The Mean of Two Symmetric 2-Densely Aggregated Compositions of n Equals the Associated Part of SRS(n).

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Notations, Definitions and Equations Relating to the Symmetric Representation of Sigma(n), SRS(n).

$n = 2^m \times q$, $m \geq 0$, q odd.

$r = \text{row}(n) = \left\lfloor \frac{1}{2}\left(\sqrt{8n+1} - 1\right)\right\rfloor$ = A003056(n).

SRS(n) denotes the area of the symmetric representation of sigma(n), i.e., the sum of the areas of its parts. The notation SRS(n) is used also for the list of areas of its (non-zero) parts.

srs(n) denotes the area of SRS(n) through the diagonal. The notation srs(n) is used also for the list of areas of its (non-zero) parts.

In order to indicate entries i in row n of a triangle in a sequence, we employ throughout the non-standard way of using two parameters with sequence numbers, e.g., A235791(n, i) and A249223(n, i).

Definitions 1

$a(n, i)$ = A235791(n, i) = $\left\lceil \frac{n+1}{i} - \frac{i+1}{2} \right\rceil$, for $1 \leq i \leq r$, and $a(n, r+1) = 0$.

A237591(n, i) = $a(n, i) - a(n, i+1)$, for $1 \leq i \leq r$.

$b(n, i)$ = A237048(n, i), for $1 \leq i \leq r$.

Lemma 1

(a) $a(n, i) = 2^m \times \frac{q}{i} - \frac{i+1}{2} + 1$, for $i|n$, $i \leq r$ and i odd.

(b) $a\left(n, 2^{m+1} \times i\right) = a\left(n, \frac{2 \times n}{j}\right) = \frac{j-1}{2} - 2^m \times \frac{q}{j} + 1$, for $j|n$, $r < j$, j odd, $q = i \times j$, and $2^{m+1} \times i \leq r$.

The entries in row n of the triangle in A237591 denote the lengths of the legs of the upper Dyck path for SRS(n) from coordinate (0, n) to the point where the path crosses the diagonal.

Proof

(a) $a(n, i) = \left\lceil \frac{n+1}{i} - \frac{i+1}{2} \right\rceil = \frac{n}{i} - \frac{i+1}{2} + \left\lceil \frac{n}{i} \right\rceil = 2^m \times \frac{q}{i} - \frac{i+1}{2} + 1.$

(b) $a\left(n, 2^{m+1} \times i\right) = a\left(n, \frac{2 \times n}{j}\right) = a\left(n, \frac{2 \times (2^m \times q)}{j}\right) = \left\lceil \frac{2^m \times q + 1}{2^{m+1} \times i} - \frac{2^{m+1} \times i + 1}{2} \right\rceil$

$= \left\lceil \frac{2^m \times q}{2^{m+1} \times i} + \frac{1}{2^{m+1} \times i} - \frac{2^{m+1} \times i}{2} - \frac{1}{2} \right\rceil = \left\lceil \frac{2^m \times j}{2^{m+1}} - \frac{1}{2} + \frac{1}{2^{m+1} \times i} - \frac{2^{m+1} \times q}{2 \times j} \right\rceil = \frac{j-1}{2} - 2^m \times \frac{q}{j} + 1.$

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Lemma 2

b(n, i) = 1, for $1 \le i \le r$, precisely when

(a) i is an odd divisor of n, or

(b) $i = 2^{m+1} \times s$ when $q = s \times t$ and $2^{m+1} \times s \le r < t$,

and b(n, i) = 0 otherwise.

Proof

b(n, i) = 1 precisely when $n = \frac{i \times (i+1)}{2} + k \times i$, for some $k \ge 0$, since column i starts at offset $\frac{i \times (i+1)}{2}$ and 1’s occur at multiples of i from that value and 0’s for all other rows in column i.

(a) Rewriting this equation yields $n = i \times \left(\frac{i+1}{2} + k\right)$ for odd i.

(b) First observe: $r < t \iff \left\lfloor \frac{1}{2}\left(\sqrt{8n+1} - 1\right)\right\rfloor < t \iff \frac{1}{2}\left(\sqrt{8n+1} - 1\right) < t$

$\iff \sqrt{8n+1} < 2 \times t + 1 \iff 8 \times 2^m \times s \times t + 1 < 4 \times t^2 + 4 \times t + 1$

$\iff 2^{m+1} \times s < t + 1 \iff 2^{m+1} \times s \le t - 1 \iff 2^{m+1} \times s \le r.$

Finally, for $i = 2^{m+1} \times s$ rewriting the equation for n yields:

$n - \frac{i}{2} = 2^m \times s \times t - 2^m \times s = 2^m \times s \times (t-1) = 2^{m+1} \times s \times \frac{(t-1)}{2} = i \times \frac{(t-1)}{2}$, that is, $i \mid (n - \frac{i}{2})$.

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Consequently, we can express the entries in row n of A237048 as follows:

$$b(n, i) = \begin{cases} 1 & \text{if } i \mid n \text{ and } i \text{ odd, or } i \mid \left(n - \frac{i}{2}\right) \text{ and } i \text{ even} \\ 0 & \text{otherwise} \end{cases}.$$

Definition 2

$c(n, i) = A249223(n, i) = \sum_{h=1}^{i} (-1)^{h+1} \times b(n, h)$, for $1 \le i \le r$; $c(n, 0) = 0$.

The entries in row n of the triangle in A249223 denote the widths between the two bounding Dyck paths that form SRS(n) up to the diagonal.

Corollary (of Lemma 2 and Definition 2)

(a) Each odd divisor $j > r$ of n is uniquely associated with the even number $2^{m+1} \times i \le r$ where $q = i \times j$, in other words, with b(n, i) = 1 also $b(n, \frac{2 \times n}{j}) = b(n, 2^{m+1} \times i) = 1$.

(b) $c(n, i+1) - c(n, i) = (-1)^{i+2} \times b(n, i+1)$.

(c) For any number $1 \le h \le r$, the count of 1’s in odd positions in row n of A237048 through index h is at least as large as the count of 1’s in even positions through h. Specifically, the two counts are equal at h when c(n, h) = 0.

(d) The value $c(n, r) \ge 0$ at the diagonal is the difference between the odd divisors d of n with $d \le r$, and the odd divisors d of n with $d > r$.

Theorem 1

(a) srs(n) = $\sum_{h=1}^{r} (a(n, h) - a(n, h+1)) \times c(n, h)$.
(b) SRS(n) = 2 × srs(n) - c(n, r)

Proof

Since SRS(n) consists of the parts of srs(n) together with their mirror parts across the diagonal, the width c(n, r) at the diagonal is being counted twice when using the formula for srs(n) for the total area of SRS(n). For a proof of the formula for SRS(n) see the links in A280851 and A264116.

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Connections Between 2-Densely Aggregated Compositions and SRS(n).

Definition 3

(Definition by Peter Munn in A384149)
We form the 2-densely-aggregated composition of sigma(n) = A000203(n) by listing the divisors of n in increasing order and assigning adjacent divisors for summation in the same aggregate if (and only if) they differ by a factor of less than or equal to 2. The ordering of the aggregate sums in the composition follows the ordering of the summed divisors.

Definition 4

The term 2-dense aggregate in n is used for the sequence of divisors of n that make up a single aggregate in the 2-densely-aggregated composition of sigma(n), and it is also used for their sum. We denote a 2-dense aggregate in n by (i_1 .. i_2), the sequence of its divisors of n.

Definition 5

(a) A complete extent in srs(n) is a closed interval [i_1, i_2], $1 \le i_1 \le h \le i_2 \le r$ satisfying c(n, h) > 0 for $i_1 \le h < i_2$, and c(n, i_1-1) = c(n, i_2) = 0.
(b) A border half-extent csrs(n) in srs(n) is a closed interval [i_1, r], $1 \le i_1 \le h \le r$ satisfying c(n, h) > 0 for $i_1 \le h \le r$, and c(n, i_1-1) = 0.

The next Lemma follows from the Corollary listed above and Definitions 5.

Lemma 3

A complete extent [i_1, i_2] starts with an odd divisor i_1 and ends with a number $2^{m+1} \times d$ where d is an odd divisor satisfying $i_1 \le d < i_2$.

Theorem 2

The area of a part of srs(n) defined by a complete extent $[i_1, i_2]$ equals one half of the sum of the 2-dense aggregate $\left(i_1 .. \frac{i_2}{2}\right)$ and of its symmetric 2-dense aggregate $\left(\frac{2 \times n}{i_2} .. \frac{n}{i_1}\right)$.

Proof

Consider the part of srs(n) defined by complete extent $[i_1, i_2]$. Using Theorem 1, the Corollary, the formulas in Lemma 1, and the fact that $c(n, i_2) = 0$, the expression of its area can be simplified as follows:

$$\sum_{i=i_1}^{i_2} ((a(n, i) - a(n, i+1)) \times c(n, i))$$

$$\begin{aligned}
= {} & a(n, i_1) \times c(n, i_1) - a(n, i_1+1) \times c(n, i_1) \\
& + a(n, i_1+1) \times c(n, i_1+1) - a(n, i_1+2) \times c(n, i_1+1) \\
& + a(n, i_1+2) \times c(n, i_1+2) - a(n, i_1+3) \times c(n, i_1+2) \\
& \ldots \\
& + a(n, i_2-2) \times c(n, i_2-2) - a(n, i_2-1) \times c(n, i_2-2) \\
& + a(n, i_2-1) \times c(n, i_2-1) - a(n, i_2) \times c(n, i_2-1) \\
& + a(n, i_2) \times c(n, i_2) - a(n, i_2+1) \times c(n, i_2) \\
= {} & a(n, i_1) \times 1 \\
& - a(n, i_1+1) \times b(n, i_1+1) \\
& + a(n, i_1+2) \times b(n, i_1+2) \\
& \ldots \\
& + a(n, i_2-1) \times b(n, i_2-1) \\
& - a(n, i_2) \times b(n, i_2) \\
- {} & a(n, i_2+1) \times 0
\end{aligned}$$

Since interval $[i_1, i_2]$ starts with width 1 after width 0 and ends with width 0, the count of odd divisors equals the count of numbers of the form $2^{m+1} \times d$. Let $i_1 = d_1 < \ldots < d_k < i_2$ be all odd divisors and let $e_1 > \ldots > e_k$ be the odd divisors satisfying $q = d_i \times e_i$. Continuing the simplification:

$$\begin{aligned}
& = \sum_{i=1}^{k} \left(2^m \times \frac{q}{d_i} - \frac{d_i+1}{2} + 1\right) - \sum_{i=1}^{k} \left(-2^m \times \frac{q}{e_i} + \frac{e_i-1}{2} + 1\right) \\
& = \sum_{i=1}^{k} 2^m \times \frac{q}{d_i} - \sum_{i=1}^{k} \frac{d_i}{2} + \sum_{i=1}^{k} 2^m \times \frac{q}{e_i} - \sum_{i=1}^{k} \frac{e_i}{2} \\
& = 2^{m+1} \times \sum_{i=1}^{k} \frac{d_i}{2} - \sum_{i=1}^{k} \frac{d_i}{2} + 2^{m+1} \times \sum_{i=1}^{k} \frac{e_i}{2} - \sum_{i=1}^{k} \frac{e_i}{2} \\
& = \frac{1}{2} \times \left(\left(2^{m+1} - 1\right) \times \sum_{d \mid q \,\&\, i_1 \le d \le i_2} d\right) + \frac{1}{2} \times \left(\left(2^{m+1} - 1\right) \times \sum_{d \mid q \,\&\, i_1 \le d \le i_2} \frac{q}{d}\right) \qquad (*)
\end{aligned}$$

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Corollary

(a) The two parenthesized terms in $(*)$ are the sums of all divisors in the respective two symmetric 2-dense aggregates $\left(i_1 .. \frac{i_2}{2}\right)$ and $\left(\frac{2 \times n}{i_2} .. \frac{n}{i_1}\right)$.

(b) The expression $(*)$ holds for all complete extents $[i_1, i_2]$ in SRS(n) regardless whether SRS(n) has an odd or even number of parts.

Theorem 3

(a) The area of a border half-segment, csrs(n), of srs(n) starting at odd divisor i_1 equals one half of the 2-dense aggregate starting at i_1.
(b) Denote by cSRS(n) the area of a part of SRS(n) that straddles the diagonal defined by a border half-extent $[i_1, \mathrm{r}]$ equals the sum of one half of the sum of the 2-dense aggregate starting at odd divisor i_1 and extending through the diagonal of srs(n) and one half of its symmetric 2-dense aggregate.

Proof

Consider the part of srs(n) defined by border half-extent $[i_1, \mathrm{r}]$. Using Theorem 1, the Corollary, the formulas in Lemma 1, and the fact that c(n, r) > 0, the expression of its area can be simplified as follows:

$$\sum_{i=i_1}^{r} ((a(n, i) - a(n, i+1)) \times c(n, i))$$

$$= a(n, i_1) \times c(n, i_1) - a(n, i_1+1) \times c(n, i_1)$$
$$+ a(n, i_1+1) \times c(n, i_1+1) - a(n, i_1+2) \times c(n, i_1+1)$$
$$+ a(n, i_1+2) \times c(n, i_1+2) - a(n, i_1+3) \times c(n, i_1+2)$$
$$\ldots$$
$$+ a(n, r-2) \times c(n, r-2) - a(n, r-1) \times c(n, r-2)$$
$$+ a(n, r-1) \times c(n, r-1) - a(n, r) \times c(n, r-1)$$
$$+ a(n, r) \times c(n, r) - a(n, r+1) \times c(n, r)$$

$$= a(n, i_1) \times 1$$
$$- a(n, i_1+1) \times b(n, i_1+1)$$
$$+ a(n, i_1+2) \times b(n, i_1+2)$$
$$\ldots$$
$$+ a(n, r-1) \times b(n, r-1)$$
$$- a(n, r) \times b(n, r)$$
$$- a(n, r+1) \times \mathrm{w}$$

Since interval $[i_1, r]$ starts with width 1 after width 0 and ends with width w = c(n, r) > 0, the count of odd divisors d exceeds the count of numbers of the form $2^{m+1} \times d$ by w. Let $i_1 = d_1 < \ldots < d_k \leq \mathrm{r}$ be all odd divisors, let $i_1 < 2^{m+1} \times d_1 < \ldots < 2^{m+1} \times d_{k-w} \leq \mathrm{r}$ be all expressions of the form $2^{m+1} \times d$, and let $e_1 > \ldots > e_k$ be the odd divisors satisfying $\mathrm{q} = d_i \times e_i$. Continuing the simplification:

$$= \sum_{i=1}^{k} \left(2^m \times \frac{q}{d_i} - \frac{d_i+1}{2} + 1\right) - \sum_{j=1}^{k-w} \left(-2^m \times \frac{q}{e_j} + \frac{e_j-1}{2} + 1\right)$$

$$= \sum_{i=1}^{k} 2^m \times \frac{q}{d_i} - \sum_{i=1}^{k} \frac{d_i}{2} + \sum_{i=1}^{k} \frac{1}{2} + \sum_{j=1}^{k-w} 2^m \times \frac{q}{e_j} - \sum_{j=1}^{k-w} \frac{e_j}{2} - \sum_{j=1}^{k-w} \frac{1}{2}$$

$$= \left(\sum_{i=1}^{k} 2^m \times \frac{q}{d_i} - \sum_{j=1}^{k-w} \frac{e_j}{2}\right) + \left(\sum_{j=1}^{k-w} 2^m \times \frac{q}{e_j} - \sum_{i=1}^{k} \frac{d_i}{2}\right) + \frac{w}{2}$$

$$= \left(\frac{1}{2} \times (2^{m+1} - 1) \times \sum_{i=1}^{k} \frac{q}{d_i}\right) + \left(\frac{1}{2} \times (2^{m+1} - 1) \times \sum_{i=1}^{k-w} \frac{q}{e_i}\right) + \sum_{j=k-w+1}^{k} \left(\frac{e_j}{2} - \frac{d_j}{2}\right) + \frac{w}{2} \qquad (**)$$

Expression (⁎⁎) is the area of the part of srs(n) defined by the interval [i_1, r] based on the odd divisors of n. Following the expression in (b) of Theorem 1 applied just to the central part of SRS(n), we add to expression (⁎⁎) its symmetric form, i.e., switching odd divisors d_j and e_j, and subtract the width w at the diagonal in order to avoid counting it twice. The resulting expression is a formula for the area of cSRS(n).

cSRS(n)

$$\begin{aligned}
&= \left(\tfrac{1}{2} \times \left(2^{m+1} - 1\right) \times \textstyle\sum_{i=1}^{k} \tfrac{q}{d_i}\right) + \left(\tfrac{1}{2} \times \left(2^{m+1} - 1\right) \times \textstyle\sum_{i=1}^{k-w} \tfrac{q}{e_i}\right) + \textstyle\sum_{j=k-w+1}^{k} \left(\tfrac{e_j}{2} - \tfrac{d_j}{2}\right) + \tfrac{w}{2} \\
&\qquad + \left(\tfrac{1}{2} \times \left(2^{m+1} - 1\right) \times \textstyle\sum_{i=1}^{k} \tfrac{q}{e_i}\right) + \left(\tfrac{1}{2} \times \left(2^{m+1} - 1\right) \times \textstyle\sum_{i=1}^{k-w} \tfrac{q}{d_i}\right) + \textstyle\sum_{j=k-w+1}^{k} \left(\tfrac{d_j}{2} - \tfrac{e_j}{2}\right) + \tfrac{w}{2} \\
&\qquad - w \\
&= \left(\tfrac{1}{2} \times \left(2^{m+1} - 1\right) \times \textstyle\sum_{i=1}^{k} \tfrac{q}{d_i}\right) + \left(\tfrac{1}{2} \times \left(2^{m+1} - 1\right) \times \textstyle\sum_{i=1}^{k-w} \tfrac{q}{e_i}\right) \\
&\qquad + \left(\tfrac{1}{2} \times \left(2^{m+1} - 1\right) \times \textstyle\sum_{i=1}^{k} \tfrac{q}{e_i}\right) + \left(\tfrac{1}{2} \times \left(2^{m+1} - 1\right) \times \textstyle\sum_{i=1}^{k-w} \tfrac{q}{d_i}\right) \\
&= \tfrac{1}{2} \times \left(2^{m+1} - 1\right) \times \left(\textstyle\sum_{i=1}^{k} d_i + \sum_{i=1}^{k-w} e_i\right) + \tfrac{1}{2} \times \left(2^{m+1} - 1\right) \times \left(\textstyle\sum_{i=1}^{k} e_i + \sum_{i=1}^{k-w} d_i\right) && (***) \\
&= \tfrac{1}{2} \times \left(2^{m+1} - 1\right) \times \textstyle\sum_{i=1}^{k} (d_i + e_i) + \tfrac{1}{2} \times \left(2^{m+1} - 1\right) \times \sum_{i=1}^{k-w} (d_i + e_i) && (****)
\end{aligned}$$

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In conclusion we summarize the above results.

Theorem 4

(a) The mean of two symmetric 2-densely aggregated compositions of n equals the associated part of SRS(n). Expressions (⁎) and (⁎⁎⁎) state the formulas for the two types of parts in SRS(n).

(b) Applying the explicit formulas (⁎) and (⁎⁎⁎) for the parts of SRS(n) shows SRS(n) = σ(n).

(c) Computing based on 2-densely aggregated compositions, i.e. all divisors, is equivalent to computing based on odd divisors only.

Corollary

(a) The first term of the sum in expression (⁎⁎⁎) is the sum of all divisors d, 2×d, ... , 2^m×d for all odd divisors d, $i_1 \le d \le r$, and all divisors e, 2×e, ... , 2^m×e where q = d×e and 2^m×d > r for $i_1 \le d \le r$. The second term of the sum represents the symmetric interval.

(b) The two sums in expression (⁎⁎⁎⁎) are aggregations of the sums of paired odd divisors d and e, d × e = q that satisfy the conditions in (a).

(c) The row lengths of the triangle in A384149 and of those in A237048, A237270, A237271, A237591, A249223 and many other related sequences are identical.

Example 1 Number of Parts in SRS(n) is Odd.

n = 45 = $3^2 \times 5$, divisors(n) = {1, 3, 5, 9, 15, 45}, r = A003056(n) = 9, SRS(n) = {23, 32, 23}.

Computations based on sequence A384149:

A384149 row n: {1, 32, 45} with aggregates (1..1), (3 .. 15), (45 .. 45)

aggregation compositions: 1 3+5+9+15 = 32 45

Computations according to expressions (∗) and (∗∗∗):

complete extent: [1, 2] expression (∗): $\frac{1}{2}\times 1\times(1) + \frac{1}{2}\times 1\times(45) = \frac{1}{2} + \frac{45}{2} = 23$

border half-extent: [3, 9] with $d_1 = 3$, $d_2 = 5$, $2\times d_1 = 6$, $d_3 = 9$ and $e_1 = 15$, $e_2 = 9$, $e_3 = 5$

expression (∗∗∗): $\frac{1}{2}\times 1\times(3+5+9+15) + \frac{1}{2}\times 1\times(15+9+5+3) = 16 + 16 = 32$

Computations based on sequences A237591 and A249223:

A237591 row n: {23, 8, 5, 2, 2, 2, 1, 1, 1}

A249223 row n: { 1, 0, 1, 1, 2, 1, 1, 1, 2}

complete extent: [1, 2] $23\times 1 + 8\times 0 = 23$

border half-extent: [3, 9] csrs(n) = $5\times 1 + 2\times 1 + 2\times 2 + 2\times 1 + 1\times 1 + 1\times 1 + 1\times 2 = 17$

entire center part: cSRS(n) = 2 × csrs(n) - c(n, r) = 2 ×17 - 2 = 32.

Example 2 Number of Parts in SRS(n) is Even.

n = 75 = 3×5^2, divisors(n) = {1, 3, 5, 15, 25, 75}, r = A003056(n) = 11, SRS(n) = {38, 24, 24, 38}.

Computations based on sequence A384149:

A384149 row n: {1, 8, 40, 75} with aggregates (1..1), (3..5), (15..25), (75..75)

aggregation compositions: 1 3+5 15+25 = 40 75

Computations according to expression (∗):

complete extents: [1, 2] $\frac{1}{2}\times 1\times(1) + \frac{1}{2}\times 1\times(75) = \frac{1}{2} + \frac{75}{2} = 38$

[3, 10] $\frac{1}{2}\times 1\times(3+5) + \frac{1}{2}\times 1\times(25+15) = \frac{8}{2} + \frac{40}{2} = 24$

Computations based on sequences A237591 and A249223:

A237591 row n: {38, 13, 7, 4, 3, 3, 2, 1, 1, 2, 1}

A249223 row n: { 1, 0, 1, 1, 2, 1, 1, 1, 1, 0, 0}

complete extent: [1, 2] $38\times 1 + 13\times 0 = 38$

[3, 10] $7\times 1 + 4\times 1 + 3\times 2 + 3\times 1 + 2\times 1 + 1\times 1 + 1\times 1 + 2\times 0 = 24$.