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THE COMBINATORIAL REPRESENTATIONS OF THE GENERALIZED FIBONACCI AND LUCAS NUMBER AND MATRIX SEQUENCES

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Summary. In this study, first of all we consider the generalized Fibonacci, generalized Lucas and Horadam numbers. Then, we define their matrix sequences which depend on two real parameters. We construct combinatorial representations of these number and matrix sequences. We present some elementary identities for these sequences.

1 INTRODUCTION

Many articles in the literature mention various special integer sequences. Horadam in [1] developed out of an interest in the Fibonacci sequence and a desire to extend the results. He also gave many properties of a certain generalized sequence of numbers in [2]. Koshy wrote a very detailed book about the Fibonacci and Lucas numbers in [3]. Catalini studied generalized bivariate polynomials, from which specifying initial values the bivariate Fibonacci and Lucas polynomials in [4]. He used a matrix approach to derive identities and inequalities. Catalani derived many identities for bivariate Fibonacci and Lucas polynomials using a matrix approach when the variables x and y are replaced by polynomials in [5]. He also gave examples of how to derive identities for Fibonacci and Lucas polynomials can be derived in [6]. Akyuz and Halici studied combinatorial identities for the generalized Fibonacci and Lucas sequences in [7]. Uygun denoted the combinatorial representation of Jacobsthal and Jacobsthal-Lucas matrix sequences and their generalized forms in [8]. The authors investigated some properties of the generalized Fibonacci, generalized Lucas and Horadam matrix sequences in [9]. Our main aim is to present new combinatoric results for the generalized Fibonacci, generalized Lucas, Horadam and their generalized matrix sequences.

Definition 1.1. Let n≥2 be any integer and $p,\ q$ real numbers and p The generalized Fibonacci , generalized Lucas $\{v_n(p,q)\}_{n\in\mathbb{N}}$, and Horadam sequence $\{h$ are defined by the following recurrence relations respectively

(1)

The characteristic equation of the recurrences (1) is The solutions are . We denote briefly

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The generalized Fibonacci, generalized Lucas and Horadam sequences satisfy the following properties [9]:

$$v_{n+1} = pu_n + 2qu_{n-1} = u_{n+1} + qu_{n-1},$$
$$(p^2 + 4q)u_{n-1} = pv_n + 2qv_{n-1},$$
$$h_{n+1} = bu_n + aqu_{n-1},$$
$$(p^2 + 4q)h_n = (bp + 2aq)v_n + q(2b - ap)v_{n-1}. \tag{2}$$

Binet formula enables us to find any elements of the generalized Fibonacci, the generalized Lucas and Horadam numbers easily. The Binet representations of the generalized Fibonacci, generalized Lucas and Horadam sequences are denoted respectively

$$u_n(p,q) = \frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2},$$
$$v_n(p,q) = r_1^n + r_2^n,$$
$$h_n(p,q) = \frac{(\mathrm{b} - \mathrm{a}r_2)r_1^{n+1} - (\mathrm{b} - \mathrm{a}r_1)r_2^{n+1}}{r_1 - r_2}.$$

Now, we want to carry the generalized Fibonacci, the generalized Lucas and Horadam sequences to matrix theory by the generalized Fibonacci $\{u_n(p,q)\}$, generalized Lucas $\{v_n(p,q)\}$, Horadam $\{h_n(p,q)\}$ sequences.

Definition 1.2. For any integer n≥2; the generalized Fibonacci matrix sequence is defined by

$$U_n(p,q) = pU_{n-1}(p,q) + qU_{n-2}(p,q),$$

with initial conditions $U_0(p,q) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $U_1 = \begin{bmatrix} p & 1 \\ q & 0 \end{bmatrix}$.

The generalized Lucas matrix sequence is defined by

$$V_n(p,q) = pV_{n-1}(p,q) + qV_{n-2}(p,q),$$

with initial conditions $V_0(p,q) = \begin{bmatrix} p & 2 \\ 2q & -p \end{bmatrix}$, and $V_1 = \begin{bmatrix} p^2 + 2q & p \\ qp & 2q \end{bmatrix}$.

The Horadam matrix sequence is given by

$$H_n(p,q) = pH_{n-1}(p,q) + qH_{n-2}(p,q),$$

with initial conditions $H_0(p,q) = \begin{bmatrix} b & a \\ aq & \frac{b-ap}{q} \end{bmatrix}$, and $H_1 = \begin{bmatrix} pb + aq & b \\ qb & qa \end{bmatrix}$.

Generalized Fibonacci $\{U_n(p,q)\}$, generalized Lucas $\{V_n(p,q)\}$; Horadam $\{H_n(p,q)\}$ matrix sequences can be demonstrated by using the elements of the generalized Fibonacci, generalized Lucas; Horadam number sequences as the follows

$$U_n = \begin{bmatrix} u_{n+1} & u_n \\ qu_n & qu_{n-1} \end{bmatrix}, \quad V_n = \begin{bmatrix} v_{n+1} & v_n \\ qv_n & qv_{n-1} \end{bmatrix}, \quad H_n = \begin{bmatrix} h_{n+1} & h_n \\ qh_n & qh_{n-1} \end{bmatrix}.$$

For any integer n≥1; the relations among generalized Fibonacci, generalized Lucas and

Horadam matrix sequences are obtained in [9] as

$$V_{n+1} = pU_n + 2qU_{n-1} = U_{n+1} + qU_{n-1},$$
$$(p^2 + 4q)U_{n-1} = pV_n + 2qV_{n-1},$$
$$H_{n+1} = bU_n + aqU_{n-1},$$
$$(p^2 + 4q)H_n = (bp + 2aq)V_n + q(2b - ap)V_{n-1}. \quad (3)$$

The Binet formulas for the generalized Fibonacci, generalized Lucas and Horadam matrix sequences are denoted respectively as

$$U_n = \frac{(U_1 - r_2U_0)r_1^{n+1}}{r_1 - r_2} - \frac{(U_1 - r_1U_0)r_2^{n+1}}{r_1 - r_2},$$
$$V_{n+1} = \frac{(V_2 - r_2V_1)r_1^{n}}{r_1 - r_2} - \frac{(V_2 - r_1V_1)r_2^{n}}{r_1 - r_2},$$
$$H_{n+1} = \frac{(H_2 - r_2H_1)r_1^{n}}{r_1 - r_2} - \frac{(H_2 - r_1H_1)r_2^{n}}{r_1 - r_2}.$$

2 COMBINATORIAL REPRESANTATIONS OF THE GENERALIZED FIBONACCI, THE GENERALIZED LUCAS AND HORADAM SEQUENCES

Lemma 2.1. For $n \in \mathbb{N}$, $\{u_n(p,q)\}$ is defined as follows provides the same recurrence relation with the generalized Fibonacci sequence as $u_n = pu_{n-1} + qu_{n-2}$ and with the initial values $u_0 = 1, \quad u_1 = p$

$$u_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} p^{n-2i} q^i$$

where $\binom{n}{k}$ should be zero when $k < 0$ or $k > n$.

Proof. If n is an even integer, then we have

$$pu_n + qu_{n-1} = p\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} p^{n-2i} q^i + q\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} p^{n-1-2i} q^i$$
$$= p^{n+1} + \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \left[\binom{n-i}{i} + \binom{n-i}{i-1}\right] p^{n+1-2i} q^i$$
$$= u_{n+1.}$$

If is an odd integer, then it is obtained that

$$pu_n + qu_{n-1} = p\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} p^{n-2i} q^i + q\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} p^{n-1-2i} q^i$$

$$= p^{n+1} + \sum_{i=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n+1-i}{i} p^{n+1-2i} q^i + \binom{n - \left\lfloor \frac{n+1}{2} \right\rfloor}{\left\lfloor \frac{n+1}{2} \right\rfloor - 1} q^{\left\lfloor \frac{n+1}{2} \right\rfloor}$$

$$= u_{n+1}$$

■

Some examples

- If $p = q = 1$, then we get the elements of the Fibonacci sequence $\{F_n\}$ as

  $F_n = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i}$.

- If $p = 1$, $q = 2$, then we get the entries of the Jacobsthal sequence $\{J_n\}$ as

  $J_n = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} 2^i$.

- If $p = 2$, $q = 1$, then we get the elements of the Pell sequence $\{P_n\}$ as

  $P_n = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} 2^{n-2i}$.

Theorem 2.2. For $n \geq 2$, $n \in \mathbb{N}$ it is obtained that for the generalized Fibonacci matrix sequences

$$U_n = u_n I_2 + u_{n-1} \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}. \tag{4}$$

Proof. The assumption is true for $n = 0$, so we have $U_0 = u_0 I_2$. Assume that the statement is true for $k < n$, For the value of $n + 1$, we have

$$U_{n+1} = pU_n + qU_{n-1}$$

$$= p\left(u_n I_2 + u_{n-1} \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}\right) + q\left(u_{n-1} I_2 + u_{n-2} \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}\right)$$

$$= (pu_n + qu_{n-1}) I_2 + (pu_{n-1} + qu_{n-2}) \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}$$

$$= u_{n+1} I_2 + u_n \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}.$$

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Corollary 2.3. The combinatoric representation of the generalized Fibonacci matrix sequences is obtained as

$$U_n = u_n I_2 + u_{n-1} \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}$$

$$= \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} p^{n-2i} q^i I_2 + \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-i}{i} p^{n-1-2i} q^i \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}.$$

Theorem 2.4. For $n \in \mathbb{N}$, $\{v_n(p,q)\}$; is defined as follows provides the same recurrence relation with the generalized Lucas sequence as $v_{n+1} = pv_n + qv_{n-1}$ and with the initial values $v_0 = 2,$ and $v_1 = p$

$$v_{n+1} = \sum_{i=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \left[\binom{n+1-i}{i} + \binom{n-i}{i-1}\right] + 1p^{n-2i}q^i.$$

Proof. By (2) and Lemma 2.1, we have

$$v_{n+1} = pu_n + 2qu_{n-1} = u_{n+1} + qu_{n-1}$$

$$= \sum_{i=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \binom{n+1-i}{i} p^{n+1-2i}q^i + q \sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-1-i}{i} p^{n-1-2i}q^i$$

$$= p^{n+1} + \sum_{i=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \binom{n+1-i}{i} p^{n+1-2i}q^i + \sum_{i=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \binom{n-i}{i-1} p^{n+1-2i}q^i$$

$$= \sum_{i=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \left[\binom{n+1-i}{i} + \binom{n-i}{i-1}\right] p^{n+1-2i}q^i.$$

■

Some examples

- If $p = q = 1$; then we get elements of the Lucas sequence

$$l_n = \sum_{i=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \left[\binom{n+1-i}{i} + \binom{n-i}{i-1}\right].$$

- If $p = 1$, $q = 2$, then we get elements of the Jacobsthal-Lucas sequence $\{c_n\}$ as

$$c_n = \sum_{i=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \left[\binom{n+1-i}{i} + \binom{n-i}{i-1}\right] 2^i.$$

- If $p = 2, q = 1$, then we get the elements of the Pell sequence $\{q_n\}$

$$q_n = \sum_{i=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \left[\binom{n+1-i}{i} + \binom{n-i}{i-1}\right] 2^{n-2i}.$$

The representations are important. Because they cover Pell, Pell-Lucas, Lucas, Fibonacci, Jacobsthal, Jacobsthal-Lucas and so much more! For example, let's substitute $p = 3$ and $q = -1$ then it will give us the sequence {2, 3, 7,18, 47,123, 322, 843, 2207, 5778, 15127, 39603,....} which is l_{2n} other Lucas number. If we look on the OEIS for this sequence, we find it at
https://oeis.org/A005248 and we discover that its binomial sum

$$l_{2n} = \sum_{i=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \left[\binom{n+1-i}{i} + \binom{n-i}{i-1}\right] 3^{n+1-2i}(-1)^i.$$

Theorem 2.5. For the generalized Lucas matrix sequence, we establish

$$V_n = v_{n+1}I_2 + v_n \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}. \tag{5}$$

Proof. By (3) and (4), it is obtained that

$$\begin{aligned} V_{n+1} &= U_{n+1} + qU_{n-1} \\ &= u_{n+1}I_2 + u_n \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix} + q\left\{u_{n-1}I_2 + u_{n-2} \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}\right\} \\ &= \ (u_{n+1} + qu_{n-1})\, I_2 + \ (u_n + qu_{n-2}) \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}. \end{aligned}$$

By the recurrence relation of the generalized Fibonacci sequence, we obtain the result.

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Corollary 2.6. The combinatoric representation of the generalized Lucas matrix sequences is also denoted as Theorem 2.4 and (5)

$$\begin{aligned} V_n &= v_{n+1}I_2 + v_n \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix} \\ &= \sum_{i=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \left[\binom{n+1-i}{i} + \binom{n-i}{i-1}\right] p^{n+1-2i} q^i\, I_2 \\ &+ \sum_{i=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \left[\binom{n-i}{i} + \binom{n-1-i}{i-1}\right] p^{n-2i} q^i \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}. \end{aligned}$$

Theorem 2.7. For the generalized Lucas matrix sequence, we also establish

$$V_n = v_n \begin{bmatrix} p & 1 \\ q & 0 \end{bmatrix} + qv_{n-1}I_2$$

Proof. The assumption is true for $n = 1$. Assume that the statement is true for $k < n$. For $k=n+1$, we have

$$\begin{aligned} V_{n+1} &= pV_n + qV_{n-1} \\ &= p\left\{v_n \begin{bmatrix} p & 1 \\ q & 0 \end{bmatrix} + qv_{n-1}I_2\right\} + q\left\{v_{n-1} \begin{bmatrix} p & 1 \\ q & 0 \end{bmatrix} + qv_{n-2}I_2\right\} \\ &= \begin{bmatrix} p & 1 \\ q & 0 \end{bmatrix} (pv_n + qv_{n-1}) + qI_2(pv_{n-1} + qv_{n-2}). \end{aligned}$$

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Corollary 2.8. The combinatorial representation for the generalized Lucas matrix sequence is also computed as

$$\begin{aligned} V_n &= \sum_{i=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \left[\binom{n+1-i}{i} + \binom{n-i}{i-1}\right] p^{n+1-2i} q^i \begin{bmatrix} p & 1 \\ q & 0 \end{bmatrix} \\ &+ \sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \left[\binom{n-i}{i} + \binom{n-1-i}{i-1}\right] p^{n-2i} q^{i+1}\, I_2. \end{aligned}$$

Theorem 2.9. For $n \in \mathbb{N}$; $\{h_n\}_{n\in\mathbb{N}}$, is defined as follows provides the same recurrence relation with Horadam sequence as $h_n = ph_{n-1} + qh_{n-2}$ and with the initial values $h_0 = a, \quad h_1 = b$

$$h_{n+1} = \sum_{i=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \left[b\binom{n-i}{i} + ap\binom{n-i}{i-1}\right] p^{n-2i} q^i.$$

Proof. By the relation with the generalized Fibonacci and Horadam sequences, we have

$$h_{n+1} = bu_n + aqu_{n-1}$$

$$= b\sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \binom{n-i}{i} p^{n-2i} q^i + aq \sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-1-i}{i} p^{n-1-2i} q^i.$$

If n is odd, then

$$h_{n+1} = bp^n + \sum_{i=1}^{\left\lfloor\frac{n}{2}\right\rfloor} \binom{n-i}{i} p^{n-2i} q^i + a \sum_{i=1}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-i}{i-1} p^{n+1-2i} q^i + a\binom{\frac{n-1}{2}}{\frac{n-1}{2}} q^{\frac{n+1}{2}}$$

$$= \sum_{i=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \left[b\binom{n-i}{i} + ap\binom{n-i}{i-1}\right] p^{n-2i} q^i.$$

If *n* is even, then we get

$$h_{n+1} = b\sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \binom{n-i}{i} p^{n-2i} q^i + aq \sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-1-i}{i} p^{n-1-2i} q^i$$

$$bp^n + \sum_{i=1}^{\left\lfloor\frac{n}{2}\right\rfloor} \binom{n-i}{i} p^{n-2i} q^i + a\sum_{i=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \binom{n-i}{i-1} p^{n+1-2i} q^i$$

$$= \sum_{i=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor} \left[b\binom{n-i}{i} + ap\binom{n-i}{i-1}\right] p^{n-2i} q^i.$$

By these findings, we get the result.

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Theorem 2.10. For Horadam matrix sequences, we obtain that

$$H_{n+1} = h_{n+1} I_2 + h_n \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}.$$

Proof. By the relation with the generalized Fibonacci and Horadam matrix sequences, we have

$$H_{n+1} = bU_n + aqU_{n-1}$$

$$= b\left(u_n I_2 + u_{n-1} \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}\right) + qa\left(u_{n-1} I_2 + u_{n-2} \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}\right)$$

$$= h_{n+1} I_2 + h_n \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}.$$

■

Corollary 2.11. The combinatorial representation of the Horadam matrix sequences is obtained as

$$H_{n+1} = h_{n+1} I_2 + h_n \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}$$

$$= \sum_{i=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \left[b \binom{n-i}{i} + ap \binom{n-i}{i-1} \right] p^{n-2i} q^i I_2 \quad 3$$

$$+ \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left[b \binom{n-1-i}{i} + ap \binom{n-1-i}{i-1} \right] p^{n-1-2i} q^i \begin{bmatrix} 0 & 1 \\ q & -p \end{bmatrix}.$$

3 CONCLUSIONS

In this paper, we investigate various combinatorial representations of generalized Fibonacci, generalized Lucas and Horadam number sequences. By using the elements generalized Fibonacci, generalized Lucas and Horadam number sequences, we get generalized Fibonacci, generalized Lucas and Horadam matrix sequences whose entries' size is 2×2. Then, we also investigate various combinatorial representations of generalized Fibonacci, generalized Lucas and Horadam matrix sequences.

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