Canadian Mathematical Olympiad 2022

A competition of the Canadian Mathematical Society.

Official Problem Set

P1. Assume that real numbers a and b satisfy

$$ab + \sqrt{ab+1} + \sqrt{a^2+b} \cdot \sqrt{b^2+a} = 0.$$

Find, with proof, the value of

$$a\sqrt{b^2+a} + b\sqrt{a^2+b}.$$

P2. Let $d(k)$ denote the number of positive integer divisors of k. For example, $d(6) = 4$ since 6 has 4 positive divisors, namely, 1, 2, 3, and 6. Prove that for all positive integers n,

$$d(1) + d(3) + d(5) + \cdots + d(2n-1) \leq d(2) + d(4) + d(6) + \cdots + d(2n).$$

P3. Let $n \geq 2$ be an integer. Initially, the number 1 is written n times on a board. Every minute, Vishal picks two numbers written on the board, say a and b, erases them, and writes either $a+b$ or $\min\{a^2, b^2\}$. After $n-1$ minutes there is one number left on the board. Let the largest possible value for this final number be $f(n)$. Prove that

$$2^{n/3} < f(n) \leq 3^{n/3}.$$

P4. Let n be a positive integer. A set of n distinct lines divides the plane into various (possibly unbounded) regions. The set of lines is called "nice" if no three lines intersect at a single point. A "colouring" is an assignment of two colours to each region such that the first colour is from the set $\{A_1, A_2\}$, and the second colour is from the set $\{B_1, B_2, B_3\}$. Given a nice set of lines, we call it "colourable" if there exists a colouring such that

(a) no colour is assigned to two regions that share an edge;

(b) for each $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$ there is at least one region that is assigned with both A_i and B_j.

Determine all n such that every nice configuration of n lines is colourable.

P5. Let $ABCDE$ be a convex pentagon such that the five vertices lie on a circle and the five sides are tangent to another circle inside the pentagon. There are $\binom{5}{3} = 10$ triangles which can be formed by choosing 3 of the 5 vertices. For each of these 10 triangles, mark its incenter. Prove that these 10 incenters lie on two concentric circles.

Important!

Please do not discuss this problem set online for at least 24 hours!

