I. Background

For $w \neq 0$ being an element in the ring $K = \mathrm{Z}[\sqrt{d}\mathrm{i}]$ for $d \in \mathrm{N}^*$ (or $K = \mathrm{Z}[\frac{1+\sqrt{d}\mathrm{i}}{2}]$ for $d \in \mathrm{N}^*, d \equiv 3(\mathrm{mod}\, 4)$), let $R(w)$ be a complete residue system over K modulo w (i.e., $R(w)$ consists of $N(w)$ elements in K such that no two elements are congruent modulo w, where $N(w)$ is the norm of w), then we intend to find the smallest possible value of $\max_{s \in R(w)} N(s)$.

Now, if we can find a set of complex numbers A such that: **(i) for any** $x \in K, r \in A, |r| \leq |r - x|$ **; (ii) every complex number** z **can be uniquely represented as** $z = x + r$**, where** $x \in K, r \in A$, then

$$S(w) := \{rw : r \in A\} \bigcap K$$

is a complete residue system modulo w **formed by choosing one element with the minimal norm in each residue class modulo** w (there may be more than one element whose norms are minimal in one residue class). This is because: (a) for any $x \in K$, $\frac{y}{w}$ can be uniquely represented as $\frac{y}{w} = x + r$, where $x \in K, r \in A$. So among all $t \in K$ such that $\frac{y-t}{w} \in K$, $t \in S(w)$ if and only if $t = rw = y - xw$. That is, for any $y \in K$, there exists a unique $t_0 \in K$ such that $\frac{y - t_0}{w} \in K$ and that $t_0 \in S(w)$. So $S(w)$ is a complete residue modulo w; (b) for any $t \in S(w)$, if $\frac{y-t}{w} \in K$, then

$$N(y) = |w|^2 |\frac{t}{w} - (-\frac{y-t}{w})|^2 \geq |w|^2 |\frac{t}{w}|^2 = N(t),$$

since that $\frac{t}{w} \in A$.

As a result, we have

$$\min \max_{s \in R(w)} N(s) = \max_{s \in S(w)} N(s),$$

disregarding the choice of the set A (as long as it satisfies (i), (ii)).

II. The case $K = \mathrm{Z}[\frac{1+\sqrt{d}\mathrm{i}}{2}]$

We first find the sets that satisfy (i). Let

$$\varepsilon_0 = 1, \varepsilon_1 = \frac{1+\sqrt{d}\mathrm{i}}{2}, \varepsilon_2 = \frac{-1+\sqrt{d}\mathrm{i}}{2}, \varepsilon_3 = -1, \varepsilon_4 = \frac{-1-\sqrt{d}\mathrm{i}}{2}, \varepsilon_5 = \frac{1-\sqrt{d}\mathrm{i}}{2},$$

Then all such sets are subsets of

$$A^+ = \{r \in \mathrm{C} : |r| \le |r - \varepsilon_j|, j = 0,1,2,3,4,5\}.$$

Note that

$$\{r \in \mathrm{C} : |r| \le |r - \varepsilon_j|, j = 0,3\} = \{\frac{u+v\sqrt{d}\mathrm{i}}{2} : u, v \in \mathrm{R}, |u| \le 1\},$$

$$\{r \in \mathrm{C} : |r| \le |r - \varepsilon_j|, j = 1,2\} = \{\frac{u+v\sqrt{d}\mathrm{i}}{2} : u, v \in \mathrm{R}, v \le -\frac{1}{d}u + \frac{d+1}{2d}\},$$

$$\{r \in \mathrm{C} : |r| \le |r - \varepsilon_j|, j = 4,5\} = \{\frac{u+v\sqrt{d}\mathrm{i}}{2} : u, v \in \mathrm{R}, v \ge \frac{1}{d}u - \frac{d+1}{2d}\},$$

so the definition of A^+ is equivalent to

$$A^+ = \{\frac{u+v\sqrt{d}\mathrm{i}}{2} : u, v \in \mathrm{R}, |u| \le 1, |v| \le -\frac{1}{d}|u| + \frac{d+1}{2d}\}.$$

Now we show for any $x \in K, r \in A^+, |r| \le |r - x|$. Let

$$r = \frac{u+v\sqrt{d}\mathrm{i}}{2}(|u| \le 1, |v| \le -\frac{1}{d}|u| + \frac{d+1}{2d}), x = \frac{a+b\sqrt{d}\mathrm{i}}{2}(a, b \in \mathrm{Z}, 2 \mid (a+b)),$$

then,

$$|r-x|^2 - |x|^2 = \frac{1}{4}(u-a)^2 + \frac{d}{4}(v-b)^2 - \frac{1}{4}u^2 - \frac{d}{4}v^2$$

$$= \frac{1}{4}(a^2 - 2au + db^2 - 2dbv) \ge \frac{1}{4}(a^2 - 2|a||u| + db^2 - 2d|b||v|)$$

$$\ge \frac{1}{4}(a^2 - 2|a||u| + db^2 - 2d|b|(-\frac{1}{d}|u| + \frac{d+1}{2d})))(\text{by } |v| \le -\frac{1}{d}|u| + \frac{d+1}{2d})$$

$$\ge \min\{\frac{1}{4}(a^2 + db^2 - 2d|b| \times \frac{d+1}{2d}), \frac{1}{4}(a^2 - 2|a| + db^2 - 2d|b| \times \frac{d-1}{2d})\}(\text{by } 0 \le |u| \le 1)$$

$$= \frac{1}{4}\min\{a^2 + db^2 - (d+1)|b|, a^2 - 2|a| + db^2 - (d-1)|b|\}$$

$$= \frac{1}{4}\min\{(|a|+1)(|a|-1) + (d|b|-1)(|b|-1), (|a|-1)^2 + (d|b|+1)(|b|-1)\}.$$

The value of the last line is nonnegative unless $(|a|,|b|) = (0,1),(1,0)$, which is not acceptable since $2 \mid (a+b)$ (and also, its value is 0 if and only if $(|a|,|b|) = (0,0),(1,1),(2,0)$), that is, $x = 0$ or $x = \varepsilon_j, j = 0,1,2,3,4,5$). So

$x \in K, r \in A^+, |r| \leq |r - x|$.

Secondly, we find the pairs (r,x) such that $x \in K, x \neq 0, r \in A^+, r - x \in A^+$. $r \in A^+ \Rightarrow |r| \leq |r - x|; r - x \in A^+ \Rightarrow |r - x| \leq |r - x - (-x)| = |r|$, so $|r| = |r - x|$. If $x \neq \varepsilon_j, j = 0,1,2,3,4,5$, then $|r| < |r - x|$, so $x = \varepsilon_j, j = 0,1,2,3,4,5$.

Let

$$E_j := \{r \mid r \in A^+, r - \varepsilon_j \in A^+\}, j = 0,1,2,3,4,5,$$

we have

$$E_j = \{r \mid r \in A^+, |r| = |r - \varepsilon_j|\}, j = 0,1,2,3,4,5.$$

This is because: if $r \in A^+, r - \varepsilon_j \in A^+$, then $|r| = |r - \varepsilon_j|$; if $|r| = |r - \varepsilon_j|$ and $r \in A^+$, then for any $x \in K$ we have $|r - \varepsilon_j - x| \geq |r| = |r - \varepsilon_j|$. Recall that

$$A^+ = \{\frac{u + v\sqrt{d}\mathrm{i}}{2} : u, v \in \mathrm{R}, |u| \leq 1, |v| \leq -\frac{1}{d}|u| + \frac{d+1}{2d}\}.$$

so,

$$\begin{aligned}
E_0 &= \{\frac{1 + v\sqrt{d}\mathrm{i}}{2} : v \in \mathrm{R}, |v| \leq \frac{d-1}{2d}\},\\
E_1 &= \{\frac{u}{2} + (-\frac{1}{d}u + \frac{d+1}{2d})\frac{\sqrt{d}\mathrm{i}}{2} : u \in \mathrm{R}, 0 \leq u \leq 1\},\\
E_2 &= \{\frac{u}{2} + (\frac{1}{d}u + \frac{d+1}{2d})\frac{\sqrt{d}\mathrm{i}}{2} : u \in \mathrm{R}, -1 \leq u \leq 0\},\\
E_3 &= \{\frac{-1 + v\sqrt{d}\mathrm{i}}{2} : v \in \mathrm{R}, |v| \leq \frac{d-1}{2d}\},\\
E_4 &= \{\frac{u}{2} - (\frac{1}{d}u + \frac{d+1}{2d})\frac{\sqrt{d}\mathrm{i}}{2} : u \in \mathrm{R}, -1 \leq u \leq 0\},\\
E_5 &= \{\frac{u}{2} - (-\frac{1}{d}u + \frac{d+1}{2d})\frac{\sqrt{d}\mathrm{i}}{2} : u \in \mathrm{R}, 0 \leq u \leq 1\},
\end{aligned}$$

Then,

$$E_0 \bigcap E_1 = \{\frac{1}{2} + \frac{d-1}{4d}\sqrt{d}\mathrm{i}\}, E_1 \bigcap E_2 = \{\frac{d+1}{4}\sqrt{d}\mathrm{i}\}, E_2 \bigcap E_3 = \{-\frac{1}{2} + \frac{d-1}{4d}\sqrt{d}\mathrm{i}\},$$
$$E_3 \bigcap E_4 = \{-\frac{1}{2} - \frac{d-1}{4d}\sqrt{d}\mathrm{i}\}, E_4 \bigcap E_5 = \{-\frac{d+1}{4}\sqrt{d}\mathrm{i}\}, E_5 \bigcap E_6 = \{\frac{1}{2} - \frac{d-1}{4d}\sqrt{d}\mathrm{i}\},$$

and $E_j \bigcap E_{j'}$ is empty unless $j' - j \equiv \pm 1 (\mathrm{mod}\, 6)$.

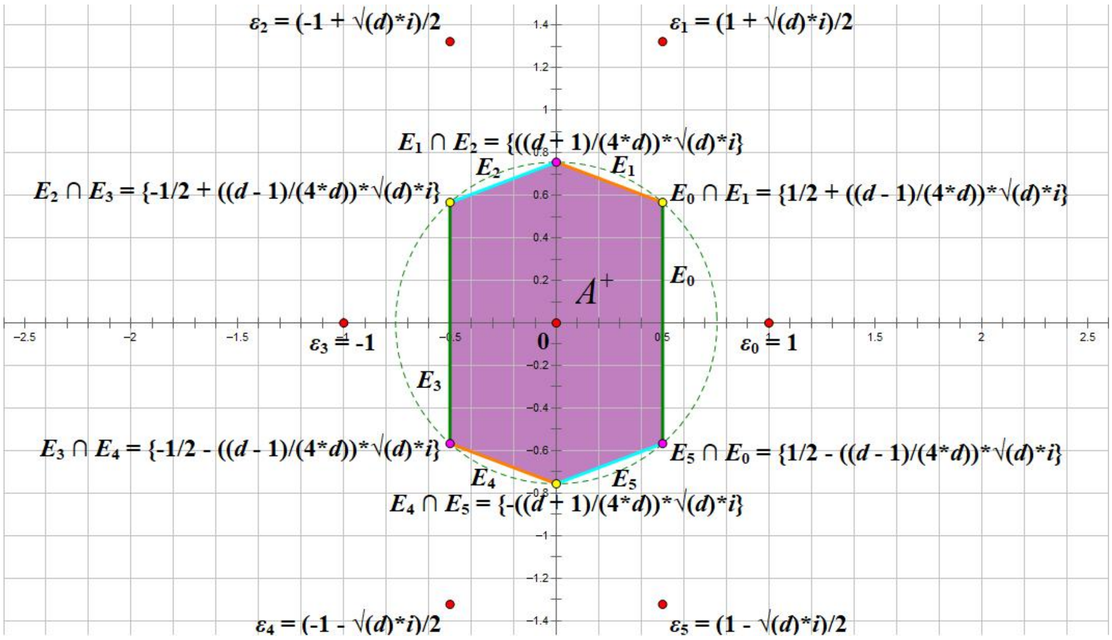

Figure 1. Illustration of the set A^+. All edges and the vertices are included, and the edges or vertices in the same color are "grouped".

Now we find the subsets of A^+ that satisfy (ii). For any complex z and any $C>0$, there are only finitely many $x\in K$ such that $|z-x|<C$, so $|z-x|$ has minimal value. Let $x_1,x_2,...,x_m\in K$ minimize $|z-x|$. By definition, $z-x_k\in A^+,k=1,2,...,m$, that is, $z-x_1+(x_1-x_k)\in A^+,k=1,2,...,m$. Here $x_1-x_k\in K,k=1,2,...,m$.

If there does not exist j such that $z-x_1\in E_j$. By definition, $z-x_1-x\in A^+$ if and only if $x=0$, so $z-x_1\in A$.

If there exists exactly one j such that $z-x_1\in E_j$. By definition, $z-x_1-x\in A^+$ if and only if $x=0$ or $x=\varepsilon_j$, so exactly one of $z-x_1$ and $z-x_1-\varepsilon_j$ is in A.

If there exists j,j' such that $z-x_1\in E_j\bigcap E_{j'}$, suppose $j'-j\equiv 1(\mathrm{mod}\,6)$, then $\{z-x_1\}=E_j\bigcap E_{j+1}$, and $\{z-x_1-\varepsilon_j\}=E_{j+2}\bigcap E_{j+3}$, $\{z-x_1-\varepsilon_{j+1}\}=E_{j+4}\bigcap E_{j+5}$, $z-x_1-x$ is not in A for other x. Here $E_{j+6}=E_j$. So exactly one of

$\pm\frac{1}{2}+\frac{d-1}{4d}\sqrt{d}\mathrm{i},-\frac{d+1}{4d}\sqrt{d}\mathrm{i}$ is in A, and exactly one of $\pm\frac{1}{2}-\frac{d-1}{4d}\sqrt{d}\mathrm{i},\frac{d+1}{4d}\sqrt{d}\mathrm{i}$ is in A.

As a result: the possible choices for the set A are as follows: for any $r\in A^+$, if r is not a member of $E_j, j=0,1,2,3,4,5$, then $r\in A$; if $r\in E_j$, but r is not in $E_{j\pm1}$, then exactly one of r and $r-\varepsilon_j$ is in A (here, $r-\varepsilon_j\in E_{j+3}$); lastly, exactly one of $\pm\frac{1}{2}+\frac{d-1}{4d}\sqrt{d}\mathrm{i},-\frac{d+1}{4d}\sqrt{d}\mathrm{i}$ is in A, and exactly one of $\pm\frac{1}{2}-\frac{d-1}{4d}\sqrt{d}\mathrm{i},\frac{d+1}{4d}\sqrt{d}\mathrm{i}$ is in A.

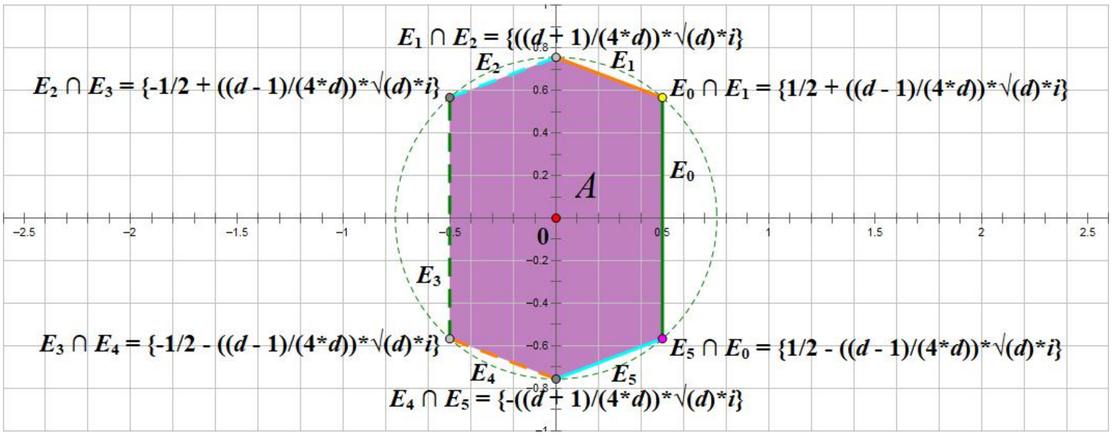

Figure 2. Illustration of the set A. The edges drawn in real lines and the colored vertices are included, while the edges drawn in dashed lines and the vertices in gray are not.

III. The case $K=\mathrm{Z}[\sqrt{d}\mathrm{i}]$

We first find the sets that satisfy (i). Let

$$\varepsilon_0=1,\varepsilon_1=\sqrt{d}\mathrm{i},\varepsilon_2=-1,\varepsilon_3=-\sqrt{d}\mathrm{i},$$

Then all such sets are subsets of

$$A^+=\{r\in\mathrm{C}:|r|\le|r-\varepsilon_j|,j=0,1,2,3\}.$$

Note that

$$\{r\in\mathrm{C}:|r|\le|r-\varepsilon_j|,j=0,2\}=\{u+v\sqrt{d}\mathrm{i}:u,v\in\mathrm{R},|u|\le\frac{1}{2}\},$$

$$\{r\in\mathrm{C}:|r|\le|r-\varepsilon_j|,j=1,3\}=\{u+v\sqrt{d}\mathrm{i}:u,v\in\mathrm{R},|v|\le\frac{1}{2}\},$$

so the definition of A^+ is equivalent to

$$A^+=\{u+v\sqrt{d}\mathrm{i}:u,v\in\mathrm{R},|u|\le\frac{1}{2},|v|\le\frac{1}{2}\}.$$

Now we show for any $x\in K,r\in A^+,|r|\le|r-x|$. Let

$$r = u + v\sqrt{d}\mathrm{i}(|u| \le \frac{1}{2}, |v| \le \frac{1}{2}), x = a + b\sqrt{d}\mathrm{i}(a, b \in \mathrm{Z}),$$

then,

$$\begin{aligned}
&|r - x|^2 - |x|^2 = (u-a)^2 + (v-b)^2 - u^2 - v^2 \\
&= a^2 - 2au + db^2 - 2dbv \ge a^2 - 2|a||u| + db^2 - 2d|b||v| \\
&\ge a^2 - |a| + d(b^2 - |b|))(\text{by } |u| \le \frac{1}{2}, |v| \le \frac{1}{2}) \ge 0.
\end{aligned}$$

(and also, its value is 0 if and only if $(|a|,|b|) = (0,0),(0,1),(1,0),(1,1)$), that is, $x = 0$ or $x = \varepsilon_j, j = 0,1,2,3$ or $x = \varepsilon_j + \varepsilon_{j+1}, j = 0,1,2,3$. Here $\varepsilon_{j+4} = \varepsilon_j$). So $x \in K, r \in A^+, |r| \le |r - x|$.

Secondly, we find the pairs (r, x) such that $x \in K, x \ne 0, r \in A^+, r - x \in A^+$. For the same reason $|r| = |r - x|$. If $x \ne \varepsilon_j, j = 0,1,2,3$ and $x \ne \varepsilon_j + \varepsilon_{j+1}, j = 0,1,2,3$, then $|r| < |r - x|$,so $x = \varepsilon_j, j = 0,1,2,3$ or $x = \varepsilon_j + \varepsilon_{j+1}, j = 0,1,2,3$.

Let

$$\begin{aligned}
E_j &:= \{r \mid r \in A^+, r - \varepsilon_j \in A^+\}, j = 0,1,2,3, \\
V_j &:= \{r \mid r \in A^+, r - \varepsilon_j - \varepsilon_{j+1} \in A^+\}, j = 0,1,2,3,
\end{aligned}$$

again,

$$\begin{aligned}
E_j &= \{r \mid r \in A^+, |r| = |r - \varepsilon_j|\}, j = 0,1,2,3, \\
V_j &= \{r \mid r \in A^+, |r| = |r - \varepsilon_j - \varepsilon_{j+1}|\}, j = 0,1,2,3.
\end{aligned}$$

Recall that

$$A^+ = \{u + v\sqrt{d}\mathrm{i} : u, v \in \mathrm{R}, |u| \le \frac{1}{2}, |v| \le \frac{1}{2}\}.$$

so,

$$\begin{aligned}
&E_0 = \{\frac{1}{2} + v\sqrt{d}\mathrm{i} : v \in \mathrm{R}, |v| \le \frac{1}{2}\}, E_1 = \{u + \frac{\sqrt{d}\mathrm{i}}{2} : u \in \mathrm{R}, |u| \le \frac{1}{2}\}, \\
&E_2 = \{-\frac{1}{2} + v\sqrt{d}\mathrm{i} : v \in \mathrm{R}, |v| \le \frac{1}{2}\}, E_3 = \{u - \frac{\sqrt{d}\mathrm{i}}{2} : r_x \in \mathrm{R}, |u| \le \frac{1}{2}\}, \\
&V_j = \{\frac{\varepsilon_j + \varepsilon_{j+1}}{2}\} = E_j \bigcap E_{j+1},
\end{aligned}$$

and $E_j \bigcap E_{j'}$ is empty unless $j' - j \equiv \pm 1 (\mathrm{mod}\, 4)$.

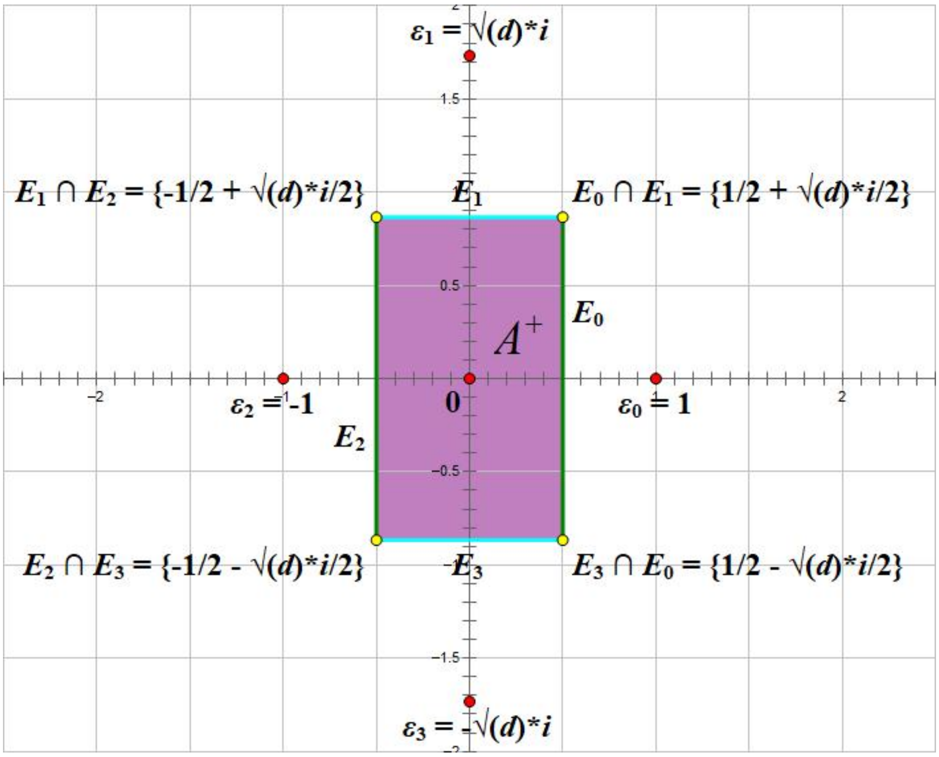

Figure 3. Illustration of the set A^+. All edges and the vertices are included, and the edges or vertices in the same color are "grouped".

Now we find the subsets of A^+ that satisfy (ii). Again, let $x_1, x_2, ..., x_m \in K$ minimize $|z - x|$. By definition, $z - x_k \in A^+, k = 1,2,...,m$, that is, $z - x_1 + (x_1 - x_k) \in A^+, k = 1,2,...,m$. Here $x_1 - x_k \in K, k = 1,2,...,m$.

If there does not exist j such that $z - x_1 \in E_j$. By definition, $z - x_1 - x \in A^+$ if and only if $x = 0$, so $z - x_1 \in A$.

If there exists exactly one j such that $z - x_1 \in E_j$. By definition, $z - x_1 - x \in A^+$ if and only if $x = 0$ or $x = \varepsilon_j$, so exactly one of $z - x_1$ and $z - x_1 - \varepsilon_j$ is in A.

If there exists j, j' such that $z - x_1 \in E_j \bigcap E_{j'}$, suppose $j' - j \equiv 1 (\mathrm{mod}\, 4)$, then $\{z - x_1\} = E_j \bigcap E_{j+1}$, and $\{z - x_1 - \varepsilon_j\} = E_{j+1} \bigcap E_{j+2}$, $\{z - x_1 - (\varepsilon_j + \varepsilon_{j+1})\} = E_{j+2} \bigcap E_{j+3}$, $\{z - x_1 - \varepsilon_{j+1}\} = E_{j+3} \bigcap E_j$, $z - x_1 - x$ is not in A for other x. Here $E_{j+4} = E_j$. So exactly one of $\pm\frac{1}{2} \pm \frac{\sqrt{d}\mathrm{i}}{2}$ is in A.

As a result: the possible choices for the set A are as follows: for any $r \in A^+$, if r is not a member of $E_j, j = 0,1,2,3$, then $r \in A$; if $r \in E_j$, but r is not in $E_{j\pm 1}$, then exactly one of r and $r - \varepsilon_j$ is in A (here, $r - \varepsilon_j \in E_{j+2}$); lastly, So exactly one of $\pm\frac{1}{2} \pm \frac{\sqrt{d}\mathrm{i}}{2}$ is in A.

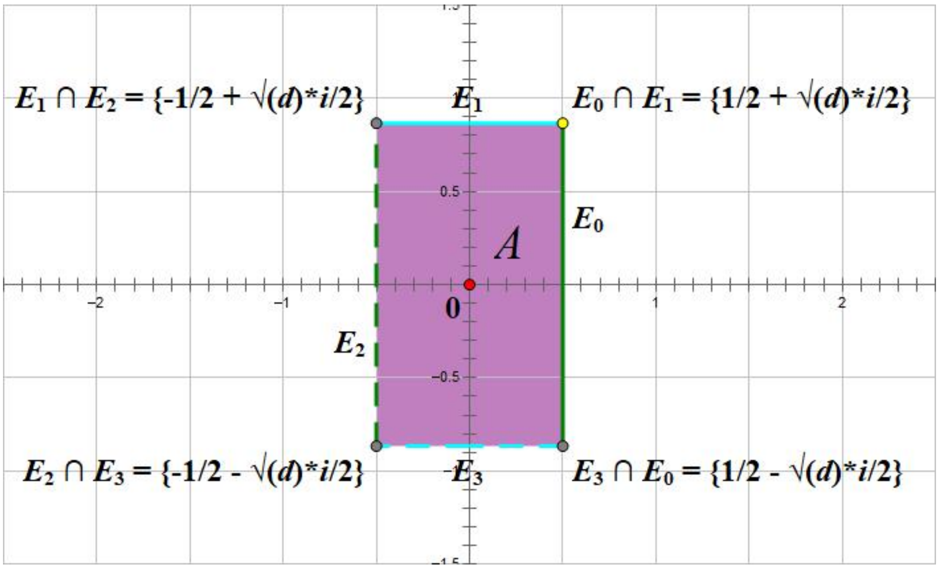

Figure 4. Illustration of the set A. The edges drawn in real lines and the colored vertices are included, while the edges drawn in dashed lines and the vertices in gray are not.

IV. Corollary

Theorem. For any $w \neq 0$ being an element in the ring $K = \mathrm{Z}[\sqrt{d}\mathrm{i}]$ for $d \in \mathrm{N}^*$ (or $K = \mathrm{Z}[\frac{1+\sqrt{d}\mathrm{i}}{2}]$ for $d \in \mathrm{N}^*, d \equiv 3 (\mathrm{mod}\, 4)$), there exists a complete residue system $R(w)$ over K modulo w such that $\max_{s \in R(w)} N(s) \leq cN(w)$, where the best choice of the constant c is $\frac{d+1}{4}$ for $K = \mathrm{Z}[\sqrt{d}\mathrm{i}]$ and $\frac{(d+1)^2}{16d}$ for $K = \mathrm{Z}[\frac{1+\sqrt{d}\mathrm{i}}{2}]$.

We first find the value of $\max_{z \in A} |z|^2$. Recall that in $K = \mathrm{Z}[\sqrt{d}\mathrm{i}]$,

$$A^+ = \{u + v\sqrt{d}\mathrm{i} : u, v \in \mathrm{R}, |u| \leq \frac{1}{2}, |v| \leq \frac{1}{2}\}.$$

as a result, for $z \in A^{+}$

$$|z|^2 = |u|^2 + d|v|^2 \leq \frac{d+1}{4},$$

with equality holds if and only if $|u| = \frac{1}{2}, |v| = \frac{1}{2}$, that is, $z = \pm\frac{1}{2} \pm \frac{\sqrt{d}\mathrm{i}}{2}$. Since that exactly one of $\pm\frac{1}{2} \pm \frac{\sqrt{d}\mathrm{i}}{2}$ is in A, we have

$$\max_{z \in A} |z|^2 = \frac{d+1}{4}.$$

In $K = \mathrm{Z}[\frac{1+\sqrt{d}\mathrm{i}}{2}]$,

$$A^{+} = \{\frac{u + v\sqrt{d}\mathrm{i}}{2} : u, v \in \mathrm{R}, |u| \leq 1, |v| \leq -\frac{1}{d}|u| + \frac{d+1}{2d}\}.$$

as a result,

$$|z|^2 = \frac{|u|^2 + d|v|^2}{4} \leq \frac{|u|^2 + d(-\frac{1}{d}|u| + \frac{d+1}{2d})^2}{4} = \frac{d+1}{4d}((2|u| - \frac{1}{2})^2 + \frac{d}{4}) \leq \frac{(d+1)^2}{16d},$$

with equality holds if and only if $|u| = 0, |v| = \frac{d+1}{2d}$ or $|u| = \frac{1}{2}, |v| = \frac{d-1}{2d}$, that is, $z = \pm\frac{1}{2} \pm \frac{d-1}{4d}\sqrt{d}\mathrm{i}, \pm\frac{d+1}{4d}\sqrt{d}\mathrm{i}$. Since that exactly one of $\pm\frac{1}{2} + \frac{d-1}{4d}\sqrt{d}\mathrm{i}, -\frac{d+1}{4d}\sqrt{d}\mathrm{i}$ is in A, and exactly one of $\pm\frac{1}{2} - \frac{d-1}{4d}\sqrt{d}\mathrm{i}, \frac{d+1}{4d}\sqrt{d}\mathrm{i}$ is in A, we have

$$\max_{z \in A} |z|^2 = \frac{(d+1)^2}{16d}.$$

In Part I, we have

$$\min \max_{s \in R(w)} N(s) = \max_{s \in S(w)} N(s),$$

where $S(w) := \{rw : r \in A\} \bigcap K$, so

$$\min \max_{s \in R(w)} N(s) = \max_{s \in S(w)} N(s) \leq \frac{d+1}{4} N(w)$$

for $K = \mathrm{Z}[\sqrt{d}\mathrm{i}]$, and

$$\min \max_{s \in R(w)} N(s) = \min \max_{s \in S(w)} N(s) \leq \frac{(d+1)^2}{16d} N(w)$$

for $K = \mathrm{Z}[\frac{1+\sqrt{d}\mathrm{i}}{2}]$.

Now, for $K = \mathrm{Z}[\sqrt{d}\mathrm{i}]$, if n is an even positive integer, then $(\pm\frac{1}{2}\pm\frac{\sqrt{d}\mathrm{i}}{2})n \in K$, so exactly one of $(\pm\frac{1}{2}\pm\frac{\sqrt{d}\mathrm{i}}{2})n$ is in $S(n)$, i.e., there are infinitely many numbers such that

$$\min \max_{s\in R(w_0)} N(s) = \max_{s\in S(w_0)} N(s) = \frac{d+1}{4} N(w_0);$$

for $K = \mathrm{Z}[\frac{1+\sqrt{d}\mathrm{i}}{2}]$, if n is a positive integer divisible by d, then $(\pm\frac{1}{2}\pm\frac{d-1}{4d}\sqrt{d}\mathrm{i})n$ and $(\pm\frac{d+1}{4d}\sqrt{d}\mathrm{i})n$ are elements in K, so exactly one of $(\pm\frac{1}{2}+\frac{d-1}{4d}\sqrt{d}\mathrm{i})n$ or $(-\frac{d+1}{4\sqrt{d}}\mathrm{i})n$ is in $S(n)$, and exactly one of $(\pm\frac{1}{2}-\frac{d-1}{4d}\sqrt{d}\mathrm{i})n$ or $(\frac{d+1}{4d}\sqrt{d}\mathrm{i})n$ is in $S(n)$, i.e., there are infinitely many numbers such that

$$\min \max_{s\in R(w_0)} N(s) = \min \max_{s\in S(w_0)} N(s) = \frac{(d+1)^2}{16d} N(w_0),$$

which gives the theorem.