Sequences that satisfy the congruence *a(n+k) == a(n)*a(k) (mod k)*

Peter Bala, July 31 2026

[1] is concerned with congruences satisfied by the OEIS sequence A028342, whose e.g.f. is given by

Sum_{n >= 0} A028342(n)*x^n/n! = exp(Sum_{m >= 1} d(m)*x^m/m),

where d(m) equals the number of positive divisors of m.

In particular, [1, Proposition 2.2] establishes the congruences

A028342(n+k) == A028342(n)*A028342(k) (mod k) ...(1)

holding for all n >= 0 and k >= 1.

Kallat's proof is a combinatorial one involving the action of cyclic groups on sets of coloured permutations.

The purpose of this note is to observe that the congruences (1) are not specific to the divisor function d(n) but hold more generally.

We present an AI generated proof of the following result.

Theorem 1. *Let {c(n) : n >= 1} be an integer sequence. Define a sequence {a(n) : n >= 0} by*

*Sum_{n >= 0} a(n)*x^n/n!*

*= exp(Sum_{n >= 1} c(n)*x^n/n).*

Then

*a(n+k) == a(n)*a(k) (mod k), for n >= 0, k >= 1.*

Proof. The proof is by induction using a recurrence derived from the e.g.f. Put F(x) = Sum_{n >= 0} a(n)*x^n/n!, so that F(x) = exp(Sum_{n >= 1} c(n)*x^n/n.

Differentiating gives

F'(x) = F(x) * Sum_{n >= 0} c(n+1)*x^n.

Hence,

Sum_{n >= 0} a(n+1)*x^n/n! =

Sum_{k >= 0} a(k)*x^k/k! * Sum_{j >= 0} c(j+1)*x^j

Comparing coefficients of x^n produces the recurrence

a(n+1) = Sum_{j = 0..n} binomial(n,j)*j!*c(j+1)*a(n-j)   ...(2)

Replacing n by n+k-1 gives

a(n+k) = Sum_{j = 0..n+k-1} binomial(n+k-1,j)*j!*c(j+1)*a(n+k-1-j)

== Sum_{j = 0..k-1} binomial(n+k-1,j)*j!*c(j+1)*a(n+k-1-j)

modulo k,   ...(3)

since j! == 0 (mod k) for j >= k.

Clearly,

binomial(n+k-1, j)*j! = Product_{i = 0..j-1} (n+k-1-i)

== Product_{i = 0..j-1} (n-1-i) (mod k)

= binomial(n-1, j)*j! (mod k)

Substituting into (3) gives

a(n+k) == Sum_{j = 0..k-1} binomial(n-1,j)*j!*c(j+1)*a(n+k-1-j)

(mod k)   ...(4)

We now prove the congruence a(n+k) == a(n)*a(k) (mod k) by induction on n.

For n = 0, a(n+k) = = a(0)*a(k) (mod k) since a(0) = 1.

Assume now that a(m+k) == a(m)*a(k) (mod k) for m < n.

In (4), n+k-1-j = (n-1-j) + k, where n-1-j < n.

Therefore, by the induction hypothesis,

a(n+k-1-j) == a(n-1-j)*a(k) (mod k).

Hence, from (4)

a(n+k) == a(k)*Sum_{j = 0..k-1} binomial(n-1,j)*j!*c(j+1)*a(n-1-j)

== a(k)*Sum_{j = 0..n-1} binomial(n-1,j)*j!c(j+1)*a(n-1-j)

== a(k)*a(n) (mod k),

by the recurrence (2), and where we were justified in extending the sum to n-1 since k divides j! for j >= k. This completes the proof of the Theorem.

For examples of the Theorem see A000085 and A244430.

Acknowledgements:

We gratefully acknowledge the use of ChatGPT (OpenAI) to prove Theorem 1. The author takes full responsibility for the final text.

References

[1] Ahaan Kallat, A Proof of Bala's Congruence Conjecture for A028342, arXiv:2607.18313 [math.CO], 2026.