A note on spanning trees

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Abstract

1 Introduction

Let $\tau_3(n)$ be the number of spanning trees in the 3rd power of a cycle of length n. We give two expressions for $\tau_3(n)$ in Section 1 and 2.

2 [1]

By [1, Theorem 1], we have the following:

Theorem 2.1 ([1, Theorem 1]). *Let*

$$T(n,z) := \cos(n \arccos(z))$$

$$z_1 := \frac{-3+\sqrt{-7}}{4}, z_2 := \frac{-3-\sqrt{-7}}{4}.$$

Then we have the following:

$$\tau_3(n) := \frac{2n}{7}(T(n,z_1)-1)(T(n,z_2)-1).$$

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3 [2]

Let $T(n)$ be the numbers in A005822. Then we have

Theorem 3.1 ([2]).

$$\begin{cases} \tau_3(n) = 2nT(n)^2 \text{ if } n \text{ is even}, \\ \tau_3(n) = nT(n)^2 \text{ if } n \text{ is odd}. \end{cases}$$

where

$$T(n+8) = 4T(n+6) + T(n+4) + 4T(n+2) - T(n).$$

Proof. The proof is similar to the discussion in [2, p.347 Theorem 9]. Let

$$f = 1 + 3x + 6x^2 + 3x^3 + x^4.$$

We denote by

$$a_1, a_2$$

its roots up to conjugate. Let

$$a(n) := \frac{(1-a_1^n)(1-a_2^n)}{\sqrt{14}\sqrt{(a_1a_2)^n}}$$

Then

$$\tau_3(n) = na(n)^2$$

and we have $a(n)$

$$a(n+4) = \sqrt{2}a(n+3) + a(n+2) + \sqrt{2}a(n+1) - a(n)$$

Then we obtain the following:

$$a(n+8) = 4a(n+6) + a(n+4) + 4a(n+2) - a(n).$$

It is easy to check that

$$\begin{cases} T(n) = \sqrt{2}a(n) \text{ if } n \text{ is even}, \\ T(n) = a(n) \text{ if } n \text{ is odd}. \end{cases}$$

□

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References

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