Proofs of some conjectures of Yanev

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For $n \in \mathbb{N}$ let $a(n) = $ A000188(n), which is defined to be the square root of the largest perfect square dividing n. Let $\chi(n) = $ A010052(n), which is defined to be the characteristic function of squares. The statement of the following theorem was conjectured in A000188 by Yanev.

Theorem 1. *Let $n \in \mathbb{N}$. Then*

$$a(n) = \sum_{k=1}^{n} \chi(nk). \tag{1}$$

Proof. By definition of $a(n)$, we may write $n = a(n)^2 b$, for some squarefree $b \in \mathbb{N}$. Let $1 \leq k \leq n$. We claim that $\chi(nk) = 1$ if and only if there exists $m \in \mathbb{N}$ such that $k = bm^2$. To see this, assume the prime factorizations

$$\begin{aligned} b &= \prod_p p^{\delta_p}, \qquad \delta_p \in \{0, 1\}, \\ a(n) &= \prod_p p^{\alpha_p}, \qquad \alpha_p \geq 0, \\ k &= \prod_p p^{\beta_p}, \qquad \beta_p \geq 0. \end{aligned}$$

Then

$$\begin{aligned} n &= \prod_p p^{2\alpha_p + \delta_p}, \\ nk &= \prod_p p^{2\alpha_p + \delta_p + \beta_p}. \end{aligned}$$

Suppose that $\chi(nk) = 1$. Then $2\alpha_p + \delta_p + \beta_p$ is even, for every prime p. Since $2\alpha_p$ is even, this is equivalent to

$$\beta_p \equiv \delta_p \pmod 2, \quad \text{for all primes } p.$$

This condition is equivalent to

$$\beta_p = \delta_p + 2\gamma_p \quad \text{for some } \gamma_p \geq 0, \quad \text{for all primes } p.$$

Set

$$m = \prod_p p^{\gamma_p}.$$

Thus, we have

$$k = \prod_p p^{\beta_p} = \prod_p p^{\delta_p} \prod_p p^{2\gamma_p} = bm^2.$$

Conversely, assume that $k = bm^2$, for some $m \in \mathbb{N}$. Then

$$nk = (a(n)^2 b)(bm^2) = (a(n)bm)^2.$$

Thus, $\chi(nk) = 1$.

It remains to count the number of $m \in \mathbb{N}$ such that $k = bm^2 \leq n$. Since $n = a(n)^2 b$, this condition is equivalent to $m \leq a(n)$. It follows that for every $m = 1, 2, \ldots, a(n)$, we have $1 \leq k = bm^2 \leq n$ and $\chi(nk) = 1$, establishing (1). □

For $n \in \mathbb{N}$ let $a(n) =$ A006519(n) be the highest power of 2 dividing n, and let $\sigma(n) =$ A000203$(n) = \sum_{d|n} d$. The statement in the following theorem was conjectured by Yanev in A006519.

Theorem 2. *Let $n \in \mathbb{N}$. Then*

$$a(n) = \frac{1}{2}\left(\frac{1}{\sigma(2n)/\sigma(n) - 2} + 1\right). \tag{2}$$

Proof. Write $n = 2^k m$, form some $k \in \mathbb{N}_0$ and odd $m \in \mathbb{N}$. By the properties of σ (e.g., [1, Section 6.1]), $\sigma(n) = \sigma(2^k)\sigma(m)$ and $\sigma(2n) = \sigma(2^{k+1})\sigma(m)$. Therefore

$$\frac{\sigma(2n)}{\sigma(n)} = \frac{\sigma(2^{k+1})\sigma(m)}{\sigma(2^k)\sigma(m)} = \frac{\sigma(2^{k+1})}{\sigma(2^k)} = \frac{2^{k+2} - 1}{2^{k+1} - 1}. \tag{3}$$

Thus,

$$\begin{aligned}
\frac{\sigma(2n)}{\sigma(n)} - 2 &= \frac{2^{k+2} - 1}{2^{k+1} - 1} - 2 \\
&= \frac{2^{k+2} - 1 - 2(2^{k+1} - 1)}{2^{k+1} - 1} \\
&= \frac{2^{k+2} - 1 - 2^{k+2} + 2}{2^{k+1} - 1} \\
&= \frac{1}{2^{k+1} - 1}.
\end{aligned}$$

It follows that

$$\frac{1}{2}\left(\frac{1}{\sigma(2n)/\sigma(n)-2}+1\right)=2^k.$$

Since $a(n)=2^k$, the proof is complete. $\square$

For $n \in \mathbb{N}$ let

$$\sigma(n) = \underline{\text{A000203}}(n) = \sum_{d|n} d,$$
$$\sigma_2(n) = \underline{\text{A001157}}(n) = \sum_{d|n} d^2,$$
$$\text{rad}(n) = \underline{\text{A007947}}(n) = \prod_{p|n} p.$$

The statement in the following theorem was conjectured by Yanev in A001157.

Theorem 3. *Let $n \in \mathbb{N}$. Then*

$$\sigma_2(n) = \frac{\sigma(n^2\text{rad}(n))}{\sigma(\text{rad}(n))}.$$

Proof. Assume the prime factorization of n is

$$n = \prod_{i\geq 0} p_i^{e_i},$$

where $e_i \in \mathbb{N}_0$, for every i. By the properties of σ_2 (e.g., [1, Exercises 22 in Section 6.1]),

$$\sigma_2(n) = \prod_{i\geq 0} \frac{p_i^{2(e_i+1)}-1}{p_i^2-1}. \qquad (4)$$

On the other hand,

$$n^2\text{rad}(n) = \left(\prod_{i\geq 0} p_i^{e_i}\right)^2 \prod_{i\geq 0} p_i = \prod_{i\geq 0} p_i^{2e_i+1}.$$

Thus, with the properties of σ (e.g., [1, Section 6.1]),

$$\sigma(n^2\text{rad}(n)) = \prod_{i\geq 0} \frac{p_i^{2e_i+2}-1}{p_i-1},$$
$$\sigma(\text{rad}(n)) = \prod_{i\geq 0}^{r} \frac{p_i^2-1}{p_i-1}.$$

It follows that

$$\frac{\sigma(n^2\mathrm{rad}(n))}{\sigma(\mathrm{rad}(n))} = \prod_{i\geq 0} \frac{p_i^{2e_i+2}-1}{p_i^2-1},$$

which is precisely the expression for $\sigma_2(n)$ in (4). □

References

[1] D. M. Burton, *Elementary Number Theory*, 6th ed., McGraw–Hill, New York, 2007.

[2] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.