1. Find all integer solutions to the equation $7x^2y^2 + 4x^2 = 77y^2 + 1260$.

   Solution:

   Notice that all coefficients are divisible by 7 except for 4, so x must be divisible by 7.

   We can rewrite and factor the equation as:

$$(x^2 - 11)(7y^2 + 4) = 1216.$$

   Notice that if $y = 0$ then $x^2 = 315$ and we have no solutions. Thus $y^2 \geq 1$. We can rewrite this equation as

$$\begin{aligned} x^2 &= \tfrac{1216}{7y^2+4} + 11 \\ &\leq \tfrac{1216}{11} + 11 \\ &< 122 \end{aligned}$$

   Since x is a multiple of 7, we only have $x = 0, \pm 7$ as possible solutions. When $x = \pm 7$ we get $y = \pm 2$ and when $x = 0$ there are no solutions.

   Thus, there are 4 solutions of the form $(\pm 7, \pm 2)$.

2. A polynomial $f(x)$ with integer coefficients is said to be *tri-divisible* if 3 divides $f(k)$ for any integer k. Determine necessary and sufficient conditions for a polynomial to be tri-divisible.

Solution:

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and suppose $f(x)$ is tridivisible. Then $f(0), f(1),$ and $f(-1)$ are all divisible by 3.

$$\begin{aligned} f(0) &= a_0 &\equiv 0 \pmod 3 \\ f(1) &= a_0 + a_1 + \cdots + a_n &\equiv 0 \pmod 3 \\ f(-1) &= a_0 - a_1 + \cdots \pm a_n &\equiv 0 \pmod 3 \end{aligned}$$

Adding all three equivalences and dividing by 2 yields $a_2 + a_4 + \cdots \equiv 0 \pmod 3$.

Subtracting the second equivalence from the third yields $a_1 + a_3 + \cdots \equiv 0 \pmod 3$.

Thus, if $(f(x)$ is tridivisible, it is necessary for the following 3 quantities to be divisible by 3: the constant term, the sum of the coefficients of terms with odd powers of x, and the sum of the coefficients of terms with even non-zero powers of x. Call a polynomial that satisfies these condition *happy*. We will show that these conditions are also sufficient by showing that every happy polynomial is tri-divisible.

Suppose $f(x)$ is happy and let n be an integer.

- If $n \equiv 0 \pmod 3$ then $f(n) \equiv a_0 \equiv 0 \pmod 3$.
- If $n \equiv 1 \pmod 3$ then $f(n) \equiv a_0 + (a_2 + a_4 + \cdots) + (a_1 + a_3 + \cdots) \equiv 0 \pmod 3$.
- If $n \equiv -1 \pmod 3$ then $f(x) \equiv a_0 + (a_2 + a_4 + \cdots) + (-a_1 - a_3 - \cdots) \equiv 0 \pmod 3$.

Thus $f(x)$ is tri-divisible, so the conditions are both necessary and sufficient.

3. Let N be a 3-digit number with three distinct non-zero digits. We say that N is *mediocre* if it has the property that when all six 3-digit permutations of N are written down, the average is N. For example, $N = 481$ is mediocre, since it is the average of $\{418, 481, 148, 184, 814, 841\}$.

   Determine the largest mediocre number.

   Solution:

   Suppose abc is a mediocre number. The 6 permutations are $\{abc, acb, bac, bca, cab, cba\}$. The sum of these numbers is $222(a + b + c$ and the average is $37(a + b + c)$. Since abc is mediocre we have $100a + 10b + c = 37(a + b + c)$.

   We can rearrange this equation to get $63a = 27b + 36c$. Notice that $63 = 27 + 36$ so we must have a strictly between b and c. Thus, $a \neq 9$.

   Notice if that $b = a + 1$ then we have $36a = 27 + 36c$ which has no integer solutions, and if $c = a + 1$ then we have $27a = 27b + 36$ which has no integer solutions. Since a is strictly between b and c, if $a = 8$ then b or c would have to be 9, but this cannot happen. Thus, $a \neq 8$.

   If $a = 7$ then either $b = 9$ or $c = 9$. This would reduce to $441 = 243 + 36c$ or $441 = 27b + 324$. Neither of these has integer solutions, so $a \neq 7$.

   If $a = 6$ then we have $378 = 27b + 36c$ which reduces to $42 = 3b + 4c$. This gives $b = 14 - \frac{4}{3}c$. To get integer digit values we must have $c = 3, 6,$ or 9 which gives corresponding b values of $10, 6,$ and 2 respectively. We can omit the first two, since 10 is not a digit and 666 is not mediocre. Thus, the largest mediocre number is 629.

4. Given an acute-angled triangle ABC whose altitudes from B and C intersect at H, let P be any point on side BC and X, Y be points on AB, AC, respectively, such that $PB = PX$ and $PC = PY$. Prove that the points A, H, X, Y lie on a common circle.

Solution:

Let E be on AC such that $AC \perp BE$ and F on AB such that $AB \perp CF$. From the problem statement, BE and CF intersect at H. Let M, N be the midpoint of BX and CY, respectively. Then $PM \perp AB$ and $PN \perp AC$. Moreover, $PM||CF$ and $PN||BE$. Hence, $BP/BC = BM/BF$ and $CP/CB = CN/CE$. Therefore,

$$\frac{BM}{BF} + \frac{CN}{CE} = \frac{BP}{BC} + \frac{CP}{CB} = 1.$$

Hence,

$$\frac{BX}{BF} + \frac{CY}{CE} = 2.$$

Hence, there is a real number r such that

$$\frac{BX}{BF} = 1 + r, \quad \frac{CY}{CE} = 1 - r.$$

By symmetry, we may assume that $r \geq 0$. Hence, X lies on ray BF pass F and Y lies on segment CE.

Hence, to show that A, H, X, Y lie on the same circle, it suffices to show that $\angle FXH = \angle EYH$. Since $\angle XFH = \angle YEH = 90°$, it suffices to show that $XF/FH = YE/EH$.

Since $BX/BF = 1 + r, r = (BX - BF)/BF = XF/BF$. Similarly, $r = YE/CE$. Therefore,

$$XF/BF = YE/CE \tag{1}$$

Note also that $\angle FBH = \angle ABE = 90 - \angle BAC = \angle ACF = \angle ECH$. Therefore,

$$HF/BF = HE/CE. \tag{2}$$

Hence, dividing (1) by (2) yield, $XF/FH = YE/EH$, as desired.

5. Let x and y be positive real numbers such that $x + y = 1$. Show that

$$\left(\frac{x+1}{x}\right)^2 + \left(\frac{y+1}{y}\right)^2 \geq 18.$$

Solution:

$$\begin{aligned}
\left(\tfrac{x+1}{x}\right)^2 + \left(\tfrac{y+1}{y}\right)^2 &= \left(\tfrac{x+1}{x}\right)^2 + \left(\tfrac{2-x}{1-x}\right)^2 && (y = 1-x) \\
&\geq 2\left(\tfrac{x+1}{x}\right)\left(\tfrac{2-x}{1-x}\right) && \text{(AM-GM inequalitty)} \\
&= 2\left(\tfrac{2+x-x^2}{x(1-x)}\right) \\
&= 2\left(\tfrac{2+x-x^2}{x(1-x)} - 9 + 9\right) \\
&= 2\left(\tfrac{2+x-x^2-9x+9x^2}{x(1-x)} + 9\right) \\
&= 2\left(\tfrac{8x^2-8x+2}{x(1-x)}\right) + 18 \\
&= 4\tfrac{(2x-1)^2}{x(1-x)} + 18 \\
&\geq 18
\end{aligned}$$

6. Let $\triangle ABC$ be a right-angled triangle with $\angle A = 90^\circ$, and $AB < AC$. Let points D, E, F be located on side BC so that AD is the altitude, AE is the internal angle bisector, and AF is the median.

   Prove that $3AD + AF > 4AE$.

   Solution:

   Notice that scaling the sides of the triangle does not change whether the inequality is true or false, so without loss of generality we may assume the length of side AC is 1. Let a be the length of side AB. By the Pythagoren Theorem, $BC = \sqrt{1+a^2}$.

   Since AD is the altitude, $AD \cdot BC = AB \cdot AC$, so $AD = \frac{a}{\sqrt{1+a^2}}$.

   Since AF is the median and ABC is right-angled at A, $AF = BF = CF = \frac{\sqrt{1+a^2}}{2}$.

   Since AE is the angle bisector, we have $\frac{BF}{CF} = \frac{BA}{CA}$, which gives that $BF = \frac{a}{\sqrt{1+a^2}}$ and $CF = \frac{1}{\sqrt{1+a^2}}$. Drop a perpendicular from F to AB at G and to AC at H. Since AE is the angle bisector of a right angle, these perpendiculars have the same length. Let this length be x. By similar triangles, $\frac{1-x}{x} = \frac{HC}{GF} = \frac{CF}{BF} = \frac{1}{a}$. Solving for x yields $x = \frac{a}{1+a}$ and so $AF = \frac{\sqrt{2}a}{1+a}$.

   Thus, the inequality we want to show is $\frac{3a}{\sqrt{1+a^2}} + \frac{\sqrt{1+a^2}}{2} > \frac{4\sqrt{2}a}{1+a}$.

   Observe that since a is positive and not equal to 1 the following inequalities are true:

$$
\begin{aligned}
& & (a-1)^4(a^2+18a+1) &> 0 \\
\Leftrightarrow & & a^6+14a^6-65a^4+100a^3+63a^2+14a+1 &> 0 \\
\Leftrightarrow & & a^6+14a^5+63a^4+100a^3+63a^2+14a+1 &> 128a^4+128a^2 \\
\Leftrightarrow & & (a^2+6a+1)^2(a+1)^2 &> 128a^2(a^2+1) \\
\Leftrightarrow & & \tfrac{(a^2+6a+1)^2}{a^2+1} &> \tfrac{128a^2}{(a+1)^2} \\
\Leftrightarrow & & \tfrac{a^2+6a+1}{\sqrt{a^2+1}} &> \tfrac{8\sqrt{2}a}{a+1} \\
\Leftrightarrow & & \tfrac{6a}{\sqrt{1+a^2}} + \tfrac{1+a^2}{\sqrt{1+a^2}} &> \tfrac{8\sqrt{2}a}{a+1} \\
\Leftrightarrow & & \tfrac{3a}{\sqrt{1+a^2}} + \tfrac{\sqrt{1+a^2}}{2} &> \tfrac{4\sqrt{2}a}{1+a}
\end{aligned}
$$

   Thus, the desired inequality is true.

7. A $(0_x, 1_y, 2_z)$-string is an infinite ternary string such that:

   - If there is a 0 in position i, then there is a 1 in position $i + x$
   - If there is a 1 in position j then there is a 2 in position $j + y$,
   - if there is a 2 in position k then there is a 0 in position $k + z$.

   For how many ordered triples of positive integers (x, y, z) with $x, y, z \leq 100$ does there exist $(0_x, 1_y, 2_z)$-string?

   Solution:

   It is clear that any $(0_x, 1_y, 2_x)$ string must contain at least one of each of the three digits. Suppose we have a 0 in position n. Then there is a 1 in position $n+x$, a 2 in position $n+x+y$ and a 0 in position $n + x + y + z$. It is similarly true that the digit in position k is the same as the digit in position $k + x + y + z$ for any k and any digit. Also note that if we know the digit in position $n + x + y + z$ then the digit in position n is the same as that.

   Given any block of $x + y + z$ consecutive digits in the string, we claim that the number of 0s, 1s, and 2s in that block must be the same. Notice that any block of $x + y + z$ digits is identical up to a cyclic reordering. Given any 0 in the block, there is a 1 that is (cyclically) x positions after it and a 2 that is (cyclically) $x + y$ positions after it. For any two different 0s in the block, the corresponding 1s and 2s are different. Thus there are at least as many 1s and 2s as there are 0s. A similar argument shows there must be the same number of each. This tells us that $x + y + z$ is divisible by 3.

   We claim that for any positive integer k, a $(0_x, 1_y, 2_z)$-string exists if and only if a $(0_{kx}, 1_{ky}, 2_{kz})$-string exists. Given a $(0_x, 1_y, 2_z)$-string, we can create a $(0_{kx}, 1_{ky}, 2_{kz})$-string by repeating each digit k times. Given a $(0_{kx}, 1_{ky}, 2_{kz})$-string we can create a $(0_x, 1_y, 2_z)$ by taking every kth digit.

   First, let us assume that $3 \not| \gcd(x, y, z)$. Then since $x+y+z = 0$ we have either $x \equiv y \equiv z \equiv 1 \pmod 3$ or $x \equiv y \equiv z \equiv 2 \pmod 3$ or (x, y, z) are equivalent to $(0, 1, 2)$ in some order.

   Consider the strings $S_1 = 012012\cdots$ and $S_2 = 021021\cdots$. It is easy to see that if $x \equiv y \equiv z \equiv 1 \pmod 3$ then S_1 is a $(0_x, 1_y, 2_z)$-string and when they are $\equiv 2 \pmod 3$ that S_2 is.

   By the above claim, if $x \equiv y \equiv z \equiv 3$ or $6 \pmod 9$ then such a string also exists. Similarly when they are all equivalent to 9 or $18 \pmod{27}$, or when they are equivalent to 27 or $54 \pmod{81}$ or when they are equivalent to $81 \pmod{243}$.

   We count the number of such triples:

x y z	mod	number
1	3	$34^3 = 39304$
2	3	$33^3 = 35937$
3	9	$11^3 = 1331$
6	9	$11^3 = 1331$
9	27	$4^3 = 64$
18	27	$4^3 = 64$
27	81	$1^3 = 1$
54	81	$1^3 = 1$
81	243	$1^3 = 1$

The total is $39304 + 35937 + 1331 + 1331 + 64 + 64 + 1 + 1 + 1 = 78034$.

We claim that when x, y, z are $0, 1, 2 \pmod 3$ in some order there are no strings. Without loss of generality, assume that $x < y$ and consider a position k in the sequence that is a 0. Then there is a 1 in position $k + x$ and a 2 in position $k + x + y$. Consider the number M that is in position $k + 2x + y$. M can't be 1, since the number in position $k + x + y$ would have to be 0 and M can't be 2, since the number in position $k + x$ would have to be 1. Thus, M is 1. By induction, we can see that for all non-negative integers i, the number in position $k + i(2x + y)$ is 1.

We can similarly show that the number in position $x + y$ must be 1. Thus, the number in position $k + y - x$ must be 1. We can also show then that for all non-negative j the number in position $k + j(y - x)$ must be 1.

Since $x \not\equiv y \pmod 3$, $2x + y$ is not a multiple of 3. Thus, we have that $\gcd(2x + y, y - x) = \gcd(2x + y, 3x)$ which must be a factor of x. Thus, there exist non-negative integers r, s so that $r(2x + y) - s(y - x) = x$. If we consider the numbers in positions $k + r(2x + y)$ and $k + s(y - x)$, they must both be 1 by the above results. However, these positions differ by x, so an 0 in position $k + s(y - x)$ would mean a 1 in position $k + r(2x + y)$. Thus, there are no solutions with x, y, z congruent to $0, 1, 2 \pmod 3$ in some order.

Therefore there are 78034 triples.

8. A magical castle has n identical rooms, each of which contains k doors arranged in a line. In room i, $1 \leq i \leq n-1$ there is one door that will take you to room $i+1$, and in room n there is one door that takes you out of the castle. All other doors take you back to room 1. When you go through a door and enter a room, you are unable to tell what room you are entering and you are unable to see which doors you have gone through before. You begin by standing in room 1 and know the values of n and k. Determine for which values of n and k there exists a strategy that is guaranteed to get you out of the castle and explain the strategy. For such values of n and k, exhibit such a strategy and prove that it will work.

   Solution:

   We will show that such a strategy exists for any value of n and k. It is clear that if $k = 1$ then it is trivial to escape the castle. In each room, label the doors in order from left to right as $0, 1, \ldots, k-1$. There are k^n different possible routes to escape from the castle, which we can express as n-digit string with digits from 0 to $k-1$ where the ith digit of the number corresponds to the door in room i. Clearly one such string is the correct way to exit the castle. We order the strings lexographically and try each string in order. In order to test a string, we need to go through the doors in the indicated order while starting from room 1. Suppose we are trying the string $d_1 d_2 \cdots d_n$. We choose a number $m \neq d_1$ and go through door m n times and then go through doors $d_1, d_2, \ldots, d_n$ in order. If door d_1 is the first door, then going through door m n times will guarantee we are in room 1. If door d_1 is not the first door, then $d_1 d_2 \cdots d_n$ is not the correct escape path, so it doesn't matter where we test it from. This gives us a way to test every possible sequence of doors and to ensure we are in room 1 if it is the correct sequence, so we are guaranteed to find our way out.