k}$$

$$\sigma_3(k) = \sum_{d|k} d^3$$

(A4009)

240 is kissing number of the lattice

Maximal Kissing Number in R^n

a “most wanted” sequence

dim.	1	2	3	4	5	6	7	8
kissing no.	2	6	12	24	40*	72*	126*	240

dim.	9	10	11	12	16	24
kissing no.	306§ (272*)	500§	582§	840§	4320	196560

○ optimal

* optimal among lattice packings

§ non-lattice, may not be optimal

(laminated lattices: A2336)

A miracle!

EXAMPLE

$$\Theta_{D_4} = 1 + 24q^2 + 24q^4 + 96q^6 + \cdots$$

(A4011)

COEFFT. OF q^N = # WAYS TO WRITE N AS A SUM OF 4 SQUARES

$$\Theta_{D_4}^{1/4} = 1 + 6q^2 - 48q^4 + 672q^6 - 10686q^8 + \cdots$$

(A108092)

HAS INTEGER COEFFTS.

Q: WHAT ARE THEY?

ROOTS OF GENERATING FUNCTIONS WITH INTEGER COEFFICIENTS

(with NADIA HENINGER & ERIC RAINS) JCT A 2005

M. SOMOS: APPARENTLY 12^{th} ROOT OF THETA SERIES OF 24-D NEBE LATTICE HAS INTEGER COEFFTS!

US: $\left(\theta_{E_8}\right)^{1/8}$ DITTO

$\left(\theta_{LEECH}\right)^{1/24}$ DITTO

RING OF FORMAL POWER SERIES

$$R = 1 + x\mathbb{Z}[[x]]$$

n-th POWERS $\mathcal{P}_n = \{g^n, g \in R\}$

First main theorem

Th. 1 Let $\mu_n = n \prod_{p \mid n} p$

$$\boxed{f \in \mathcal{O}_n \iff (f \bmod \mu_n) \in \mathcal{O}_n}$$

Proof Claim: %'s in $f^{1/n}$ are integers iff %'s in $(f + \mu_n x^k)$ are.

Well, let $\phi(f) = f^{1/n}$. Taylor $\Rightarrow$

$$\phi(f + \mu_n x^k) = \sum_{r=0}^{\infty} \frac{(\mu_n x^k)^n}{r!} \phi^{(n)}(f)$$

$$= \sum_{r=0}^{\infty} \frac{(\mu_n x^k)^r}{r!} \, r! \binom{\frac{1}{n}}{r} f^{\frac{1}{n} - r}$$

$$= f^{\frac{1}{n}} \sum_{r=0}^{\infty} \underbrace{\mu_n^r \cdot \binom{\frac{1}{n}}{r}}_{c} \frac{x^{kr}}{f^r}$$

Proof of Theorem 1 (cont.)

LET $|t|_p$ = HIGHEST POWER OF p THAT DIVIDES t

i) IF $p \mid n$, $|c|_p = r|\mu_n|_p - r|n|_p - |r!|_p \geq 0$

SINCE $|r!|_p < \frac{r}{p-1}$

ii) IF $p \nmid n$, $\frac{1}{n}$ IS INVERTIBLE MOD p

AND $|c|_p \geq 0$

$\therefore (f + \mu_n x^k)^{1/n} = f^{1/n} \cdot g$

$g \in R$. $\square$

Cor 1. IF $f \equiv 1 \mod \mu_n$, THEN f IS n^{th} POWER

THE 8-DIM. E_8 LATTICE

$$\theta_{E_8} = 1 + 240 \sum_{1}^{\infty} \sigma_3(m) x^m$$
$$= 1 + 240x + 2160x^2 + \dots$$
(A4009)

$n = 8 \quad \mu_n = 8.2 = 16$

$\theta_{E_8} \equiv 1 \mod 16 \therefore$ IN $\mathcal{S}_8$

$$\theta_{E_8}^{1/8} = 1 + 30x - 2880x^2 + 416640x^3 - 69178110x^4 + \dots$$
(A108091)

Q. WHAT ARE THESE COEFFTS ?

Th. 2 THETA SERIES OF EXTREMAL EVEN UNIMOD. LATTICE IN $\mathbb{R}^d$ IS IN $\mathcal{S}_n$, WHERE GET n BY DROPPING ALL PRIMES > 5 FROM d.

So Leech^(1/24) has integer coefficients!

Suppose Θ_Λ has integer coefficients

CONJECTURE

IF $\Theta_\Lambda^{1/n}$ HAS INT. COEFFTS.,

THEN $\dim \Lambda \geq n$ (?)

[PERHAPS $n \mid \dim \Lambda$??]

ANALOGUES FOR CODES FALSE!

HOWEVER:

Th. 3 WEIGHT ENUM. $RM(r,m)$

$\in \mathcal{C}_{2^r}$

CONJECTURE

ANALOG FOR BCH CODES.

The Recaman Hypothesis

Every number appears in Recaman's sequence (?)

Recamán's Sequence

0	1	2	3	4	5	6	7	8	9	...
0	1	3	6	2	7	13	20	12	21	...

$$a_n = a_{n-1} \; - \; n$$

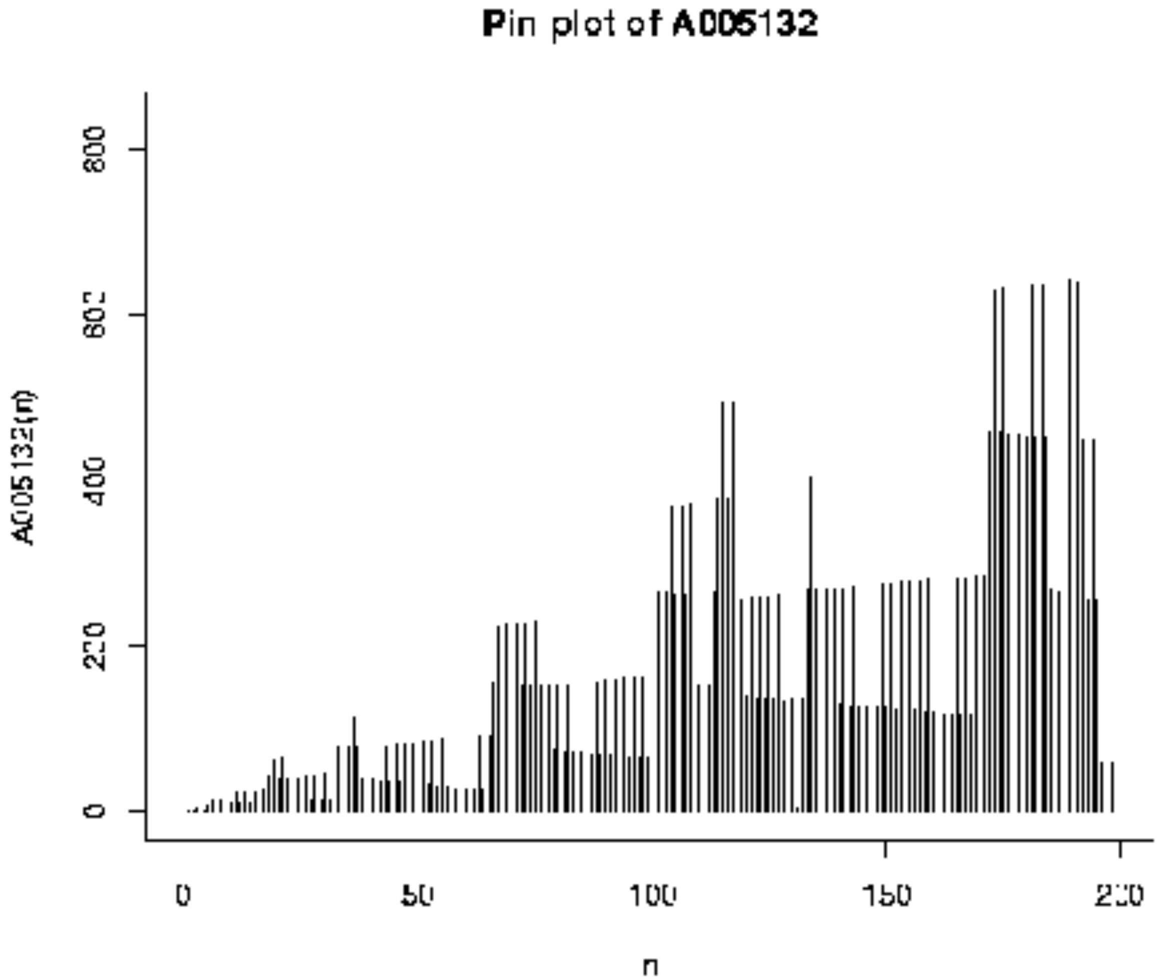
(A5132)

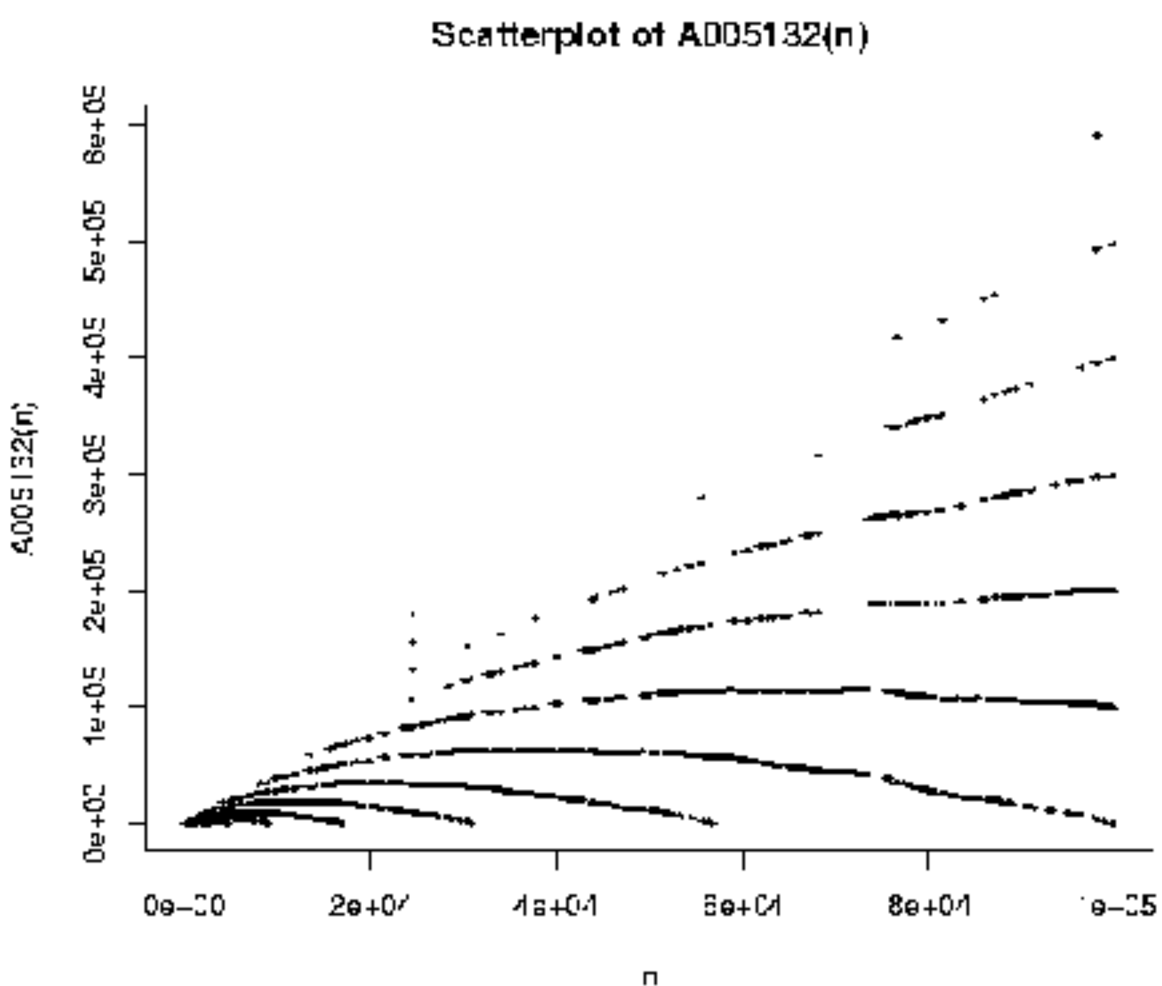
if positive and new, otherwise

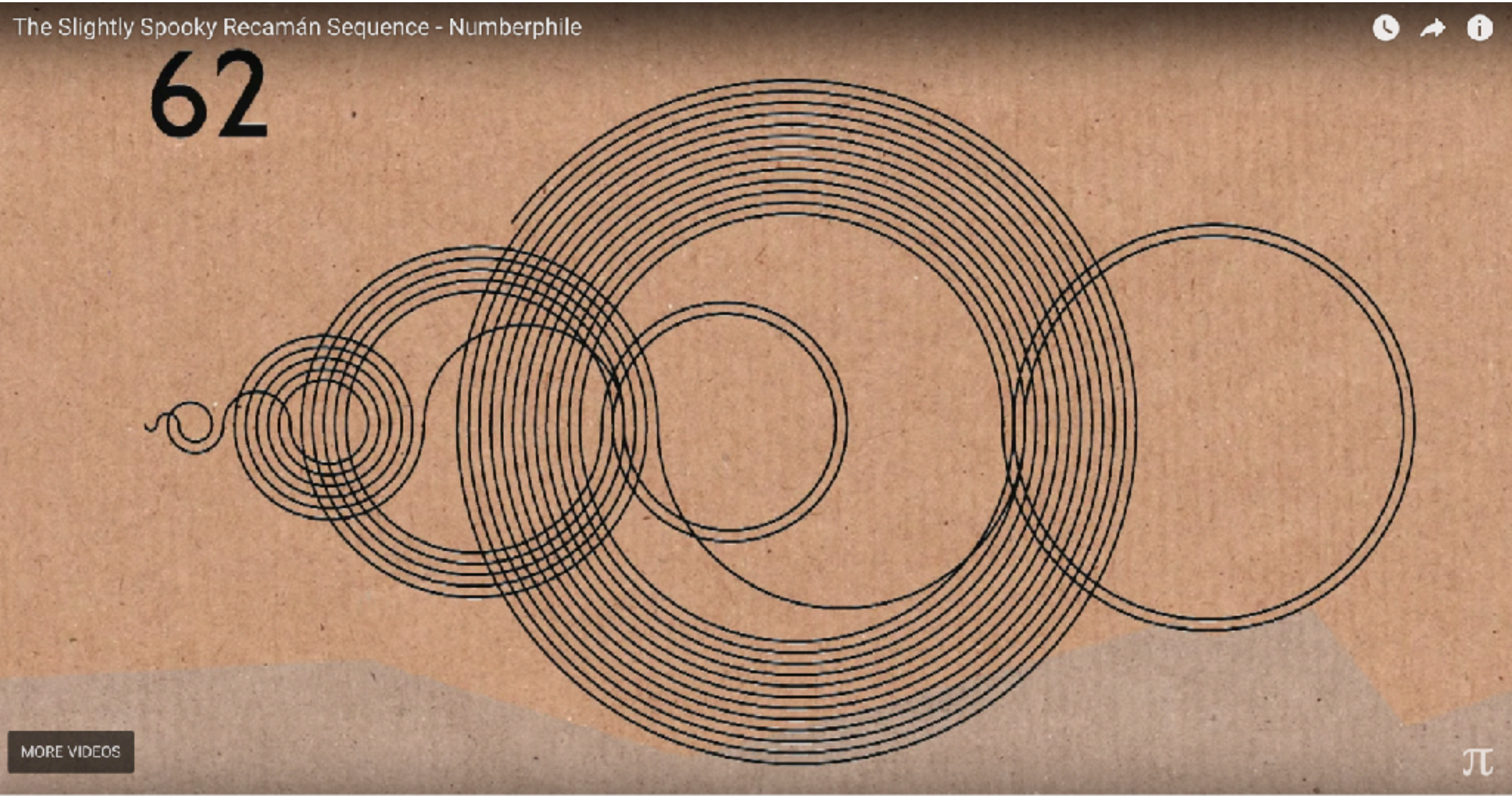
$$a_n = a_{n-1} \; + \; n$$

- from Bernardo Recamán Santos (Colombia), circa 1991

Recamán, continued







Drawn by Edmund Harriss, 2018

Recamán, continued

Numbers that take a record number of steps to appear:

1	1
2	4
4	131
19	99,734
61	181,653
879	328,002
1355	325,374,625,245
2406	394,178,473,633,984
852655	> **10^230**

Benjamin Chaffin, January 2018

(A64228) (A64227)

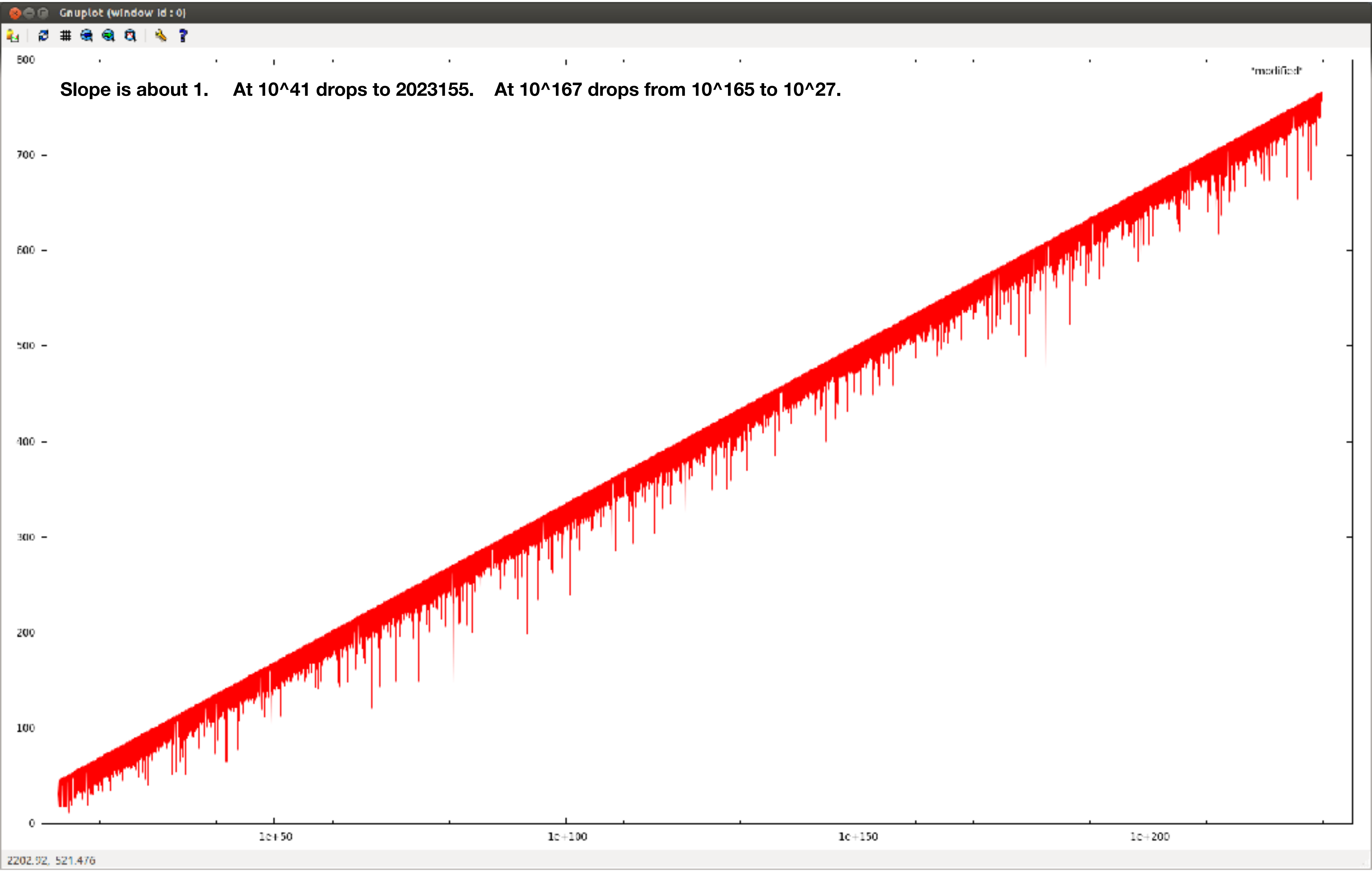
Allan Wilks, Nov. 2001

Bell Labs Talk,
“How to Wreck a Man’s Life”

Computed 10^15 terms
All numbers below 852,655 had appeared, but
852,655 = 5 * 31 * 5501
was missing

Ben Chaffin, 2018: After 10^230 terms,
852,655 is still missing

Ben Chaffin’s “Paint-dripping” log-log plot of 10^230 terms of Recaman’s sequence



Easy Recaman

Can subtract as long as don't go negative

Repeats are OK when subtracting (or `adding`)

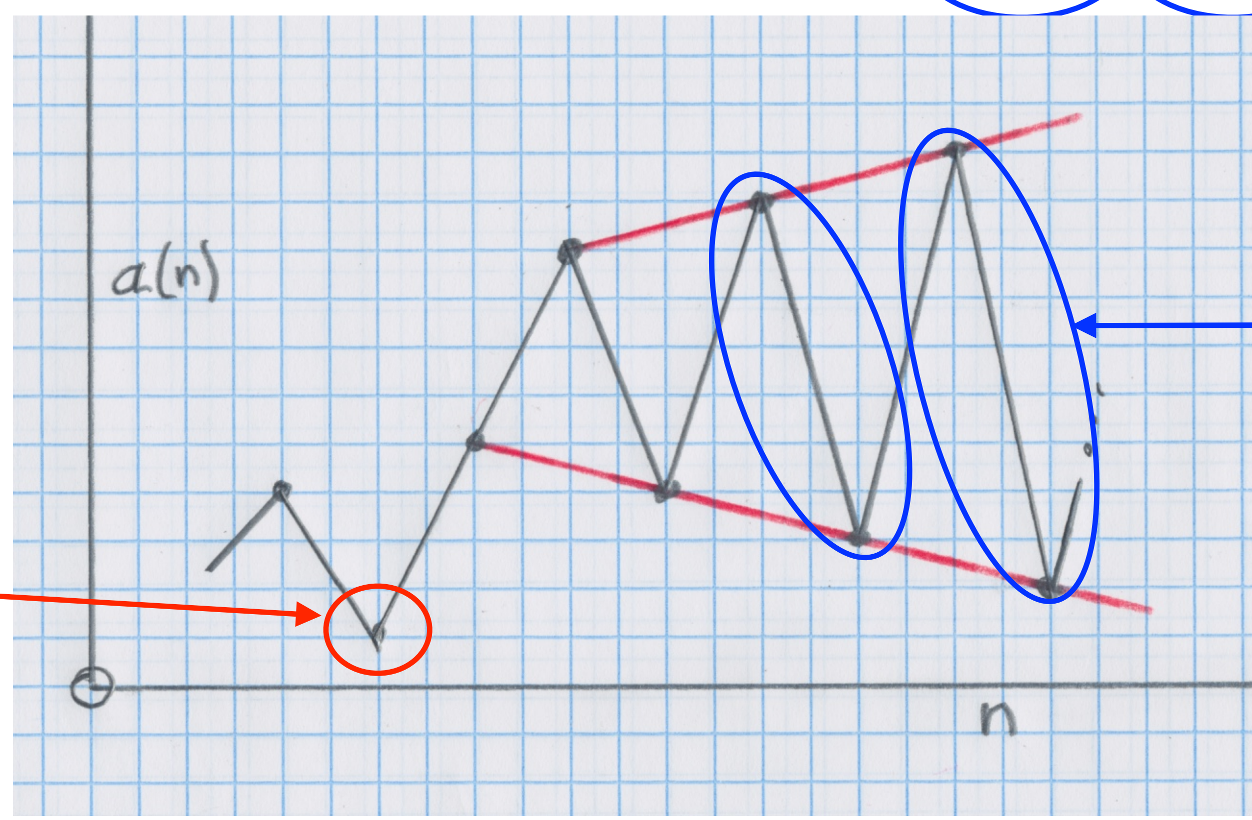
Easy Recaman (cont.)

Given sequence p(n), n >= 1, and starting term s, define a(n), n >= 0, by

a(0)=s,
a(n) = a(n-1) - p(n)   if that is >= 0
a(n) = a(n-1) + p(n)   otherwise

If p(n) = n, s = 0

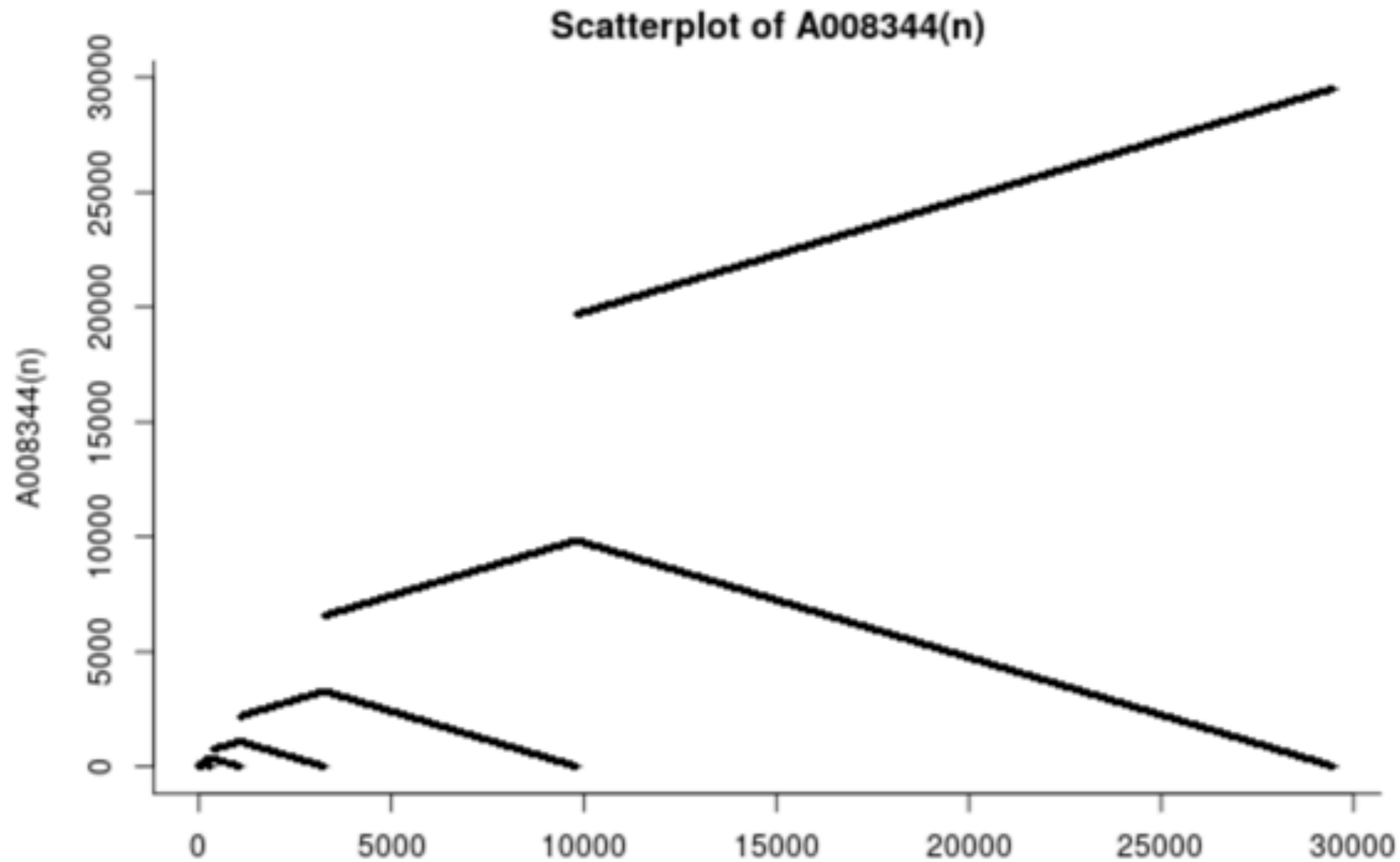
0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19
0, 1, 3, 0, 4, 9, 3, 10, 2, 11, 1, 12, 0, 13, 27, 12, 28, 11, 29, 10   A8344



Drop of
p(n+1)-p(n)
= DELTA(p(n))
every 2 steps

Low point
(down, up, up)

Easy Recaman (cont.)



LINEAR CASE p(n) = n A008344

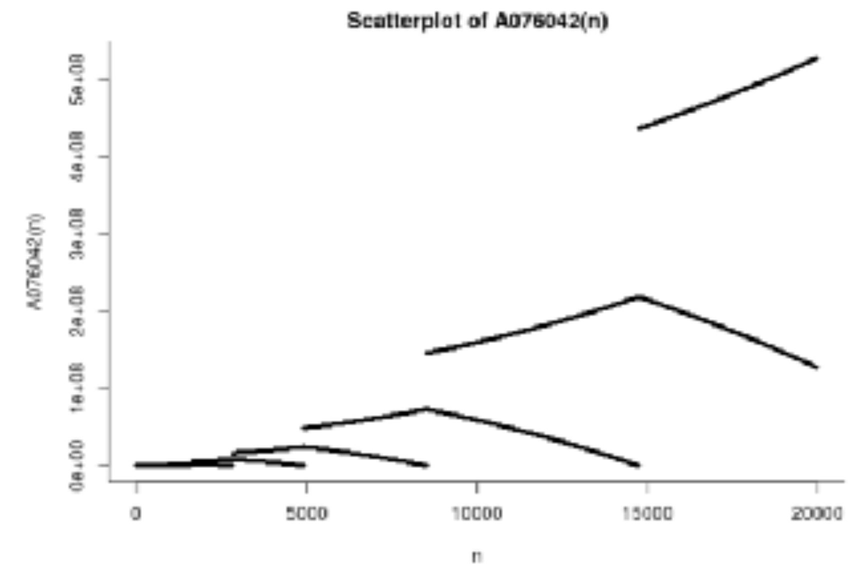
Low points at n = 0, 3, 12, 39, 120, … (3^k-3)/2

Values at low points: 0, 0, 0, 0, …

Epochs have lengths 3, 9, 27, 81, 243, …
(gaps between low points)

Easy Recaman (cont.)

SQUARES p(n) = n^2



0, 1, 5, 14, 30, 5, 41, 90, 26, 107, 7, 128, 272, 103, 299, 74 **A076042**

Low points at 0, 5, 10, 19, 34, 59, 104, 181, 314, 545, 946,

Low values 0, 5, 7, 4, 19, 104, 74, 193, 515, 725, 241, 1948

Epoch lengths 5, 5, 9, 15, 25, 45, 77, 133, 231, 401, 693, 1201,

Tomas Rokicki: a(n) > 0 for 0 < n < 2^25000

Conjecture: a(n) NEVER returns to 0

Easy Recaman (cont.)

PRIMES: p(n) = prime(n)

a(n)	0, 2, 5, 0, 7, 18, 5, 22, 3, 26, 55, 24, 61, 20, 63, 16, 69,
Low points at	0, 3, 8, 21, 56, 145, 366, 945, 2506, 6633, 17776, 48521
Low values	0, 0, 3, 2, 1, 2, 3, 2, 7, 2
Epoch lengths	3, 5, 13, 35, 89, 221, 579, 1561, 4127, 11143, 30745, 84585,
Zeros at	0, 3, 369019, 22877145

Conjecture: infinitely many zeros (?)

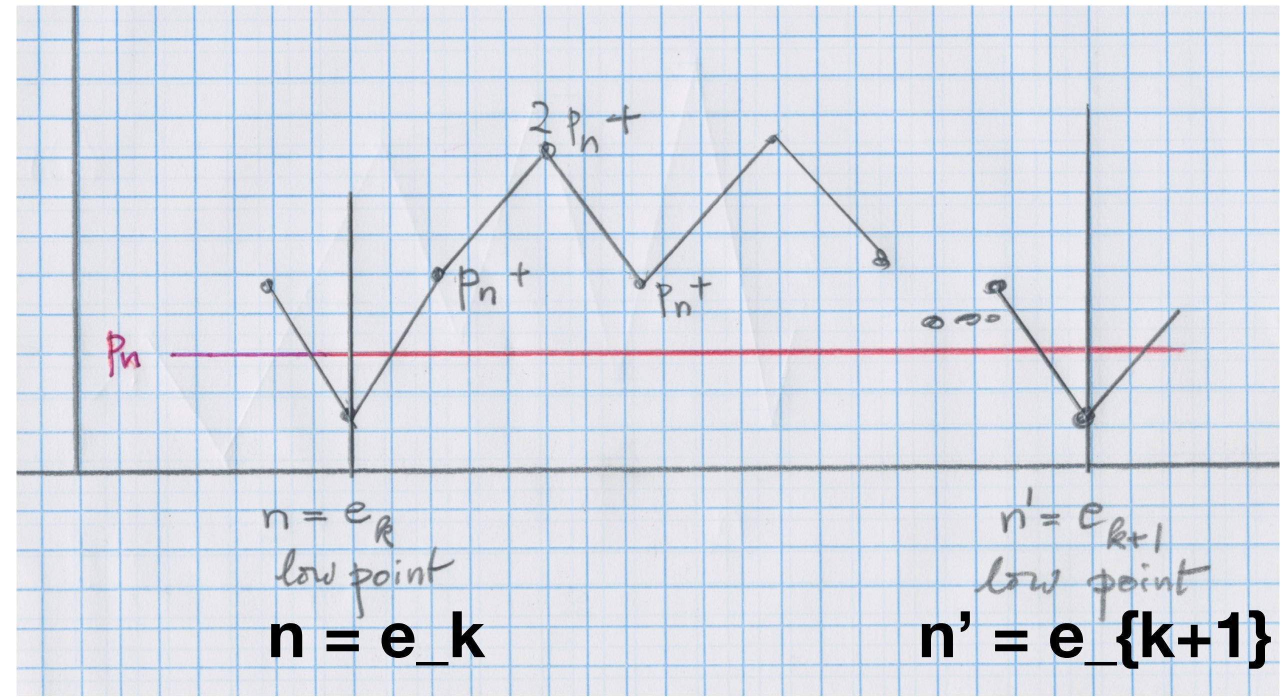
Easy Recaman (cont.)

Analysis (primes case)

p(n) = prime(n) ~ n log n, DELTA(p(n)) = p(n+1) - p(n) ~ log n

Epoch(k) starts at n = e(k) with a(n) = m (small)

a(n+1) = p(n) + m ~ e(k) log e(k)
DELTA = log e(k)
So 2e(k) steps to get down to 0
so e(k+1) = e(k) + 2e(k) = 3e(k)
e(k) = c.3^k



Probability of actually hitting 0 = 1/DELTA = 1/log 3^k = 1/k

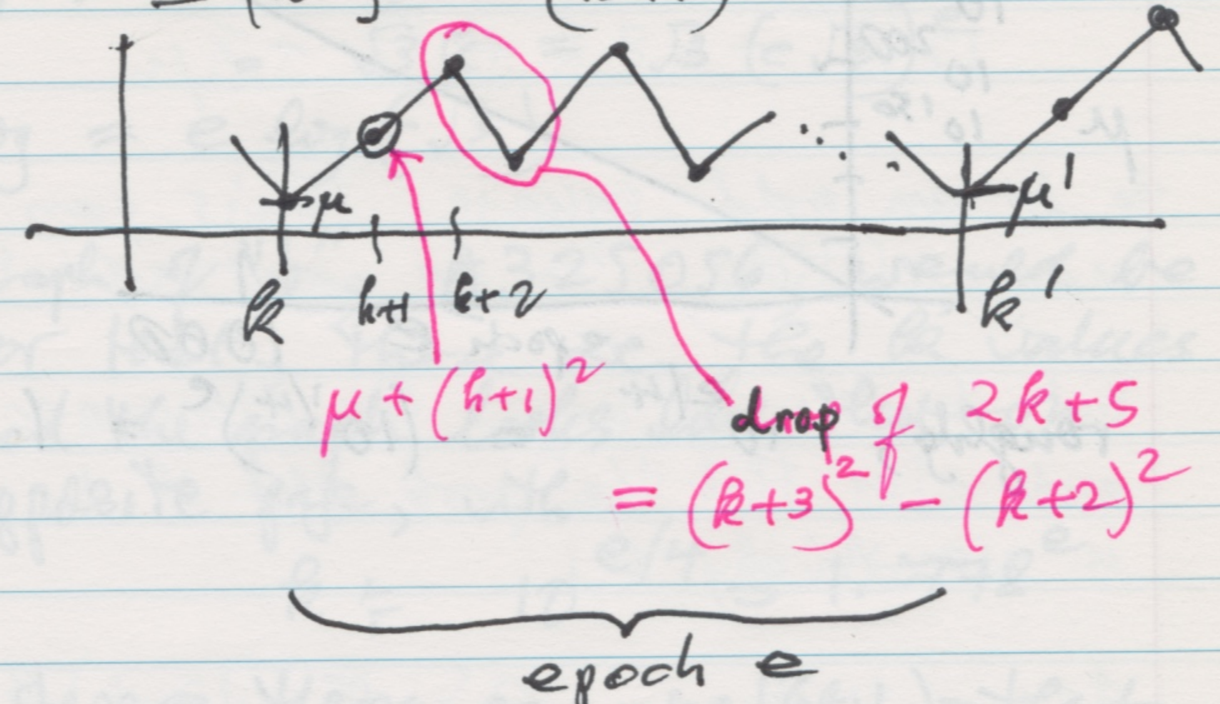
Expected number of 0's = Sum 1/k = oo

So 0, 3, 369019, 22877145, … may contain more terms!

Easy Recaman (cont.)

Analysis (squares case)

$$\Delta[k^2] = (k+1)^2 - k^2 = 2k+1$$



Exact number of drops to hit 0 is

$$\text{“}\lambda\text{”} = \frac{1}{2}\left(-k - \frac{3}{2} + \sqrt{\left(k+\frac{3}{2}\right)^2 + 2(k+1)^2 + 2\mu}\right)$$

Must be an integer, so use

$$\lambda = \left\lfloor -\frac{k}{2} - \frac{3}{4} + \frac{1}{2}\sqrt{3k^2 + 7k + \frac{17}{4} + 2\mu}\right\rfloor \quad (67.1)$$

and get

$$k' = k + 1 + 2\lambda \quad (67.2)$$

$$\mu' = \mu + (k+1)^2 - (2k\lambda + 3\lambda + 2\lambda^2) \quad (67.3)$$

I ran the recurrence (67.1), (67.2), (67.3), start @ $k=5$, $\mu=5$, for 10000 steps. μ' never zero.

$a(n)$ = A76042, low points at k = A325056, values of low points = A324791.

Easy Recaman (cont.)

Analysis (squares case, cont.)

To get exactly down to 0, need

$$12k^2 + 28k + 17 + 8\mu$$

to be an odd square. This can happen, many solutions, so no contradiction there

But it does not happen for 10000 epochs, and after that the prob. that it happens is essentially 0.

Asymptotically: $2\lambda \cong (\sqrt{3}-1)k$

$$k' \cong k + 2\lambda \cong \sqrt{3}\,k$$

$$k = c(\sqrt{3})^e \text{ at epoch } e$$

Last drop $\cong 2\sqrt{3}\,k$

Probability of 0 $= \dfrac{1}{2\sqrt{3}\,k} = \dfrac{1}{c'(\sqrt{3})^e}$

Expected no. of 0's $= \sum_e \dfrac{1}{c'(\sqrt{3})^e} < \infty$

So at most finitely many 0's.

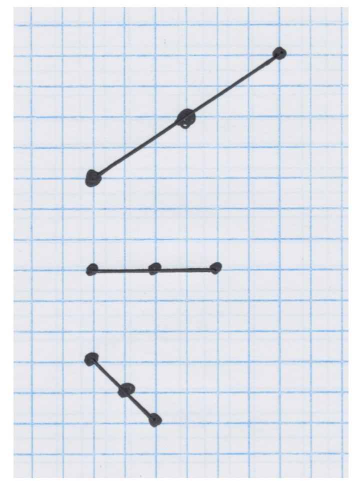
Conjecture: None.

The Forest Fire

A229037   Jack Grahl, 2013

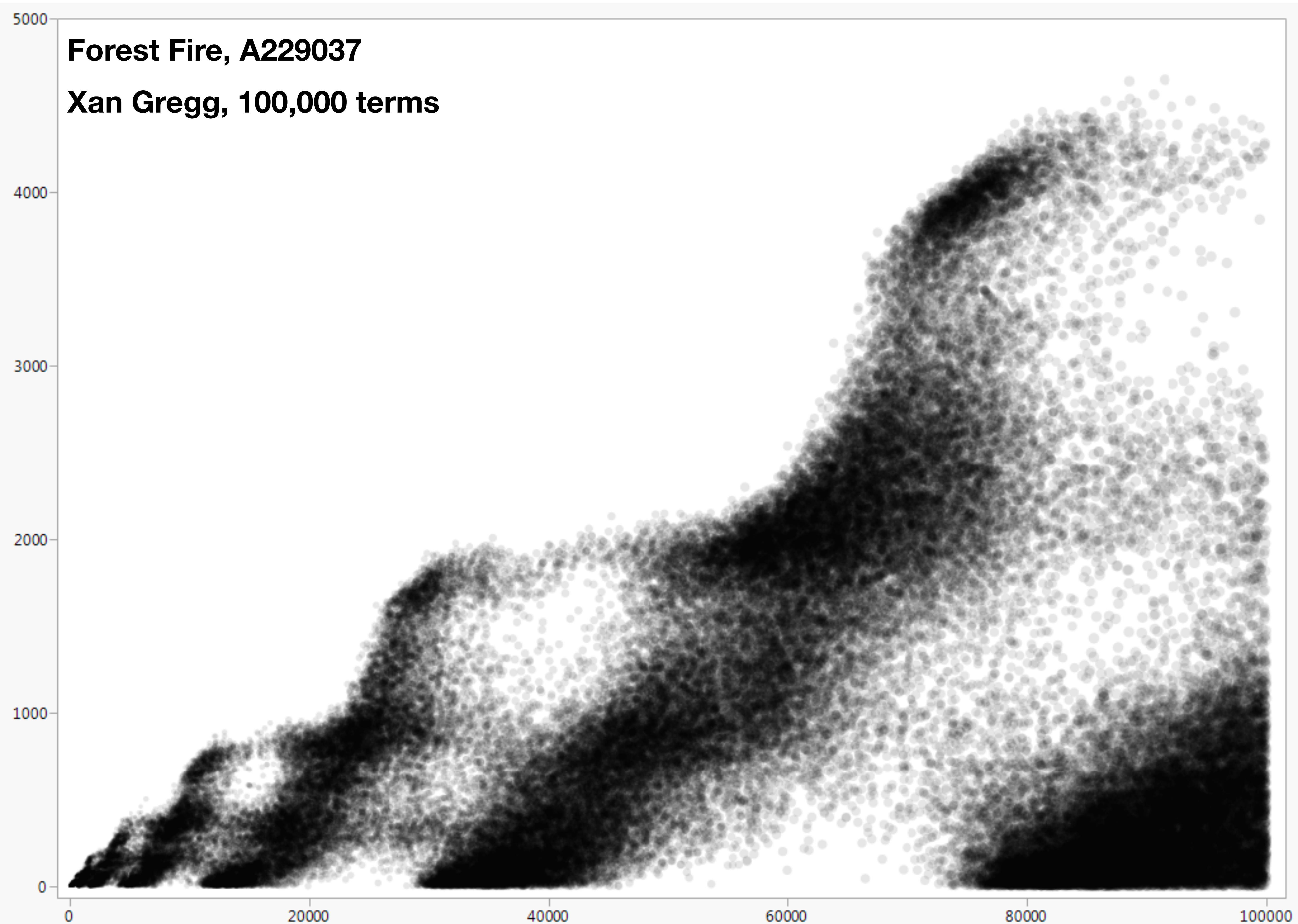
Forest Fire, A229037

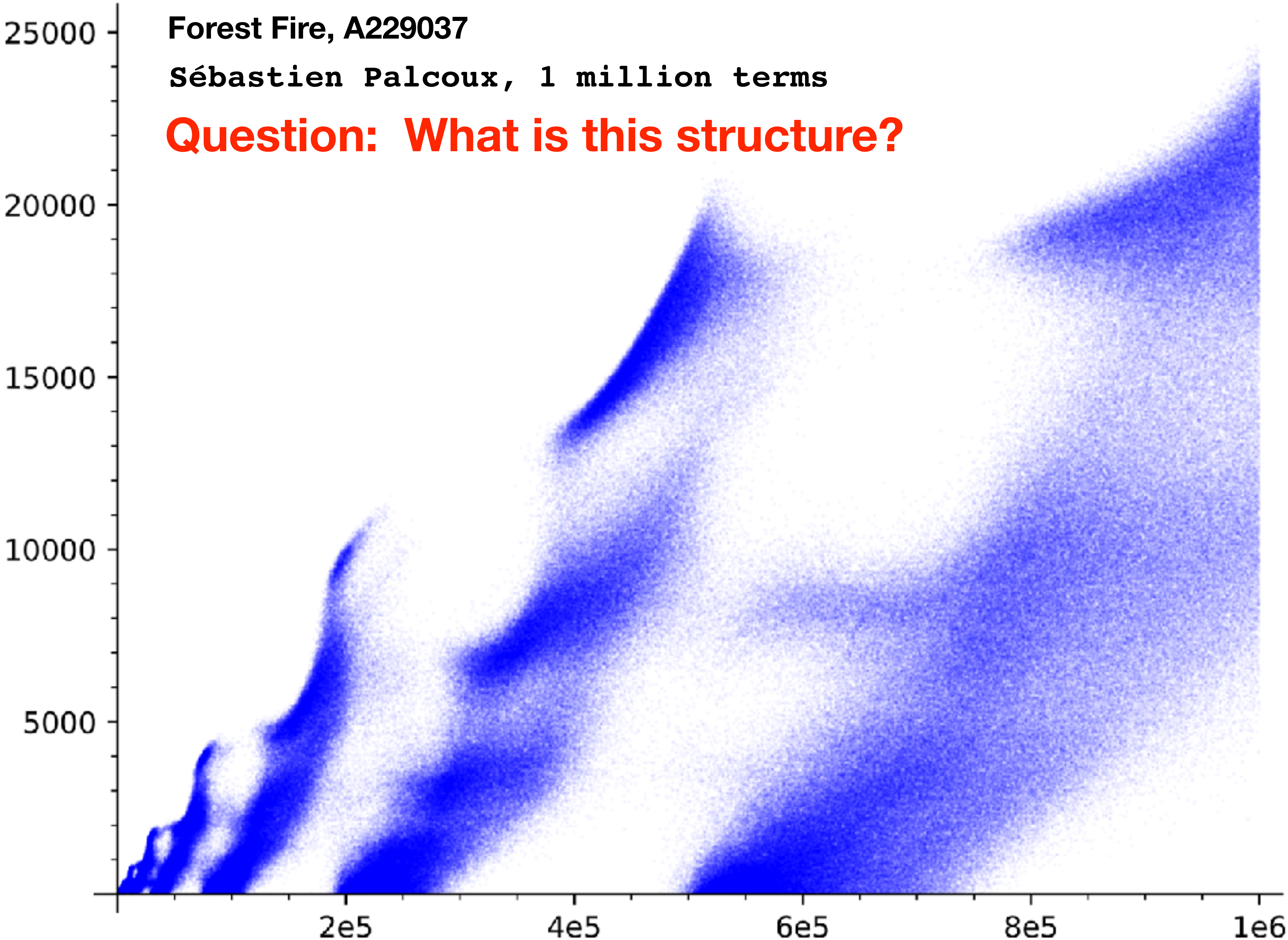
Smallest pos. number such that
NOT $a(j), a(j+k), a(j+2k)$ in arithmetic progression

NOT 

n	1	2	3	4	5	6	7	8	9	10
a(n)	1	1	2	1	1	2	2	4	4	1

not 1 **not 1** **not 1, 2, 3**

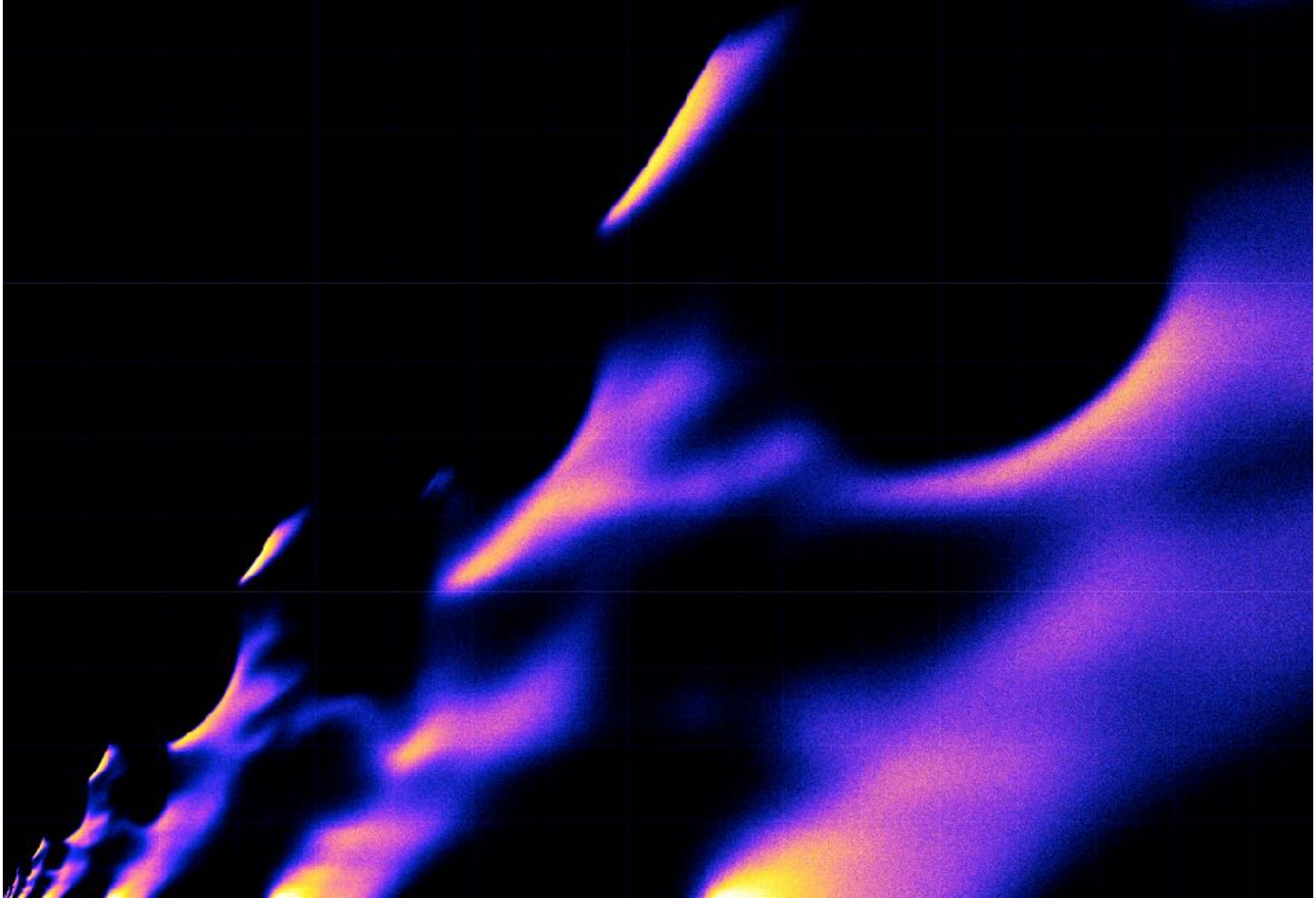

Forest Fire, A229037
Xan Gregg, 100,000 terms
5000
4000
3000
2000
1000
0
0
20000
40000
60000
80000
100000



Forest Fire, A229037
Sébastien Palcoux, 1 million terms
Question: What is this structure?
25000
20000
15000
10000
5000
2e5
4e5
6e5
8e5
1e6

Forest Fire, A229037

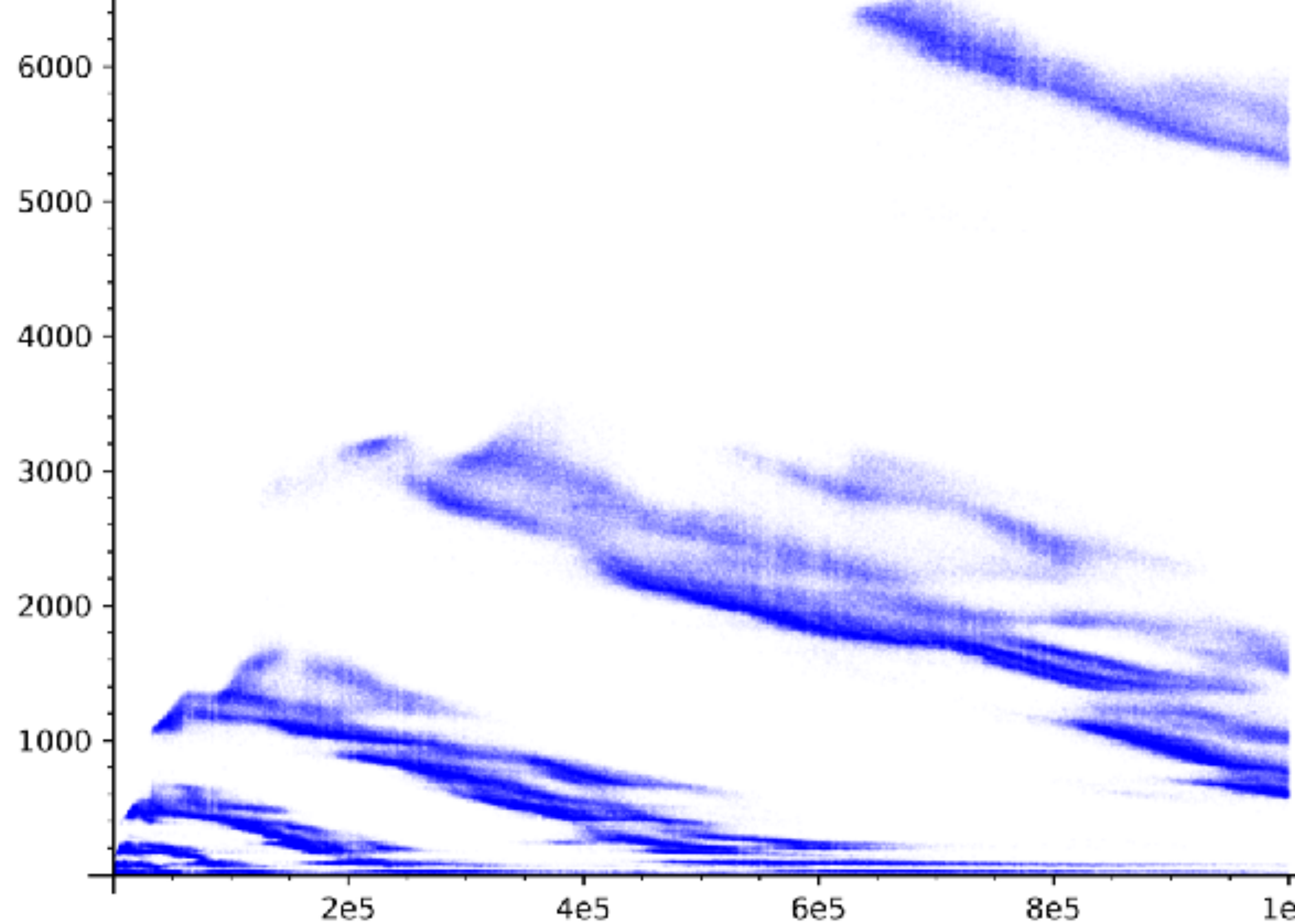
“garnet420”, 16 million terms



Forest Fire variants

Richard Stanley, A309890, August 2019
allow 3-term DECREASING APS

`Sébastien Palcoux, 1 million terms`



Aaron Kemats, Sept 2019, A309108
Values a(j)*a(j+k)*a(j+2k) distinct

1, 1, 1, 2, 3, 2, 5, 6, 7, 4, 10, 9, 7, 11, 12, 8, 13, …

Covering with Geometric Progressions

Suggested by a problem on the All-Russian Mathematical Olympiad 1995

Dmitry Kamenetsky, July 2019, A309095

a(n) = max k such that can cover (1,2,3,…,k) with n GPs (with rational ratios)

n = 1	k = 2	(1,2)
n = 2	k = 5	(1,2,4), (3,5)
n = 3	k = 8	(1,2,4,8), (3,5), (6,7)
n = 6	k = 16	(1,2,4,8,16), (3,6,12), (5,7), (9,10), (11,13), (14,15)

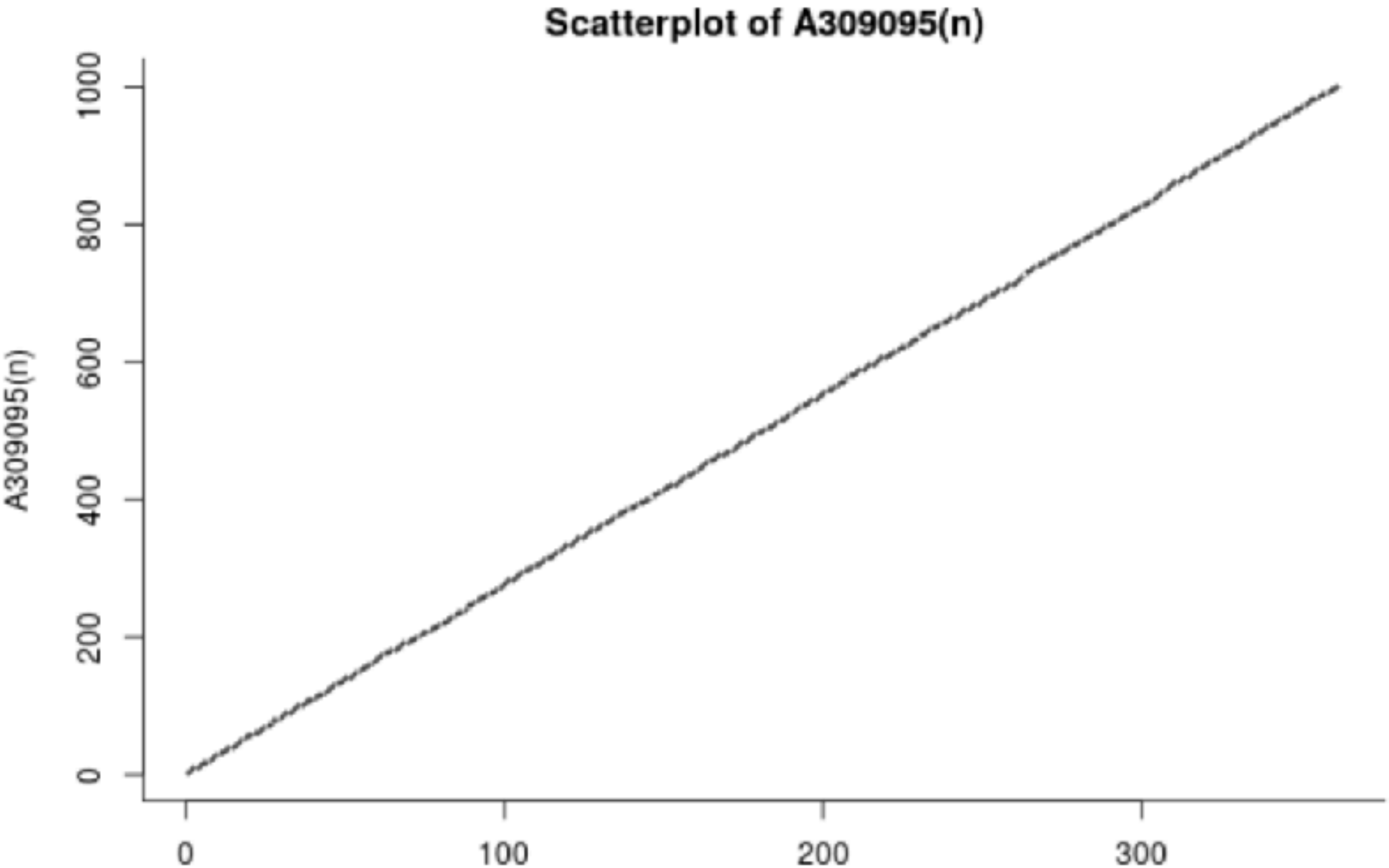
2, 5, 8, 10, 13, 16, 18, 21, 25, 28, 30, 33, 35, 37, 40

A309095

Rob Pratt found 362 terms, getting k out to 1000.
Graph is essentially a straight line, slope 2.766…

Conjecture on Math StackExchange is slope = e

Graph of Rob Pratt's values



Differences mostly 2', 3's, 4's. Max difference = 6.

All Differences Distinct

All Differences Distinct

1, 3, 9, 5, 12, 10, 23, 8, ...

2, 6, -4, 7, -2, 13, -15, ...

4, -10, 11, -9, 15, -28, ...

-14, 21, -20, 24, -43, ...

35, -41, 44, -67, ...

-76, 85, -111, ...

161, -196, ...

-357, ...

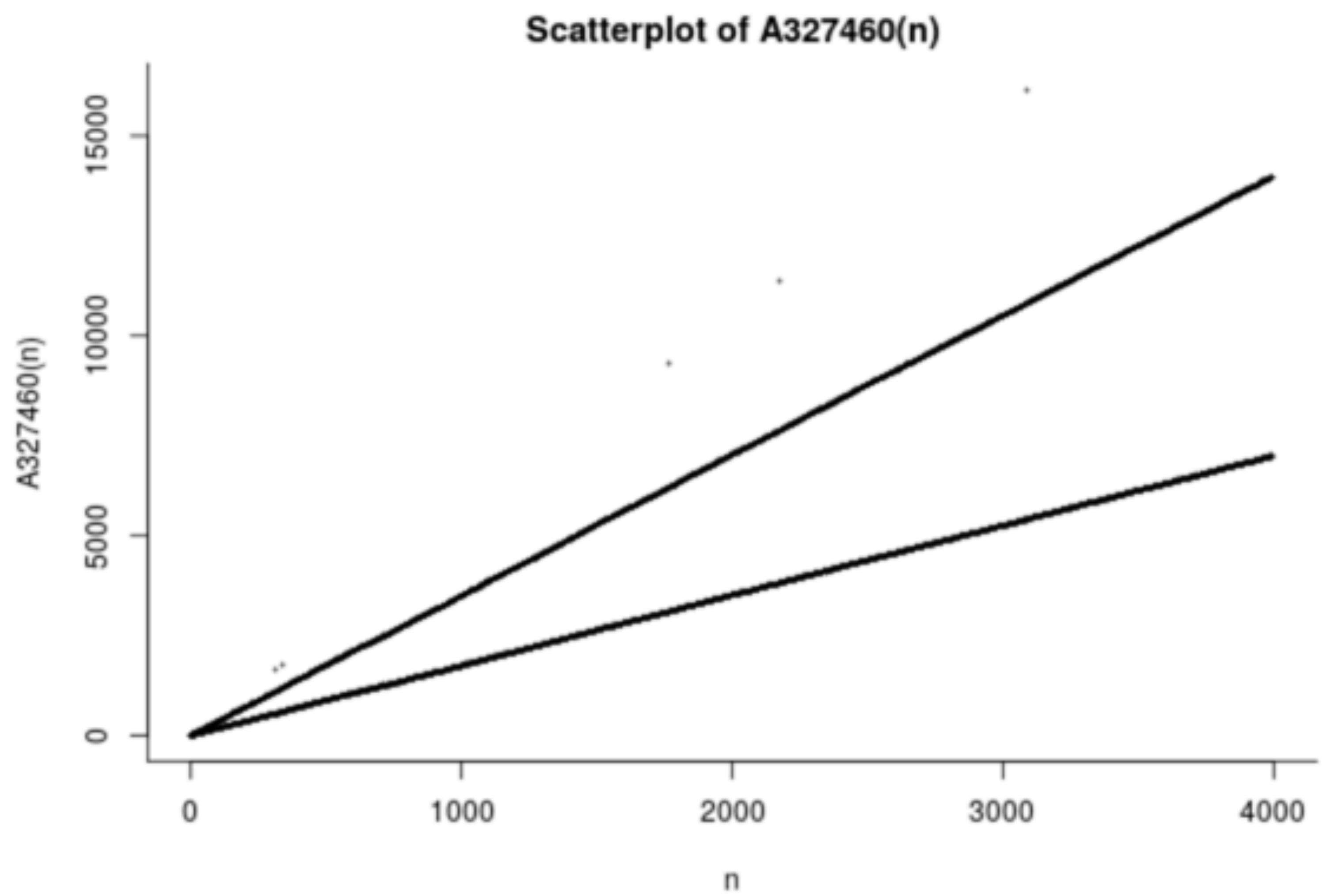
Triangle of differences

All visible numbers distinct !

Lexicographically earliest

Explain!

The sequence A327460



Peter Kagey, Rémy Sigrist, NJAS, Sept. 20129

A Curious Property of 909

909!

A326344, Max Tohline, Sep 11 2019

R = Reverse, R(10)=1
P = next Prime, P(10)=11
C = next composite, C(10)=12

a(1) = 1; if n is prime, a(n) = R(P(a(n-1))); if n is composite, a(n) = R(C(a(n-1)))

1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,	14,	15,	16,	17,	18,	19,
1,	2,	3,	4,	5,	6,	7,	8,	9,	1,	2,	4,	5,	6,	8,	9,	11,	21,	32,

Theorem (Andrew Weimholt): a(n) <= 909

Proof: The only transitions from a 1-digit number to a 2-digit number are a(n-1) = 8 or 9 to a(n) = 11 with n prime.

(Continued on next slide)

Proof, continued

Suppose a(n) = 11 with n prime.
Two cases, n == 1 or -1 mod 6.
Two trees, here is one of them:

n mod 6	a(n)
1	11
2	21
3	22
4	42
5	44 34
0	54 53
1	55 95 45 95
…….	…….

The only 3-digit number reached is 101 for n ==1 mod 6

There is now a single tree starting at 101,
and the max 3-digit number reached is 909.

Robert Dougherty-Bliss (Rutgers): In base 3, a(n) <= 20 (A326894),
and in base 7, a(n) <= 310 (A327241).

and in bases b=2,3,4,… the max value is (A327701)

1, 20, 3, 107, 5, 310, 7, 668, 909, 1253, 11, 2082, 13, 3224, 3880, 4670, 17, 6558, 19

Éric Angelini

Éric Angelini (with Jean-Marc Falcoz), A309529, August 5 2019

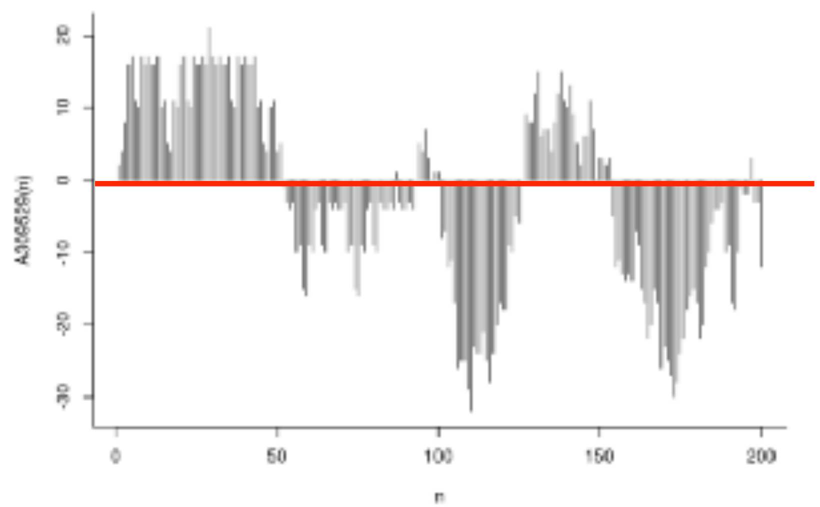
n:	1	2	3	4	5	6	7	8	9	10	11
a(n):	2	4	8	16	17	11	10	17	16	16	17

a(1)=2, then

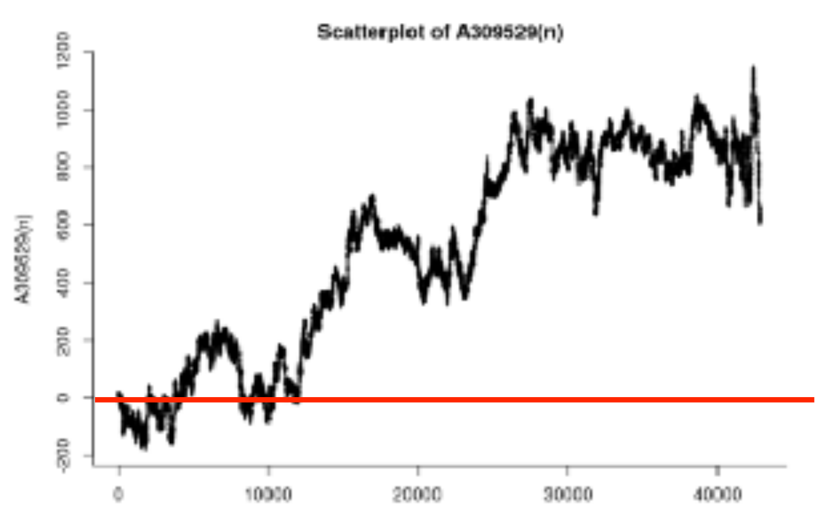
a(n+1) = a(n) + n-th digit if a(n) is even

a(n+1) = a(n) - n-th digit if a(n) is odd

Coin-tossing paradox, infinitely many sign changes, at longer intervals (Feller)



200 terms



4200 terms

The OEIS at Age 55 and the Future

Summary of 55 Years

Begun at Cornell in 1964; books in 1973, 1995; on web since 1996.
The OEIS Foundation, 2009; Wiki in 2011.
Today 327,000 sequences; 8000 citations.

Typical citation:
Matthias Franz, The cohomology rings of homogeneous spaces, arXiv:1907.04777 [math.AT], July 10, 2019. "The connection between tensor products of A_∞-maps and hypercubes (Remark 4.2) was discovered by consulting the OEIS"

Thanks to the

Editors, Systems Administrators, Trustees

who keep the OEIS running

Especially

David Applegate, Joerg Arndt, Jon Awbrey, Bruno Berselli, Lars Blomberg, David Corneth, Russ Cox, Susanna Cuyler, Harvey Dale, Michael De Vlieger, Amiram Eldar, Georg Fischer, Felix Fröhlich, Olivier Gérard, Ron Graham, Charles Greathouse, Maximilian Hasler, Alois Heinz, Sean Irvine, Peter Luschny, Michel Marcus, Richard Mathar, Omar Pol, Martin Pedersen, Robert Price, Jon Schoenfield, Rémy Sigrist, Hugo van der Sanden, Gus Wiseman, ChaiWah Wu, and dozens more.

Thanks also to the Pro Bono Partnership (attorneys helping non-profit organizations)

33 new sequences accepted every day; 170 edits every day.

We need more editors

We are swamped with submissions

No pay, but lots of fun

Work as much or as little as you like

Contact   njasloane@gmail.com

Requirements: Familiarity with Math, English, and the OEIS

The Future

Hire a manager?

- **Either raise endowment of \$1M (at least)**
- **Or every year raise \$40K at least from some foundation**