Proof of a conjecture stated in A007051

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Let n be a natural number and let $a_n = \text{A007051}_n = \frac{3^n+1}{2}$. Define a sequence $(s_n)_{n\in\mathbb{N}}$ by

$$s_n = \begin{cases} 2, & \text{if } n = 1, \\ \frac{2s_{n-1}+1}{s_{n-1}+2}, & \text{if } n > 1. \end{cases}$$

The purpose of this note is to prove a conjecture stated (with what seems to be a typo) in A007051 by Gary Detlefs. We shall exploit the fact stated in A007051 that a_n satisfies the recurrence $a_n = 3a_{n-1} - 1$.

Theorem 1. *Let $n \in \mathbb{N}$. Then*

$$s_n = \frac{a_n}{a_n - 1}.$$

Proof. We proceed by induction. The base case holds since $a(1) = 2$ and $\frac{a(1)}{a(1)-1} = \frac{2}{2-1} = 2 = s_1$. Assume that the assertion holds for $n-1$. Then

$$s_n = \frac{2s_{n-1}+1}{s_{n-1}+2} = \frac{2\frac{a_{n-1}}{a_{n-1}-1}+1}{\frac{a_{n-1}}{a_{n-1}-1}+2} = \frac{\frac{3a_{n-1}-1}{a_{n-1}-1}}{\frac{3a_{n-1}-2}{a_{n-1}-1}} = \frac{3a_{n-1}-1}{3a_{n-1}-2} = \frac{a_n}{a_n-1},$$

proving the induction step. □

Remark. The original conjecture read

$$s_n = \frac{a_n}{a_{n-1} - 1}.$$

It is easily verified that this is wrong. For example, for $n = 2$ we have

$$s_2 = \frac{2\cdot 2+1}{2+2} = \frac{5}{4}.$$

On the other hand, since $a_1 = 2$ and $a_2 = 5$, we have

$$\frac{a_2}{a_1 - 1} = \frac{5}{2-1} = 5.$$

References

[1] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., https://oeis.org.