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NOTE ON COREFUL DIVISOR FUNCTIONS

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Summary. We say that a divisor d of n is a coreful divisor if it has the same set of distinct prime factors as n. The number and sum of such divisors of n are denoted respectively by $\tau^{(c)}(n)$ and $\sigma^{(c)}(n)$. In the present paper, we discuss some properties of these functions and characterize coreful perfect numbers, defined as integers n such that $\sigma^{(c)}(n) = 2n$.

1 INTRODUCTION

Throughout this note, we let $p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$ denote the canonical decomposition of the integer $n > 1$. An integer d is called an exponential divisor (or briefly, e-divisor) of n if $d = p_1^{d_1} p_2^{d_2} \cdots p_r^{d_r}$ with $d_i \mid n_i$ $(1 \le i \le r)$. Let $\tau^{(e)}(n)$ and $\sigma^{(e)}(n)$ denote, respectively, the number and sum of exponential divisors of n, where $\tau^{(e)}(1) = \sigma^{(e)}(1) = 1$ by convention. It is evident that $\tau^{(e)}(n)$ and $\sigma^{(e)}(n)$ are multiplicative functions, and hence

$$\tau^{(e)}(n) = \prod_{i=1}^{r} \tau(n_i)$$

and

$$\sigma^{(e)}(n) = \prod_{i=1}^{r} \sum_{d_i \mid n_i} p_i^{d_i},$$

where $\tau(n)$ stands for the number of divisors of n.

Similar to the notion of e-divisor, we say that the positive integer d is a coreful divisor (or c-divisor) of n, if $d = p_1^{d_1} p_2^{d_2} \cdots p_r^{d_r}$ with $1 \le d_i \le n_i$ for all i. The number and sum of such divisors of n are denoted respectively by $\tau^{(c)}(n)$ and $\sigma^{(c)}(n)$. By convention, 1 is a coreful divisor of itself, so that $\tau^{(c)}(1) = \sigma^{(c)}(1) = 1$. It should be noted that the notion of coreful divisors was introduced by Hardy and Subbarao [1].

Easily one can show that both $\tau^{(c)}(n)$ and $\sigma^{(c)}(n)$ are multiplicative functions and one can have

$$\tau^{(c)}(n) = \prod_{i=1}^{r} n_i$$

and

$$\sigma^{(c)}(n) = \prod_{i=1}^{r} \sum_{k=1}^{n_i} p_i^k.$$

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Moreover, the set $D^{(c)}(n)$ of all c-divisors of n forms a commutative monoid with the identity element $\prod_{i=1}^{r} p_i$ (respectively n) under the gcd operation (respectively under the lcm operation).

Many authors have studied the properties of the e-divisors (see, for example, [2-5]).

For a maximal order of $\tau^{(e)}(n)$, Erdős [4, Theorem 6.2] showed that:

$$\limsup_{n\to\infty} \frac{\log \tau^{(e)}(n) \log_2 n}{\log n} = \frac{\log 2}{2},$$

where $\log_m x$ denotes the m-fold iterated logarithm. Straus and Subbarao [3] proved several results related to exponentially perfect numbers, that is, integers n for which $\sigma^{(e)}(n) = 2n$ (see the sequence A054979 in the OEIS [6]), and they settled a question raised in [4] by proving that there are no odd exponentially perfect numbers.

In this paper, inspired by above results, we prove several results concerning the coreful divisor functions. Additionally, we investigate and characterize coreful perfect numbers, defined as integers n such that $\sigma^{(c)}(n) = 2n$.

2 MAIN RESULTS

The following properties are elementary, but still worth mentioning.

Theorem 2.1

1) n is prime if and only if $\tau^{(c)}(n) = 1$.
2) $\tau^{(c)}(n)$ is prime if and only if $n = p^q k$, where p and q are primes and k is square-free integer.
3) We have

$$\tau^{(c)}(n^k) = k^{w(n)}\, \tau^{(c)}(n) \text{ for all } k \in \mathbb{R}^*,$$

where $w(n)$ denotes the number of distinct prime divisors of n.

4) For all positive integers m and n we have

$$\tau^{(c)}(mn) \geq \tau^{(c)}(m)\, \tau^{(c)}(n).$$

Proof. Items 1, 2, and 3 follow at once from the definition. Now, we prove item 4. Clearly, the equality holds when $\gcd(m,n) = 1$, since $\tau^{(c)}$ is multiplicative. If $\gcd(m,n) > 1$, then we can write m and n as:

$$n = \left(\prod_{i=1}^{t} p_i^{n_i}\right)\left(\prod_{i=t+1}^{r} p_i^{n_i}\right) \text{ and } m = \left(\prod_{i=1}^{t} p_i^{m_i}\right)\left(\prod_{i=t+1}^{s} q_i^{m_i}\right).$$

Thus,

$$\tau^{(c)}(mn) = \prod_{i=1}^{t} (n_i + m_i)\left(\prod_{i=t+1}^{r} n_i\right)\left(\prod_{i=t+1}^{s} m_i\right).$$

Notice that

$$\prod_{i=1}^{t} (n_i + m_i) \geq \left(\prod_{i=1}^{t} n_i\right) + \left(\prod_{i=1}^{t} m_i\right).$$

From which,

$$\tau^{(c)}(mn) \geq \left(\left(\prod_{i=1}^{t} n_i\right) + \left(\prod_{i=1}^{t} m_i\right)\right)\left(\prod_{i=t+1}^{r} n_i\right)\left(\prod_{i=t+1}^{s} m_i\right)$$
$$\geq \left(\prod_{i=1}^{r} n_i\right)\left(\prod_{i=1}^{s} m_i\right) = \tau^{(c)}(m)\,\tau^{(c)}(n).$$

The proof is finish.

Let k be a non-negative integer. We define E_k to be the Diophantine equation

$$E_k\colon\ \varphi(n) - \tau^{(c)}(n) = k,$$

where φ denotes the well known Euler's totient function (see, e.g. [7]). Let S_k be the set of solutions of E_k. In what follows, we will prove some facts about E_k, beginning with the following remark.

Remark 2.2 Let m be an odd positive integer. If $m \in S_k$, then $2m \in S_k$ as well. This follows from the fact that:

$$\varphi(2) = \tau^{(c)}(2) = 1,$$

and from the multiplicative properties of both functions.

Theorem 2.3 We have $S_0 = \{1, 2, 4\}$, and $\varphi(n) > \tau^{(c)}(n)$ for all $n \geq 5$.

Proof. Clearly, $n = 1$ is a solution of E_0. By Remark 2.2, $2 \in S_0$ as well.

Now, consider powers of 2. For $\alpha \geq 1$, it can be shown by induction that $2^{\alpha-1} \geq \alpha$ with equality only when $\alpha = 1$ or $\alpha = 2$. Therefore, for these values:

$$\varphi(2^\alpha) = 2^{\alpha-1} = \alpha = \tau^{(c)}(2^\alpha),$$

holds for $\alpha = 1$ and $\alpha = 2$, so that $4 \in S_0$.

On the other hand, if $p \geq 3$ is prime, then

$$p^{\alpha-1}(p-1) \geq 2 \cdot 3^{\alpha-1} > \alpha \text{ for all } \alpha \geq 1.$$

Thus $\varphi(n) > \tau^{(c)}(n)$ for all odd $n \neq 1$.

Finally, assume that $n = 2^\alpha m$, with $m > 1$ and $\gcd(2, m) = 1$. Then

$$\varphi(n) = 2^{\alpha-1}\varphi(m) > 2^{\alpha-1}\,\tau^{(c)}(m) \geq \alpha\tau^{(c)}(m) = \tau^{(c)}(n).$$

Consequently, $S_0 = \{1, 2, 4\}$, and $\varphi(n) > \tau^{(c)}(n)$ for all $n \geq 5$.

Theorem 2.4 We have

1) $S_1 = \{3, 6, 8\}$, and $\varphi(n) > \tau^{(c)}(n) + 1$ for all $n \geq 9$.
2) $S_2 = \{12\}$, and $\varphi(n) > \tau^{(c)}(n) + 2$ for all $n \geq 13$.
3) $S_3 = \{5, 10\}$, and $\varphi(n) > \tau^{(c)}(n) + 3$ for all $n \geq 11$.
4) $S_4 = \{9, 16, 18\}$, and $\varphi(n) > \tau^{(c)}(n) + 4$ for all $n \geq 19$.
5) $S_5 = \{7, 14, 24\}$, and $\varphi(n) > \tau^{(c)}(n) + 5$ for all $n \geq 25$.
6) $S_6 = \{20\}$, and $\varphi(n) > \tau^{(c)}(n) + 6$ for all $n \geq 21$.
7) $S_7 = \{15, 30\}$, and $\varphi(n) > \tau^{(c)}(n) + 7$ for all $n \geq 31$.
8) $S_8 = \{36\}$, and $\varphi(n) > \tau^{(c)}(n) + 8$ for all $n \geq 37$.
9) $S_9 = \{11, 22\}$, and $\varphi(n) > \tau^{(c)}(n) + 9$ for all $n \geq 23$.
10) $S_{10} = \{28\}$, and $\varphi(n) > \tau^{(c)}(n) + 10$ for all $n \geq 29$.

Proof. The proof proceeds in two main steps, which together allow us to solve the equation E_k.

Step 1: Find a positive integer n_k, such that

$$\varphi(n) > \tau^{(c)}(n) + k \text{ for all } n \geq n_k.$$

Such an integer n_k allways exists, due to the following observations:

a) There exists $\mu_k \in \mathbb{N}$ such that for all $\alpha \geq \mu_k$, we have

$$\varphi(2^\alpha) = 2^{\alpha-1} > \alpha + k = \tau^{(c)}(2^\alpha) + k.$$

b) For any prime $p \geq 3$ there exists $\eta_k \in \mathbb{N}$ such that for all $\alpha \geq \eta_k$,

$$\varphi(p^\alpha) = p^{\alpha-1}(p-1) \geq 2 \cdot 3^{\alpha-1} > \alpha + k = \tau^{(c)}(2^\alpha) + k.$$

For example, from Theorem 2.3, we can take $\mu_0 = 3, \eta_0 = 1$, and $n_0 = 5$.

Step 2: Use a computational tool (in our case, SageMath) to check whether there exists any integer $n \leq n_k$ that satisfies the equation E_k.

Theorem 2.5 One has, for any $\varepsilon > 0$,

$$\tau^{(c)}(n) \leq O_\varepsilon(n^\varepsilon).$$

Proof. We have

$$\frac{\tau^{(c)}(n)}{n^\varepsilon} = \prod_{i=1}^{r} \frac{n_i}{p_i^{n_i\varepsilon}}$$

$$= \left(\prod_{i\leq r, p_i < 2^{1/\varepsilon}} \frac{n_i}{p_i^{n_i\varepsilon}}\right)\left(\prod_{i\leq r, p_i \geq 2^{1/\varepsilon}} \frac{n_i}{p_i^{n_i\varepsilon}}\right).$$

In the second product, $p_i \geq 2^{1/\varepsilon}$, so that $p_i^{n_i\varepsilon} \geq 2^{n_i}$ and

$$\frac{n_i}{p_i^{n_i\varepsilon}} \leq \frac{n_i}{2^{n_i}} \leq 1.$$

From which we must estimate the first product. Since

$$n_i\varepsilon \log 2 \leq e^{n_i\varepsilon \log 2} = 2^{n_i\varepsilon} \leq p^{n_i\varepsilon},$$

we have

$$\frac{n_i}{p_i^{n_i\varepsilon}} \leq \frac{1}{\varepsilon \log 2}.$$

Thus,

$$\frac{\tau^{(c)}(n)}{n^\varepsilon} \leq \left(\prod_{p<2^{1/\varepsilon}} \frac{1}{\varepsilon \log 2}\right).$$

In particular, $\tau^{(c)}(n) \leq O_\varepsilon(n^\varepsilon)$ as desired.

The following theorem gives the maximal order of $\tau^{(c)}(n)$.

Theorem 2.6 One has

$$\limsup_{n\to\infty} \frac{\log \tau^{(c)}(n) \log\log n}{\log n} = \frac{\log 3}{3}.$$

The proof of this theorem based on the following lemma.

Lemma 2.7 [8] Let F be a multiplicative function with $F(p^a) = f(a)$ for every prime power p^a, where f is positive and satisfies $f(n) = O(n^\alpha)$ for some fixed $\alpha > 0$. Then

$$\limsup_{n\to\infty} \frac{\log F(n) \log\log n}{\log n} = \sup_{m\geq 1} \frac{\log F(m)}{m}.$$

Proof of Theorem 2.6. Apply Lemma 2.7 for $F(n) = \tau^{(c)}(n)$, $\alpha = 1$, and $f(a) = a$. Note that $\sup_{m\geq 1} \frac{\log m}{m} = \frac{\log 3}{3}$.

We devote the remainder of this paper to introducing and characterizing coreful perfect numbers. Let us start by the definition of coreful perfect numbers.

Definition 2.8 We call coreful perfect number any positive integer verifies

$$\sigma^{(c)}(n) = 2n.$$

The smallest coreful perfect number is 36, since its coreful divisors are 6, 12, 18, and 36, whose sum is $72 = 2 \cdot 36$. The sequence of coreful perfect numbers appears as sequence A307958 in the OEIS [6].

Since $\sigma^{(c)}(p) = p$ for any prime p and $\sigma^{(c)}$ is multiplicative, then if n is a coreful perfect number, mn is also a coreful perfect number for any squarefree number m coprime to n. Thus, there are infinitely many coreful perfect numbers, and all of them can be generated from what we call the sequence of *primitive coreful perfect* numbers, i.e., powerful coreful perfect numbers, the sequence of primitive coreful perfect numbers appears as sequence A307959 in the OEIS [6]. The following theorem gives the form of all even primitive coreful perfect numbers.

Theorem 2.9 A number N is an even primitive coreful perfect number if and only if it is of the form:

$$N = 2^p(2^p - 1)^2,$$

where $2^p - 1$ is Mersenne prime number.

Proof. Let $2^p - 1$ be a Mersenne prime number. We aim to show that $N = 2^p(2^p - 1)^2$ is a primitive coreful perfect number. We start by noting that if $2^p - 1$ is prime, then the exponent p must also be prime. Thus, we can write the coreful divisors of N as follows:

$$2^a(2^p - 1)^b, \ \ (a, b) \in \{1, 2, \dots, p\} \times \{1, 2\}.$$

Hence,

$$\begin{aligned}
\sigma^{(c)}(N) &= \sum_{b=1}^{2}\sum_{a=1}^{p} 2^a(2^p - 1)^b \\
&= 2(2^p - 1)^2 + 2(2^p - 1)^2(2^p - 1) \\
&= 2^{p+1}(2^p - 1)^2 \\
&= 2N.
\end{aligned}$$

Conversely, assume that N is an even primitive coreful perfect number. Note that $\sigma^{(c)}(2^a) = 2(2^a - 1) < 2^{a+1}$ for all a, which means that N cannot be a power of two. Thus, N has at least two factors, and we can write it as $N = 2^a l$, where $a \geq 2$ and l is an odd powerful number. Since $\sigma^{(c)}$ is multiplicative, we have

$$\sigma^{(c)}(N) = \sigma^{(c)}(2^a l) = 2(2^a - 1)\, \sigma^{(c)}(\, l\,).$$

On the other hand, as N is coreful perfect, we have

$$\sigma^{(c)}(N) = 2N = 2^{a+1} l.$$

By comparing the two formulas obtained for $\sigma^{(c)}(N)$, we deduce that

$$(2^a - 1)\, \sigma^{(c)}(\, l\,) = 2^a\, l. \tag{1}$$

This identity shows that 2^a divides $\sigma^{(c)}(\, l\,)$ since $\gcd(2^a,\ 2^a - 1) = 1$. Therefore, there exists a positive integer r such that $\sigma^{(c)}(\, l\,) = 2^a\, r$. Substituting this into (1), we obtain $l = (2^a - 1)r$. Thus, we get

$$l + r = (2^a - 1)r + r = 2^a\, r = \ \sigma^{(c)}(\, l\,),$$

from which r is a coreful divisor of l, i.e., l has only two coreful divisors: l and r. The only way this is possible is if l is a square of a prime number q. This implies that $(2^a - 1) = r = q$, which means that $N = 2^a\, q^2$ with $q = (2^a - 1)$ is a Mersenne prime. The proof is complete.

We conclude this note by presenting the following theorem, which provides conditions for the existence of odd coreful perfect numbers.

Theorem 2.10 An odd primitive coreful perfect number, if it exists, must have the form:

$$N = p^\alpha \prod_{i=1}^{s} p_i^{\alpha_i},$$

where $p \equiv 1 \pmod 4$, $\alpha \geq 2$, $\alpha \equiv 2 \pmod 4$, $\gcd(p, p_i) = 1$, $\alpha_i \equiv 1 \pmod 2$, and $\alpha_i \geq 3$ for all i.

Proof. Let $N = \prod_{i=1}^{r} p_i^{a_i}$ be an odd primitive coreful perfect number. Note that $\sigma^{(c)}(p^a) = \frac{p}{p-1}(p^a - 1) < 2p^a$ for all odd primes p and for all a, which means that N cannot be a power of any odd prime. Therefore, we necessarily have $r \geq 2$. Since N is a coreful perfect number and $\sigma^{(c)}$ is multiplicative, we have

$$\begin{aligned} \sigma^{(c)}(N) &= \sigma^{(c)}\left(p_1^{a_1}\right)\sigma^{(c)}\left(p_2^{a_2}\right) \cdots \sigma^{(c)}\left(p_r^{a_r}\right) \\ &= 2N = 2p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}. \end{aligned} \tag{2}$$

It is worth noting that $\sigma^{(c)}(p_i^{a_i})$ is even only when a_i is even. According to (2), $\sigma^{(c)}(N)$ is limited to having at most one factor of two. Thus, all a_i are odd, except for one. Without loss of generality, let a_1 be even.

Now, observe that $\sigma^{(c)}\left(p_1^{a_1}\right) \equiv 1 \pmod 4$, and either $p_1 \equiv 1 \pmod 4$ or $p_1 \equiv -1 \pmod 4$. The latter cannot be true, since

$$\begin{aligned} \sigma^{(c)}\left(p_1^{a_1}\right) &= p_1 + p_1^2 + \cdots + p_1^{a_1} \\ &\equiv \ -1 + 1 - 1 + \cdots + 1 \equiv 0 \pmod 4, \end{aligned}$$

which is a contradiction. Thus, it must be true that $p_1 \equiv 1 \pmod 4$. This congruence implies that

$$\sigma^{(c)}(p_1^{a_1}) \equiv 1 + 1 + \cdots + 1 \equiv a_1 \pmod 4$$
$$\equiv 2 \pmod 4.$$

Since N is primitive, we must have $a_1 \geq 2$ and $a_i \geq 3$ for $i \neq 1$. All that remains is to set $p = p_1$, $\alpha = a_1$, $\alpha_i = a_{i+1}$, and $s = r - 1$. This completes the proof.

3 CONCLUSIONS

In this work, we explored key properties of the core divisor function $\tau^{(c)}$ and $\sigma^{(c)}$, including its behavior on primes, powers, and multiplicative structure. We examined the Diophantine equation $\varphi(n) - \tau^{(c)}(n) = k$, determined complete solution sets for small values of k. We also established upper bounds and asymptotic behavior for $\tau^{(c)}(n)$. Finally, we introduced and characterized *coreful perfect numbers*, proving that there are infinitely many and identifying forms for both even and potential odd cases.

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REFERENCES

[1] G. E. Hardy and M.V.Subbarao, "Highly powerful numbers", *Congr. Numer.* **37**, 277-307 (1983).
[2] A. Smati and J. Wu, "On the exponential divisor function", *Publ. Inst. Math. (Beograd) (N.S.)*, 61, 21-32 (1997).
[3] E. G. Straus and M. V. Subbarao, "On exponential divisors", *Duke Math. J.*, 41, 465-471 (1974).
[4] M. V. Subbarao, "On some arithmetic convolutions, in The Theory of Arithmetic Functions", Lecture Notes in Mathematics, Vol. 251, Springer, 247-271 (1972).
[5] J. Wu, "Problème de diviseurs exponentiels et entiers exponentiellement sans facteur carré", *J. Théor. Nombres Bordeaux*. **7**, 133-141 (1995).
[6] N. J. A. Slone et al. "The *On-Line Encyclopedia of Integer Sequences*". Publish electronically at https://oeis.org/, (last modified on 3 February 2025, at 12:25).
[7] T. M. Apostol, "*Introduction to Analytic Number Theory*", Springer-Verlage, New York, (1976).
[8] D. Suryanarayana and R. Sita Rama Chandra Rao, "On the true maximum order of a class of arithmetical functions", *Math. J. Okayama Univ*. **17**, 95-101 (1975).

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