Proof that the number of isomorphism classes of groups $|G| = 2^n p$ when G contains a unique Sylow p-subgroup and the maximal 2^m dividing $p-1$ is such that $2^m \geq 2^n$ is the same for all p.

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Outline: the gist of the proof is to show that groups $|G| = 2^n p$ with a unique Sylow p-subgroup all have a semidirect product representation $C_p \rtimes H_{2^n}$, which imposes an upper bound on the number of isomorphism classes. That the maximal 2^m dividing $p-1$ is greater or equal than 2^n ensures that the lower bound is equal to the upper bound.

Theorem

Let G be a group of order $2^n p$ for prime p, $n \geq 0$. If p satisfies the two following congruence conditions:

(1) $2^m \equiv 1$ *(mod p), 2^m* ***minimal****, and $2^m > 2^n$,*

(2) $p \equiv 1$ *(mod 2^k), 2^k* ***maximal****, and $2^k \geq 2^n$,*

then the number of isomorphism classes of $|G| = 2^n p$ is the same for all p, distinct for n.

Proof. Condition **(1)** allows us to establish that for $|G| = 2^n p$, the prime cyclic group C_p is a unique Sylow subgroup and thus normal in G. By the Sylow theorems, if n_p is the number of Sylow p subgroups, then n_p divides $[G : C_p]$ and $n_p \equiv 1(\text{mod } p)$. Since $|G : C_p| = 2^n$, therefore $n_p = 2^r$, $0 \leq r \leq n$. If condition **(1)** is met, then the least 2^m such that $2^m \equiv 1$ (mod p) is greater than 2^n, and $n_p \equiv 1(\text{mod } p)$ only when $n_p = 1$. Hence for p satsifying this condition, C_p is a unique normal Sylow subgroup of G.

By the Schur-Zassenhaus theorem, every G is necessarily a semidirect product $C_p \rtimes H$, where H is an arbitrary 2-group of order 2^n. This means that determining the number of groups reduces to determining the number of semidirect products. And since the number of direct products does not depend on p, we only have to consider the proper semidirect products.

H defined as above, a semidirect product $C_p \rtimes H$ exists when there exists a homomorphism $\sigma : H \to \text{Aut}(C_p) \cong C_{p-1}$. Since C_{p-1} is a nonprime cyclic, σ exists when H contains a normal subgroup N such that $H/N \cong \text{im}(\sigma)$ is cyclic and $|\text{im}(\sigma)|$ divides $p-1$. Condition **(2)** ensures that the number of homomorphisms is the same for every such p. Let $p \equiv 1$ (mod 2^k) and $q \equiv 1$ (mod 2^m), and let $n \leq m < k$ (therefore satisfying condition **(2)**). Since C_{q-1} and C_{p-1} are cyclic, each contains a unique subgroup for every divisor of the group order, and therefore C_{q-1} contains m subgroups of every order $2^{1 \leq m}$, and C_{p-1} contains k subgroups of every order $2^{1 \leq k}$. Let Σ_p be the total number of homomorphisms $\sigma : H \to \text{Aut}(C_p)$, and similarly Σ_q for $\pi : H \to \text{Aut}(C_q)$. Since $|H| = 2^n$, $\max(\Sigma_p) = \max(\Sigma_q) = n$. But whether H contains a subgroup N such that H/N is cyclic is independent of C_p or C_q: if the homomorphism $\sigma : H \to \text{Aut}(C_p)$ exists, then so does the homomorphism $\pi : H \to \text{Aut}(C_q)$. Since both Σ_p and Σ_q have a maximum of n, therefore the number of homomorphisms is the same for both p and q, and consequently the number of semidirect products is the same.