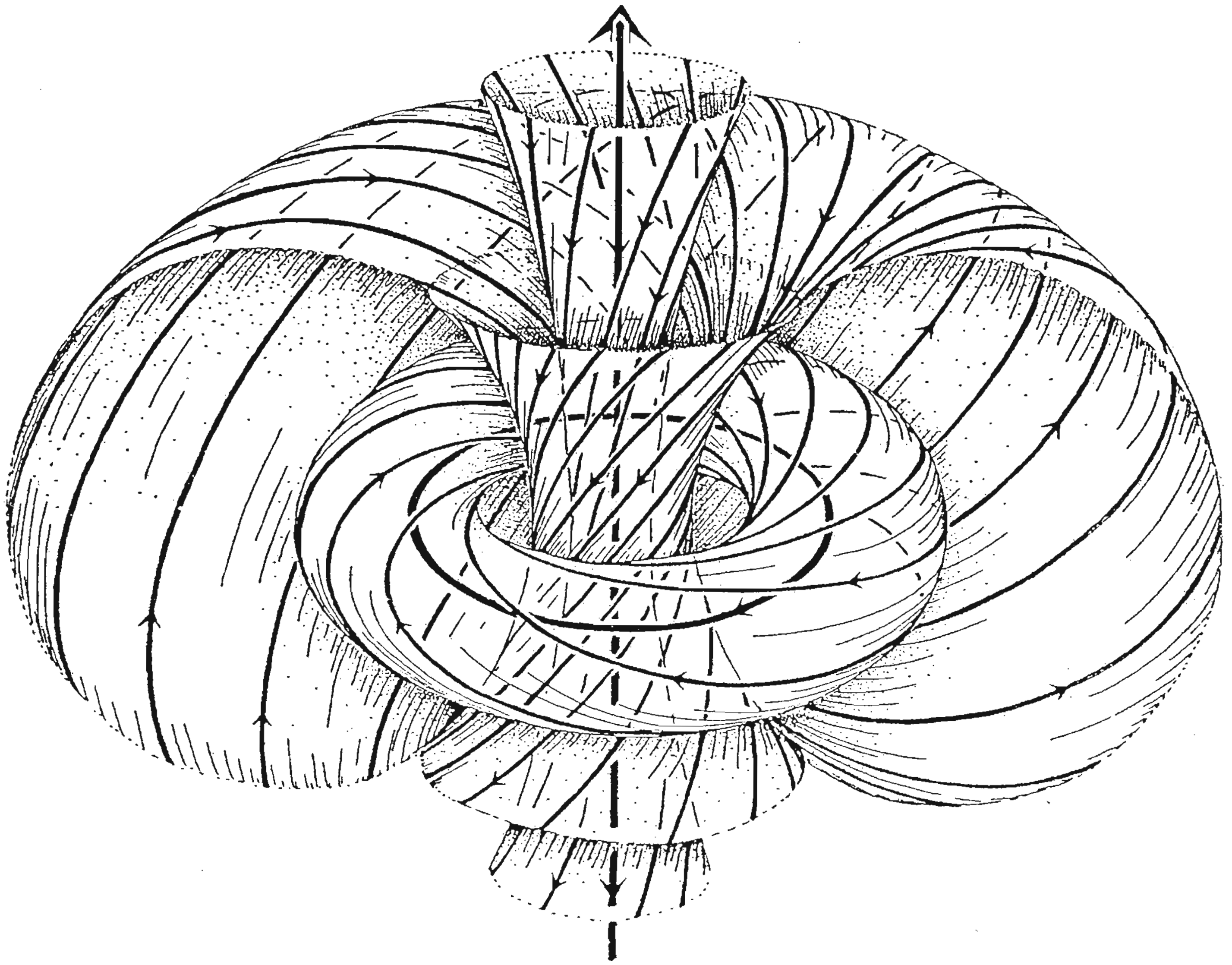
Twistor Newsletter

(no 41: 26 Aug. 1996)



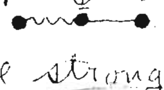
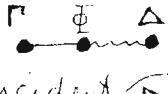
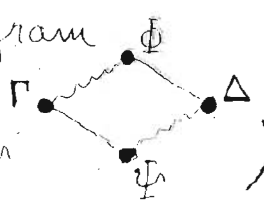
Mathematical Institute

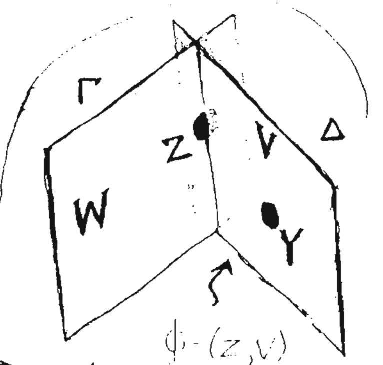
Oxford, England

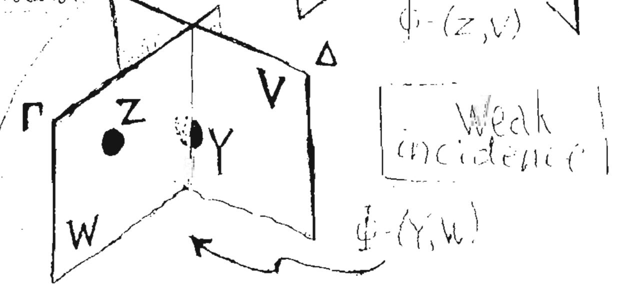
Incidence between Complex Null Rays

The incidence relations between complex null rays exhibit certain curious properties. I shall try to illuminate these here. The structures that arise have relevance to the geometry of ambitwistor space. The discussion will be phrased in terms of compactified complex Minkowski space $\mathbb{C}\mathbb{M}^{\#}$ and its associated (projective) twistor space $\mathbb{PT}$, in the first instance, with some generalizations to curved space-time indicated at the end.

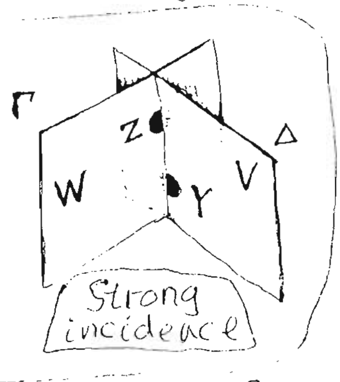
By a <u>ray</u> I mean a (generally complex) null geodesic. A ray γ in $\mathbb{C}\mathbb{M}^{\#}$ has a twistor interpretation in terms of a pair $\Gamma = (Z, W)$, where $Z \in \mathbb{PT}$, $W \in \mathbb{PT}^*$ with Z incident with W (i.e. $Z^\alpha W_\alpha = 0$). I shall refer to Γ as a <u>pencil</u>, in reference to the plane pencil of lines lying in the plane $W \subset \mathbb{PT}$ and passing through the point $Z \in \mathbb{PT}$, these lines representing the points of the ray $\gamma \subset \mathbb{C}\mathbb{M}^{\#}$. The point $Z \in \mathbb{PT}$ is called the <u>vertex</u> of the pencil and W is its plane.

There are various kinds of incidence relation holding between rays in $\mathbb{C}\mathbb{M}^{\#}$, which I shall call "weak incidence", "strong incidence", "α-incidence", and "β-incidence". Two pencils $\Gamma = (Z, W)$ and $\Delta = (Y, V)$ are <u>α-incident</u> if $Y = Z$ (diagram ); they are <u>β-incident</u> if $V = W$ (diagram ); they are <u>weakly incident</u> if the join $Y \sqcup Z$ of Y and Z meets the intersection $V \cap W$ of V and W (diagram  or  for some Φ); they are <u>strongly incident</u> if $Y \sqcup Z = V \cap W$ (diagram  for some Φ, Ψ).





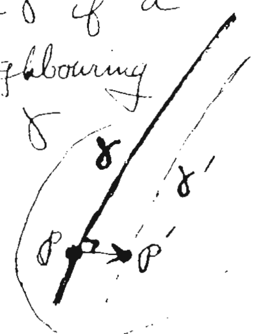
Strong incidence between Γ and Δ is the condition that the corresponding rays γ and δ in $\mathbb{C}M^{\#}$ have a point in



common. (See also the discussion by T.R.F. in TN 38 pp. 30–34.)

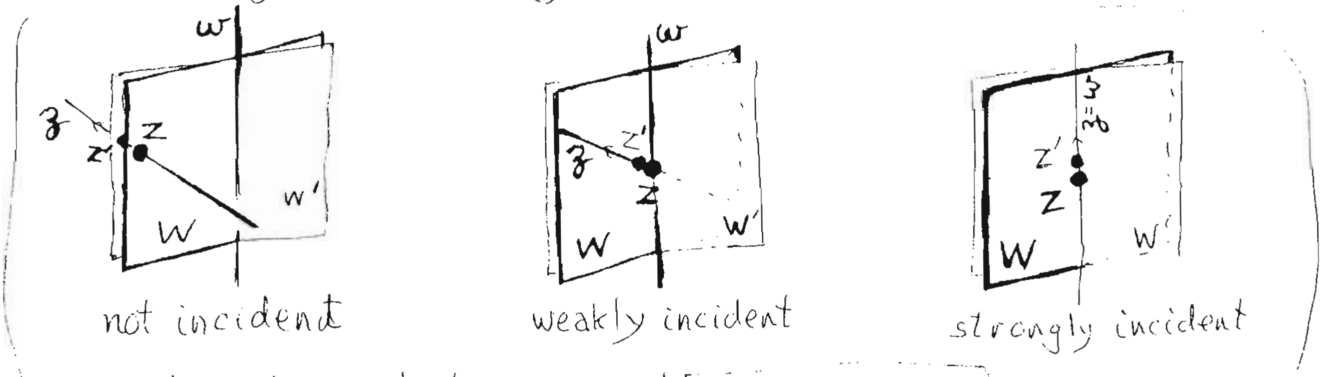
Note that the space $\mathcal{D}$ of those Δ that are weakly incident with a fixed Γ is the union of two smooth manifolds $\mathcal{D}_\alpha$, $\mathcal{D}_\beta$ (respectively), each being a $\mathbb{CP}^2$ bundle over $\mathbb{CP}^2$. This arises because each of the spaces of pencils Φ α-incident or β-incident with Γ is a $\mathbb{CP}^2$, and we apply one step in each to arrive at weak incidence. The condition of strong incidence does <u>not</u> provide a smooth manifold, however, as we shall be seeing shortly.

Consider, next, weak and strong incidence with γ of a ray γ′ <u>neighbouring</u> to γ (i.e. we are concerned with infinitesimal variations of rays away from γ). We find that the notion of weak incidence between γ′ and γ is equivalent to <u>abreastness</u>. We say that γ′ is abreast with γ if a connecting vector from a point p on γ to a neighbouring point p′ of γ′ is <u>orthogonal</u> to the direction of γ (cf. Penrose & Rindler, Spinors & Space-time Vol. 2 p. 176). This property is independent of the choice of p on γ and of the neighbouring point p′ on γ′.



An easy way to see that **weak** incidence between neighbouring rays implies abreastness is to observe that a way of stating the condition for two rays γ and δ to be weakly incident is that there be a ray meeting each of γ and δ orthogonally. This is the ray φ represented by the pencil Φ in Γ–Φ–Δ or Γ–Φ–Δ. (Note that any two rays in the same α-plane or in the same β-plane are orthogonal at their intersection point.) Hence, when δ becomes γ′, neighbouring to γ, with p the intersection of φ with γ, we see that the condition for abreastness is satisfied. For the converse, we may simply note that weak incidence is a codimension 1 condition, so the neighbourhood of $\mathcal{D}$ at Γ cannot be smaller than the space of γ′ abreast with γ, abreastness being an irreducible condition.

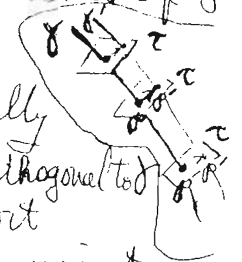
In terms of $\mathbb{PT}$, weak incidence between Γ and Γ' is understood as the condition that the join $z = Z \sqcup Z'$ meet the intersection $w = W \cap W'$. Unlike the case of finitely differing pencils, the condition now becomes symmetrical, w and z just being two lines in the plane W passing through the point Z.

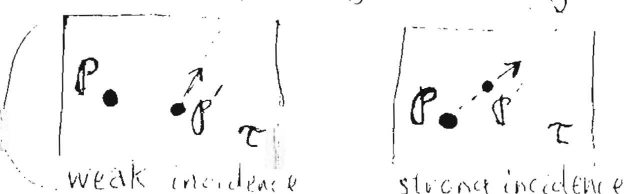


Incidence between neighbouring pencils.

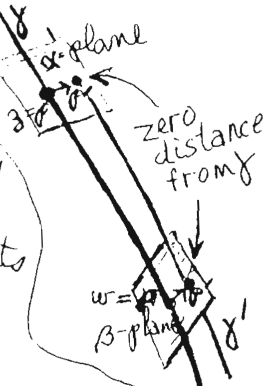
What about strong incidence? This is the condition that γ' actually "meets" γ', so that the pencils Γ and Γ' have a line in common. This line must be both z and w, so we have $z = w$ as the condition for strong incidence. Recall that with finitely differing pencils Γ, Δ, if both forms of weak incidence hold together, then strong incidence must hold. It is curious that for infinitesimally differing pencils Γ, Γ', the situation is quite different. The two forms of weak incidence for Γ, Γ' coincide, and the simultaneous holding of both forms is insufficient to imply strong incidence. I shall come to the explanation of this curious fact shortly.

In terms of space-time geometry, we can think of γ' as given by a Jacobi field of connecting vectors along γ. Referring this to a family of parallelly propagated 2-plane elements τ, at the points of γ, orthogonal to γ, all identified with one another by parallel transport along γ, we find that, for an abreast ray γ', the point ρ' in this plane executes a straight line in τ, uniformly with the affine parameter on γ.





If γ' is strongly incident with γ, this line passes through ρ; in the general case of weak incidence, it does not.



In the case of strong incidence, the point where p' coincides with p is the point $z \in \gamma$, represented by the line $z = Z \sqcup Z' = w = W \cap W'$ in $\mathbb{PT}$. In the general case of weak incidence, the points z and w are distinct, and they represent the locations of p at which the point p' reaches zero distance from p. These are imaginary points, when the rays are real, and they occur where γ' intersects the α-plane and the β-plane through γ, respectively.

In terms of twistor notation, we can write

$$Y^\alpha = Z^\alpha + \delta Z^\alpha, \quad V_\alpha = W_\alpha + \delta W_\alpha$$

where

$$Z^\alpha W_\alpha = 0, \quad Y^\alpha V_\alpha = 0$$

whence

$$W_\alpha \delta Z^\alpha + Z^\alpha \delta W_\alpha + \delta Z^\alpha \delta W_\alpha = 0.$$

To first order, we have

$$W_\alpha \delta Z^\alpha = -Z^\alpha \delta W_\alpha.$$

The condition for weak incidence is either $Z^\alpha V_\alpha = 0$, i.e.

$$Z^\alpha \delta W_\alpha = 0$$

or $Y^\alpha W_\alpha = 0$, i.e.

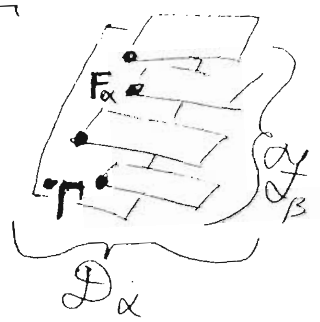
$$W_\alpha \delta Z^\alpha = 0$$

which, to first order, are indeed equivalent. To recognise strong incidence we must go to second order, the condition being that, in addition, the quadratic relation

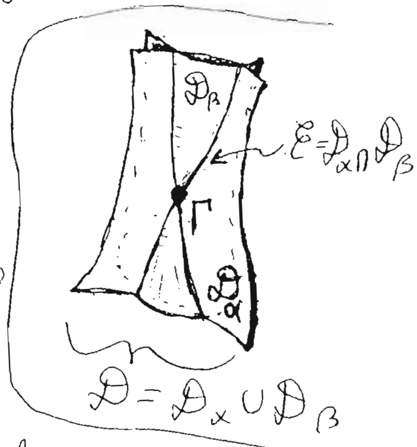
$$\delta Z^\alpha \delta W_\alpha = 0$$

must hold. (For real light rays the discussion is the same, but with $W_\alpha = \bar{Z}_\alpha$, $\delta W_\alpha = \delta \bar{Z}_\alpha$.)

Let us try to understand this curious situation in terms of the geometry of ambitwistor space $\mathbb{A}$, for $\mathbb{C}M^\#$. ($\mathbb{A}$ is the space of Γs, i.e. of complex rays in $\mathbb{C}M^\#$.) This space is 5-dimensional. The space $\mathcal{D}$ of points of $\mathbb{A}$ that are weakly incident with Γ is, as we have seen, the union of two smooth manifolds $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$. We can think of $\mathcal{D}_\alpha$ as constructed as follows. There is a foliation $\mathcal{F}_\alpha$ of $\mathbb{A}$ by families of points of $\mathbb{A}$ that are α-incident with each other and another foliation $\mathcal{F}_\beta$ by families of points of $\mathbb{A}$ that are β-incident with each other. To construct $\mathcal{D}_\alpha$, we fix the member F_α of $\mathcal{F}_\alpha$ through Γ and sweep out a region of $\mathbb{A}$ by taking all the members of $\mathcal{F}_\beta$ through points of F_α. In an exactly similar way, we can construct $\mathcal{D}_\beta$, now by fixing F_β through Γ and allowing the $\mathcal{F}_\alpha$ members through points of F_β to vary.



Note that the difference between $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$, when constructed in this way is a <u>commutator</u>, and this difference is thus of second order. In other words, $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ have the same tangent plane at Γ. This explains the symmetry between the two forms of weak incidence when we look only to first order. However, the <u>strong</u> incidence locus $\mathcal{E}$ is the intersection of $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$. Since $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ touch at Γ, their intersection has a <u>conical singularity</u> at Γ. The equation, locally, of this cone is the quadratic relation $\delta Z^\alpha \, \delta W_\alpha = 0$.



These ideas can also be applied in a general curved space-time $\mathcal{M}$ in reference to a choice of hypersurface $\mathcal{H}$ (possibly $\mathcal{H} = \mathcal{I}^+$, for example). In this case $\mathbb{PT}$ and $\mathbb{PT}^*$ are, respectively, the α-curves and β-curves in $\mathbb{C}\mathcal{H}$. Incidence between α-curves and β-curves is the condition that they <u>intersect</u>. The other notions of incidence are built up from this. The geometry described above is virtually unchanged. (Of course all this is <u>relative</u> to the choice of $\mathcal{H}$. If $\mathcal{H}$ is moved within $\mathcal{M}$, the structure changes.)

Further details are to appear in an article in honour of Andrzej Trautman's 64th birthday.

~ R.P.

6

Coherent States and Fubini-Study Geometry

Timothy R. Field
Mathematical Institute, Oxford, U.K.
email: trfield@maths.ox.ac.uk

In this article I derive the geometry of the submanifold of coherent states for a bosonic quantum field theory. The approach taken is to view the coherent state submanifold, which we shall denote by $\mathcal{C}$, as embedded in an infinite dimensional complex projective space $CP^{\aleph_0}$ – the quantum-mechanical state space, or projective Fock space. The metric on the ambient state space is the familiar Fubini-Study metric, and we use this to calculate the induced metric on $\mathcal{C}$. There are some technicalities associated with the fact that in physics one is dealing typically with an infinite dimensional Fock space of states, which itself is built up from an infinite dimensional single particle Hilbert space $\mathcal{H}^1$. However it is in practice reasonable to assume that the underlying single particle Hilbert space is separable, that is to say any vector can be decomposed along countably and possibly infinitely many basis states, as for example occurs in the Fourier series analysis of an oscillator with boundary conditions. In the case of a state not built up in this way one can always argue via continuity in the relevant function space.

We begin with the basic notation and definitions.

1. Fock Space and Abstract Index Notation

Let V denote the Hilbert space of real solutions to some classical linear field equation, and define the single particle Hilbert space as

$$\mathcal{H} = V \otimes C.$$

Now introduce the notation for the n-fold tensor product of $\mathcal{H}$ with itself,

$$\mathcal{H}^n \equiv \mathcal{H}^{\otimes^n}$$

where $\otimes$ denotes the symmetric tensor product for bosons and antisymmetric tensor product for fermions. Then $\mathcal{H}^n$ is said to be the n-particle Hilbert space, and we define Fock space as the Hilbert space

$$\mathcal{F} = C \oplus \mathcal{H} \oplus \mathcal{H}^2 \oplus \ldots \oplus \mathcal{H}^n \oplus \ldots$$

in which C represents the vacuum state or $\mathcal{H}^0$. Now we define $\mathcal{H}^n_{\pm}$ to be the positive and negative frequency n-particle Hilbert spaces respectively,

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

giving rise to the associated positive and negative frequency Fock spaces $\mathcal{F}_\pm$. We shall use the abstract index notation for elements of $\mathcal{H}$ so that we write

$$\xi^a \in \mathcal{H}.$$

We take $\mathcal{H}$ to have a *countably* infinite basis, so that the index on ξ can be thought of as running over the natural numbers. For a positive frequency field we shall use an unprimed Greek index so that for example $\xi^\alpha \in \mathcal{H}^1_+$.

Now a state vector in Fock space $\mathcal{F}_+$ can be written

$$|\xi\rangle = (\xi, \xi^\alpha, \xi^{\alpha\beta}, \ldots)$$

where $\xi^{\alpha\beta} \in \mathcal{H}^2_+$ and so on.

The evaluation of the squared Hilbert space norm of a vector $|\xi\rangle$ in $\mathcal{F}$ is according to

$$\|\xi\|^2 = \xi\bar{\xi} + \xi^\alpha\bar{\xi}_\alpha + \xi^{\alpha\beta}\bar{\xi}_{\alpha\beta} + \ldots$$

(for further details of this index notation and the relation to the complex structure involved in the frequency splitting see Geroch 1971). For any $\sigma^\alpha \in \mathcal{H}^1_+$ we define creation and annihilation operators $\hat{C}_\alpha$, $\hat{A}^\alpha$ respectively, according to

$$\hat{C}_\alpha \sigma^\alpha |\xi\rangle \equiv \hat{C}(\sigma)\psi = (0, \sigma^\alpha \xi, \sqrt{2}\sigma^{(\alpha}\xi^{\beta)}, \sqrt{3}\sigma^{(\alpha}\xi^{\beta\gamma)}, \ldots)$$

$$\hat{A}^\alpha \bar{\sigma}_\alpha |\xi\rangle \equiv \hat{A}(\bar{\sigma})\psi = (\xi^\mu \bar{\sigma}_\mu, \sqrt{2}\xi^{\mu\alpha}\bar{\sigma}_\mu, \sqrt{3}\xi^{\mu\alpha\beta}\bar{\sigma}_\mu, \ldots)$$

and these obey the CCR

$$[\hat{C}(\sigma), \hat{C}(\sigma')] = 0$$

$$[\hat{A}(\bar{\sigma}), \hat{A}(\bar{\sigma}')] = 0$$

$$[\hat{A}^\alpha, \hat{C}_\beta] = \delta^\alpha_\beta \iff [\hat{A}(\bar{\sigma}), \hat{C}(\sigma)] = (\sigma \cdot \bar{\sigma})I.$$

The creation and annihilation operators are adjoints of each other, so that for any vector $|\phi\rangle \in \mathcal{F}$ we have

$$\langle \hat{C}(\sigma)\psi, \phi\rangle = \langle \psi, \hat{A}(\bar{\sigma})\phi\rangle.$$

2. Coherent States

Now we define the space of coherent states. There are many different characterizations of coherent states within the context of quantum field theory. To begin with we give the definition in terms of *exponentiation* of the single particle Hilbert space. We begin with a vector

$$\xi^a \in \mathcal{H}^1$$

and we wish to infer from this, uniquely, an element of state space $P\mathcal{F}$, said to be the state *coherent* to ξ^a and which we shall denote $P|\xi_c\rangle$. Here P is the quotient map by complex scalar multiples. The exponential map e is defined by

$$\xi^a \mapsto e(\xi^a) = (1, \xi^a, \xi^a\xi^b/\sqrt{2!}, \ldots, \xi^a\xi^b \ldots \xi^d/\sqrt{n!}, \ldots) =: |\xi_c\rangle \in \mathcal{F}$$

in which the term containing $\sqrt{n!}$ has n indices. Now $|\xi_c\rangle$ is said to be a coherent state *vector* and P takes this to the associated complex ray in Fock space. It is crucial to understand that

$$P \circ e(\lambda\xi^a) = P \circ e(\mu\xi^a)$$

for $\xi^a \neq 0$ holds if and *only* if $\lambda = \mu$. For the vacuum parts of both vectors $e(\lambda\xi^a)$ and $e(\mu\xi^a)$ are unity and so if these vectors are proportional then we must have $(\mu - \lambda)\xi^a$ vanishing, which implies the property claimed. Note however, that although $\lambda\xi^a$, $\mu\xi^b$ define different vectors in $\mathcal{H}^1$ for $\lambda \neq \mu$, they define the same single particle states for all non-zero values of λ, μ. This is because the single particle states are elements of $P\mathcal{H}^1$ and not $\mathcal{H}^1$. In summary we must beware of the following property,

Changing the phase or scale of a single particle state vector changes its associated coherent state.

In terms of fibre bundles, for an $(n+1)$-dimensional single particle Hilbert space with corresponding state space CP^n we consider the *universal bundle* $\mathcal{U}$

$$\Pi : \mathcal{U} \to P\mathcal{H}^1$$

over the state space, defined so that the fibre above any point or state is precisely the ray in the Hilbert space it represents. Now the map e defined above is a map from the *bundle* to Fock space

$$e : \mathcal{U} \to \mathcal{F}$$

and this is non constant along each fibre $\Pi^{-1}(s)$ for all $s \in P\mathcal{H}^1$. At first sight this may seem a little pedantic but this idea is essential to the discussion which follows.

We now examine the action of the creation and annihilation operators acting on coherent states. For $\tau^\alpha \in \mathcal{H}^1_+$ and $|\psi_c\rangle$ defined as before we have

$$\hat{A}(\bar{\tau})|\psi_c\rangle = (\xi \cdot \bar{\tau})|\psi_c\rangle$$

or equivalently

$$\hat{A}^\alpha|\psi_c\rangle = \psi^\alpha|\psi_c\rangle$$

so that *coherent states are eigenstates of the annihilation operator* . The action of the creation operator is via differentiation with respect to $\mathcal{H}^1$. For any $\sigma^\alpha \in \mathcal{H}^1_+$ we have

$$\hat{C}(\sigma)|\psi_c\rangle = (0, \sigma^\alpha, \sqrt{2}\,\sigma^{(\alpha}\psi^{\beta)}, \ldots, \sqrt{n}/\sqrt{(n-1)!}\,\sigma^{(\alpha}\psi^\beta \ldots \psi^{\sigma)}, \ldots)$$

where n factors appear in the general term, or equivalently

$$\hat{C}_\alpha|\psi_c\rangle = \frac{d|\psi_c\rangle}{d\psi^\alpha}.$$

For convenience we shall set

$$\Lambda := \xi^\alpha \bar{\xi}_\alpha \equiv \|\xi\|^2.$$

Then we have

$$\langle\psi_c|\psi_c\rangle = e^\Lambda$$

and we must divide by this factor to calculate the expected value of any operator from its matrix element with a coherent state vector.

Another important fact about coherent states is that they provide a *resolution of unity*

$$\int_{t\in\mathcal{A}} p_t|c_t\rangle\langle c_t| = \mathbf{1} \quad (*)$$

for some index set $\mathcal{A}$, and that the above equation determines p_t *uniquely* (see for example Klauder and Skagerstam 1985). This uniqueness property is in spite of the fact that the coherent states are not mutually orthogonal, and are indeed in a sense over complete. In fact, in our Hilbert space notation we have

$$\langle\xi_c|\psi_c\rangle = e^{\psi\cdot\bar{\xi}}$$

and clearly this can never equal zero. The unique resolution of unity implies that we can expand any state vector as a superposition of coherent states in a unique way. This implies the following.

Lemma 1. *The submanifold of coherent states is non-linear inside state space $P\mathcal{F}$. That is to say, given any two distinct coherent state vectors $|\xi_1\rangle$, $|\xi_2\rangle$, the superposition*

$$\lambda|\xi_1\rangle + \mu|\xi_2\rangle$$

is a coherent state vector if and only if exactly one of λ or μ vanishes.

Proof. This follows immediately from the uniqueness of decomposition of any state vector into coherent states.

3. The Fubini-Study Geometry

The projective form of the Fubini Study metric on CP^n which one most usually encounters in the context of quantum theory (see for example Hughston 1996) is given by

$$ds^2 = 8k^{-1}\frac{Z^{[\alpha}dZ^{\beta]}\bar{Z}_{[\alpha}d\bar{Z}_{\beta]}}{(Z^\gamma\bar{Z}_\gamma)^2}$$

where Z^α are homogeneous coordinates for CP^n with $\alpha = 0, 1, ..., n$, and k is the holomorphic sectional curvature, which in subsequent calculations we shall take to equal one. For our purposes in the application to coherent states it will be useful also to have this metric expressed in *non*-homogeneous coordinates. This is because if we set Z^0 to be the coordinate of the vacuum part of a coherent state then, as follows from above this coordinate is necessarily non-vanishing. Thus the coherent states form a submanifold of the *affine* part of the projective Fock space, the latter consisting of all elements of the form

$$\{(1, \xi^a, \xi^{ab}, ...)\} =: \mathcal{A} \supset \mathcal{C} \cong C^\infty$$

and whose *compactification* is

$$\{P(0, \psi^a, \psi^{ab}, ...)\} =: \mathcal{B} \cong CP^\infty,$$

that is to say states for which the probability of no quanta being present is zero. Note here that the image of any state vector $|\psi\rangle$ under the creation operator $\hat{C}(\sigma)$ always lies within the compactification,

$$\hat{C}(\sigma)|\psi\rangle \in \mathcal{B}_+ \;\; \forall\, |\psi\rangle \in \mathcal{F}_+,\; \sigma^a \in \mathcal{H}_+.$$

From now on we deal with coherent states viewed as forming a submanifold of $\mathcal{A}$ and we shall make use of the following standard result.

Lemma 2. *The equivalent non-projective form of the Fubini Study metric on CP^n is given by*

$$ds^2 = 4\frac{(1+\zeta^\alpha\bar{\zeta}_\alpha)d\zeta^\alpha d\bar{\zeta}_\alpha - (\zeta^\alpha d\bar{\zeta}_\alpha)(\bar{\zeta}_\alpha d\zeta^\alpha)}{(1+\zeta^\alpha\bar{\zeta}_\alpha)^2}$$

where $\zeta^\alpha = Z^\alpha/Z^0$ for $\alpha = 1, 2, ..., n$.

We shall also require the following standard

Definition 1. *A complex manifold M is said to be Kähler if it comes equipped with an Hermitian metric $h_{\alpha\beta'}$ with $ds^2 = h_{\alpha\beta'}dz^\alpha \otimes d\bar{z}^{\beta'}$ such that the associated real 2-form*

$$\Omega = ih_{\alpha\beta'}dz^\alpha \wedge d\bar{z}^{\beta'}$$

is closed. Then Ω is said to be a Kähler form for M.

This is equivalent to the existence on M of a Kähler scalar function K which is real valued, such that

$$\Omega = i\partial\bar{\partial}K \longleftrightarrow h_{\alpha\beta'} = \partial_\alpha\bar{\partial}_{\beta'}K \quad (*).$$

Thus we have the consequence that a complex submanifold N of a Kähler manifold M is itself Kähler, since one simply restricts the function K down to N to find the induced Kähler metric according to (*). In the case of the Fubini-Study metric on CP^n the Kähler scalar function takes the form

$$K = 4k^{-1}\log[1 + k(|\zeta^1|^2 + |\zeta^2|^2 + ... + |\zeta^n|^2)]$$

where ζ^α are inhomogeneous coordinates on CP^n, that is $\zeta^\alpha = Z^\alpha/Z^0$ as above.

The following result also will be of interest with regard to the theorem to follow.

Lemma 3. *Let Ω be a positive $(1,1)$-form on a complex manifold M. Then Ω is a Kähler form for M if and only if for all $x_0 \in M$ there exist holomorphic 'Euclidean' coordinates $z^1, ...z^n$ around x_0 such that*

$$\Omega = ih_{\alpha\beta'}dz^\alpha \wedge d\bar{z}^{\beta'}$$

$$h_{\alpha\beta'} = \delta_{\alpha\beta'} + O(|z|^2) \quad \text{at } x_0,$$

and thus the Kähler metric osculates to the flat Euclidean metric to second order.

Proof. The implication towards Ω being a Kähler form is clear. In order to prove the reverse implication, begin with holomorphic coordinates $z^1, ..., z^n$ such that $dz^1, ..., dz^n$ give an orthonormal basis of T^*_{M,x_0}, the dual tangent space to M at x_0. Then this implies that

$$\Omega = i\tilde{h}_{\alpha\beta'} dz^\alpha \wedge d\bar{z}^{\beta'}$$

where

$$\begin{aligned}\tilde{h}_{\alpha\beta'} &= \delta_{\alpha\beta'} + O(|z|)\\ &= \delta_{\alpha\beta'} + \sum_{1\leq\gamma\leq n} (a_{\gamma\alpha\beta'} z^\gamma + a'_{\gamma'\alpha\beta'} \bar{z}^{\gamma'}) + O(|z|^2).\end{aligned}$$

That Ω is real implies

$$a'_{\gamma'\alpha\beta'} = \bar{a}_{\gamma'\beta'\alpha}.$$

Furthermore the Kähler condition

$$\frac{\partial h_{\alpha\beta'}}{\partial z^\gamma}|_{x_0} = \frac{\partial h_{\gamma\beta'}}{\partial z^\alpha}|_{x_0}$$

implies that

$$a_{\alpha\gamma\beta'} = a_{(\alpha\gamma)\beta'}.$$

Now we define our holomorphic Euclidean coordinates $\hat{z}^\alpha$ to be

$$\hat{z}^\beta := z^\beta + \frac{1}{2} \sum_{\gamma,\alpha} a_{\gamma\alpha\beta'} \delta^{\beta\beta'} z^\gamma z^\alpha$$

which completes the proof of the lemma. qed

4. Geometry of Coherent States

We are now ready to state our main result. This is a global geometrical property which, from Lemma 3, is a special case of a flatness property which applies locally to any Kähler manifold.

Theorem 1. *The metric induced on the coherent state submanifold from the ambient Fubini-Study metric on the quantum mechanical state space is intrinsically flat. Moreover the coordinates ξ^a on the single particle Hilbert space $\mathcal{H}^1$ are complex Euclidean coordinates for the coherent state submanifold.*

Suppose instead we take the point of view that we begin with the coherent state submanifold and decide a priori to place on it the complex Euclidean metric, giving us $\mathcal{C}_E$ say. Then we have the following equivalent result.

Theorem 2. *The Euclidean coherent state submanifold has an isometric embedding into the Fubini-Study state manifold*

$$\mathcal{C}_E \overset{i}{\hookrightarrow} \mathcal{F}_{F.S.},$$

where i is the inclusion map, and is in this case an isometry.

We note that the theorems is independent of the details of the single particle Hilbert space $\mathcal{H}^1$. We give three proofs of this result, each of which uses a different technique.

Coordinate Proof. From the way that we defined coherent state vectors, we may regard $\xi^a \in \mathcal{H}^1$ as complex coordinate functions for the coherent state submanifold. It will be helpful to introduce some further notation. We define

$$\xi^{(n)} := \frac{\xi^{(\alpha}\xi^\beta \ldots \xi^{\delta)}}{\sqrt{n!}}$$

so that $\xi^{(n)}$ is the tensor contribution to the coherent state vector $|\xi_c\rangle$ which lies in $\mathcal{H}^n$, there being n factors in the symmetrized tensor product above. Similarly we define $\bar{\xi}_{(n)}$. Then setting $\Lambda = \xi^\alpha \bar{\xi}_\alpha$ we calculate

$$\xi^{(n)} \bar{\xi}_{(n)} = \frac{1}{n!} \Lambda^n.$$

Now *restricted* down to the coherent state submanifold $\mathcal{C}$ we can calculate the tangent vector to a coherent state induced by an element $d\xi$ of $T^*\mathcal{H}$. The component in $\mathcal{H}^n$ of the (dual) tangent vector is given by

$$d\xi^{(n)} = \frac{n}{\sqrt{n!}} d\xi^{(\alpha}\xi^{\beta}\xi^{\gamma}\ldots\xi^{\delta)}$$

and similarly for the complex conjugate. Now to insert into the Fubini-Study line element we shall need to find the *coordinate* inner product of a tangent vector with itself. Note that the contribution to this of any pair of vectors lying in distinct $\mathcal{H}^n$ vanishes, as follows from our expression for the Hilbert space norm given earlier. The essential point here is that when evaluating the inner product of two Fock space vectors in the abstract index notation, one contracts over vectors and their conjugates with the same number of indices. Thus we have

$$d\xi^{(n)}d\bar{\xi}_{(m)} = \delta^{(n)}_{(m)} \cdot \frac{1}{(n-1)!}[\Lambda^{n-1}d\xi^{\alpha}d\bar{\xi}_{\alpha} + (n-1)\Lambda^{n-2}|\xi^{\alpha}d\bar{\xi}_{\alpha}|^2]$$

for all $n \geq 1$. We shall also need the coordinate inner product

$$\bar{\xi}_{(m)}d\xi^{(n)} = \delta^{(n)}_{(m)} \cdot \frac{\bar{\xi}_{(\alpha}\bar{\xi}_{\beta}\ldots\bar{\xi}_{\delta)}}{\sqrt{n!}} \cdot \frac{n}{\sqrt{n!}} d\xi^{(\alpha}\xi^{\beta}\xi^{\gamma}\ldots\xi^{\delta)}$$

$$= \delta^{(n)}_{(m)} \cdot \frac{1}{(n-1)!}(\bar{\xi}_{\alpha}d\xi^{\alpha})\Lambda^{n-1}$$

and similarly

$$\xi^{(n)}d\bar{\xi}_{(m)} = \delta^{(n)}_{(m)} \cdot \frac{1}{(n-1)!}(\xi^{\alpha}d\bar{\xi}_{\alpha})\Lambda^{n-1}$$

for all $n \geq 1$. Now we refer back to our expression for the Fubini-Study line element given in non-homogeneous coordinates derived above. In this expression a vector ξ lies in the Fock space and so is given by the collection $\{\xi^{(n)}\}$ for *all* values of n. Thus to evaluate the line element induced on the coherent state submanifold $\mathcal{C}$ we must sum over all $1 \leq m, n \leq \infty$ in the above identities. In doing this we obtain

$$\sum_{m,n} d\xi^{(n)}d\bar{\xi}_{(m)} = e^{\Lambda}(d\xi^{\alpha}d\bar{\xi}_{\alpha} + |d\xi^{\alpha}\bar{\xi}_{\alpha}|^2)$$

and

$$\sum_{m,n} \bar{\xi}_{(m)}d\xi^{(n)} = e^{\Lambda} \cdot \bar{\xi}_{\alpha}d\xi^{\alpha}$$

together with its complex conjugate. Also for the denominator in the Fubini-Study line element we have simply

$$1 + \sum_{m,n} \xi^{(n)}\bar{\xi}_{(m)} = e^{\Lambda}.$$

Hence our induced line element becomes

$$ds^2 = \frac{4}{e^{2\Lambda}} \cdot [e^{2\Lambda}(d\xi^{\alpha}d\bar{\xi}_{\alpha} + |d\xi^{\alpha}\bar{\xi}_{\alpha}|^2) - e^{2\Lambda}|\xi^{\alpha}d\bar{\xi}_{\alpha}|^2]$$

which is simply

$$ds^2 = 4d\xi^{\alpha}d\bar{\xi}_{\alpha}.$$

This completes the coordinate proof. ∎

Algebraic Proof. This is somewhat more general than the coordinate proof above in that it is valid for any quantum field theory including the possible presence of interactions. For we shall assume only the CCR for the creation and annihilation operators, together with the fact that the coherent states are eigenstates of the annihilation operator, for this defines the coherent states uniquely (Klauder and Skagerstam 1985). For a complicated Lagrangian including interactions the precise details of A^{α} and C_{β} of course will change but the fundamental algebraic relation (CCR) between them and the defining properties of the coherent states are always those given above.

We shall adopt the Dirac notation for state vectors according to

$$Z^\alpha \leftrightarrow |\psi\rangle, \quad \bar{Z}_\alpha \leftrightarrow \langle\psi|.$$

In this notation the Fubini-Study line element becomes

$$ds^2_{F.S.} = 4\{\frac{\langle d\psi|d\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\psi|d\psi\rangle\langle d\psi|\psi\rangle}{\langle\psi|\psi\rangle^2}\}.$$

Now we abbreviate so that $|\psi\rangle \in \mathcal{F}$ denotes the state vector coherent to $\psi^\alpha \in \mathcal{H}^1$. Then recall that

$$A^\alpha|\psi\rangle = \psi^\alpha|\psi\rangle \longleftrightarrow \langle\psi|\bar{\psi}_\alpha = \langle\psi|C_\alpha$$

together with

$$C_\alpha|\psi\rangle = \frac{|d\psi\rangle}{d\psi^\alpha} \longleftrightarrow \langle\psi|A^\alpha = \frac{\langle d\psi|}{d\bar{\psi}_\alpha}.$$

It follows that

$$|d\psi\rangle = C_\alpha|\psi\rangle d\phi^\alpha$$

and correspondingly

$$\langle d\psi| = d\bar{\phi}_\beta\langle\psi|A^\beta.$$

Now using the CCR

$$[A^\alpha, C_\beta] = \delta^\alpha_\beta$$

we calculate

$$\begin{aligned}\langle d\psi|d\psi\rangle &= d\bar{\phi}_\beta d\phi^\alpha\langle\psi|A^\beta C_\alpha|\psi\rangle \\ &= d\bar{\phi}_\beta d\phi^\alpha\langle\psi|[A^\beta, C_\alpha] + C_\alpha A^\beta|\psi\rangle \\ &= [d\phi^\alpha d\bar{\phi}_\alpha + (\phi^\beta d\bar{\phi}_\beta)(\bar{\phi}_\alpha d\phi^\alpha)]\langle\psi|\psi\rangle.\end{aligned}$$

Similarly we calculate

$$\begin{aligned}\langle\psi|d\psi\rangle &= \langle\psi|C_\alpha|\psi\rangle d\phi^\alpha = \langle\psi|\bar{\phi}_\alpha|\psi\rangle d\phi^\alpha \\ &= \langle\psi|\psi\rangle(\bar{\phi}_\alpha d\phi^\alpha).\end{aligned}$$

Therefore the line element induced on $\mathcal{C}$ reduces simply to

$$ds^2_{F.S.} = 4(d\phi^\alpha d\bar{\phi}_\alpha)^2$$

as required. ∎

Kählerian Proof. We recall the Kähler scalar function for CP^n where now we take n to be countable infinity, $\aleph_0$. For $\mathcal{A} \subset \mathcal{F}$ defined in section 3 we have

$$K = 4\log(1 + |\xi^{(1)}|^2 + \ldots + |\xi^{(j)}|^2 + \ldots \text{ ad inf.})$$

where $\xi^{(j)} = \xi^{\alpha_1 \ldots \alpha_j}$ as before. Now for $\mathcal{C}$ we have for the coherent state vector associated to $\psi^\alpha \in \mathcal{H}^1$

$$\psi^{(j)} = \frac{(\psi^\alpha)^{\otimes^j_S}}{\sqrt{j!}}$$

and therefore

$$|\psi^{(j)}|^2 = \frac{\Lambda^j}{j!}$$

where as before $\Lambda = \psi^\alpha \bar{\psi}_\alpha$. Summing over j all the way to infinity (to sum to infinity is in fact necessary for flatness) we obtain the remarkably simple relation

$$K|_{\mathcal{C}} = 4\Lambda$$

and then the induced metric on $\mathcal{C}$ is given by

$$h_{\alpha\beta'} = 4\partial_\alpha \bar{\partial}_{\beta'}(\psi^\gamma \bar{\psi}_\gamma) = 4\delta_{\alpha\beta'}$$

as required. ∎

We have remarked earlier that the coherent state submanifold $\mathcal{C}$ is non-linear, in the sense that the complex projective line joining two distinct coherent states lies entirely in the complement of $\mathcal{C}$ except at its two intersection points which are the coherent states themselves. This is an algebraic result whose proof relies upon the uniqueness of decomposition of any given state into coherent states. It suggests the following geometrical property of the coherent state submanifold.

Proposition 1. *Given any two distinct coherent states the complex projective line joining them intersects $\mathcal{C}$ transversally, that is to say, the line joining the two coherent states does not lie in the tangent space to $\mathcal{C}$ at either intersection point.*

Proof. We make essential use of the main theorem. Since $\mathcal{C}$ is homogeneous we may assume that one of the coherent states is the vacuum state, that is $P|0\rangle$ where $|0\rangle$ is the element of Fock space which is the exponential of the origin in the vector space $\mathcal{H}^1$. Then from the main theorem the *intrinsic* geodesic distance s from $P|0\rangle$ to $P|\xi_c\rangle$, $\xi \neq 0$ is given by

$$s = 2\Lambda^{1/2}.$$

Now recall (cf. Hughston 1996) that the geodesic distance θ between the two states in $P\mathcal{F}$ with respect to the ambient Fubini-Study metric on $P\mathcal{F}$ is determined by

$$\frac{1}{2}(1+\cos\theta) = \frac{\langle\xi_c|0\rangle\langle 0|\xi_c\rangle}{\langle 0|0\rangle\langle\xi_c|\xi_c\rangle}$$

where we take θ to be the principal value determined from the above equation,

$$0 \le \theta \le \pi$$

so that the cross ratio expression above fixes θ uniquely. Now clearly

$$\langle 0|0\rangle = \langle 0|\xi_c\rangle = \langle\xi_c|0\rangle = 1$$

and therefore

$$\theta = \cos^{-1}(2e^{-\Lambda} - 1), \quad 0 \le \theta \le \pi$$

from which it follows that

$$\frac{d\theta}{d\Lambda} = (e^\Lambda - 1)^{-1/2}.$$

Hence we obtain

$$\frac{d\theta}{ds} = (\frac{\Lambda}{e^\Lambda - 1})^{1/2}.$$

Thus clearly $d\theta/ds$ is a monotone decreasing function beginning at $d\theta/ds = 1$, where $\Lambda = 0$, and decaying to zero as Λ tends to infinity. Now tangency at $P|\xi_c\rangle$ would require $d\theta/ds = 1$ for some $\Lambda \neq 0$ and this is clearly not possible from the form of the function $d\theta/ds$. This completes our proof.

The method above also proves another result which one expects intuitively from the non-linear picture of $\mathcal{C}$.

Corollary. *The geodesic distance along the projective line in $P\mathcal{F}$ joining two distinct coherent states is always strictly greater than the intrinsic geodesic distance within $\mathcal{C}$.*

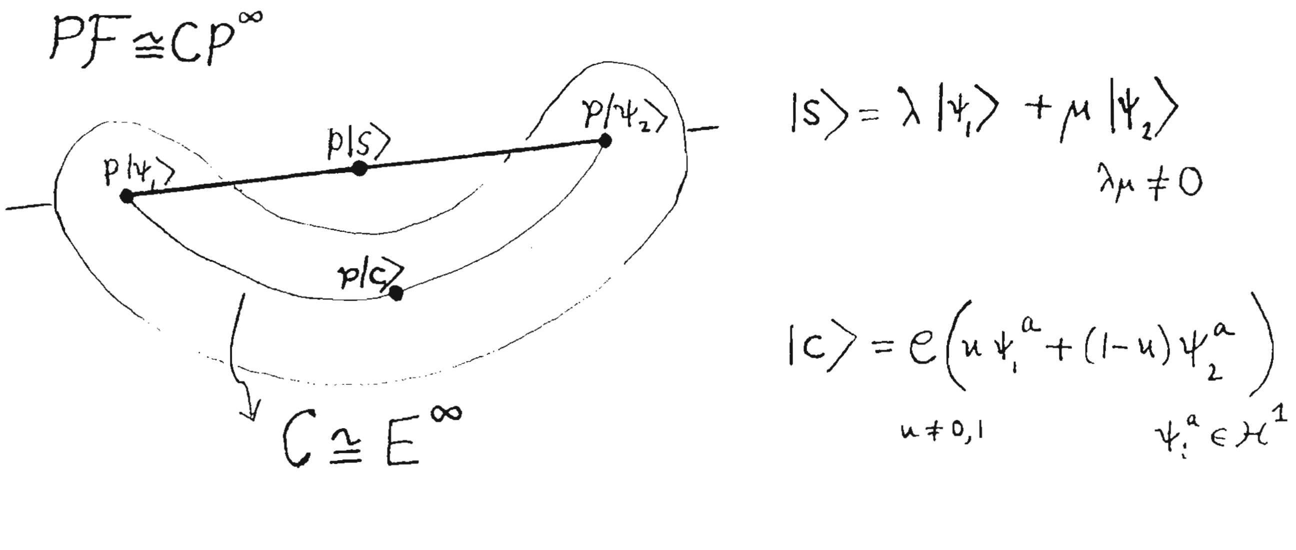
The theorem also proves the following simple geometrical properties of $\mathcal{C}$.

Lemma. *The intrinsic $\mathcal{C}$ geodesic distance between two coherent states is given by the Hilbert space norm of the difference field of the two corresponding vectors in $\mathcal{H}^1$. Thus the $\mathcal{C}$ distance of a coherent state from the vacuum state is equal to its Hilbert space norm.*

Lemma. *The overlap $\langle \xi_c | \psi_c \rangle$ of two normalized coherent state vectors is equal to the cosine of the angle these states subtend at the vacuum state.*

Discussion

These results illustrate the geometrical character of two emergent linear structures in quantum theory. One is able to simply add solutions of a classical linear field equation to obtain a new classical field, and as we have seen the geometry of the associated coherent states is Euclidean, with the vector solutions in fact serving as Euclidean coordinates. On the other hand one can take any two distinct coherent states and superpose them in the quantum mechanical sense of joining them with the unique complex projective line in the ambient Fubini-Study geometry of the underlying state space. Then we have seen that this superposition is necessarily non-coherent, or in familiar language non-classical. Such features as these are notably fully present even in a linear theory of gravity, and this is in accordance with common experience, since one does not observe classical superpositions of weak gravitational fields. Thus a preferred basis of states is given, namely the coherent states, together with a unique probability distribution for the associated resolution of unity, and this in turn gives rise to unique coherence transition probabilities. The geometry clearly illustrates how a quantum superposition of distinct classical geometries is outside the classical domain.



An interesting application of this geometry to a problem in stochastic state vector reduction is currently being pursued in collaboration with L. P. Hughston, to whom I express thanks for valuable discussions in respect of the work presented here.

References

Geroch 1971. *Ann. Phys.* **62**, 582.
Hughston 1996. Geometry of stochastic state vector reduction. *Proc. Roy. Soc. Lond.* **A452**, 953-979.
Klauder and Skagerstam 1985. Coherent States. *World Scientific* Ch. 1.

Quantum Electrodynamic Birdtracks

by Louis H. Kauffman <Kauffman@uic.edu>

I. Introduction

This paper provides a diagrammatic version of the Feynman – Dyson derivation of non-commutative electromagnetism from quantum mechanical formalism [1], [2]. Our spin-network formalism lays bare the structure of this result.

Here is a statement of the main result in conventional notation. $X = (X_1, X_2, X_3)$ denotes a vector of non-commutative coordinates, each a differentiable function of time t. Let $\dot{X} = (\dot{X}_1, \dot{X}_2, \dot{X}_3)$ denote the vector of time derivatives of these functions. Let κ be a non-zero scalar. Assume the axioms below:

$$\left\{ \begin{array}{ll} (1) & [X_i, X_j] = 0 \text{ for all } i, j. \\ (2) & [X_i, \dot{X}_j] = \kappa \delta_{ij} \end{array} \right\}$$

Here $[X, Y] = XY - YX$ and

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Then there exist fields E and H such that

(a) $\ddot{X} = E + \dot{X} \times H$

(b) $\nabla \cdot \mathbf{H} = 0$

(c) $\dfrac{\partial H}{\partial t} + \nabla \times E = 0.$

In fact, we can take $H = \frac{1}{\kappa} \dot{X} \times \dot{X}$ where this denotes the non-commutative vector cross product.

In this paper we give a new proof of this result. These methods apply equally well to the discrete framework employed in [2]. Section II reviews notation and defines the non-commutative vector calculus via abstract tensor diagrams. Section III contains the promised derivation. Section IV discusses problems and questions arising from this work.

II. Vectors, Abstract Tensors and The Epsilon

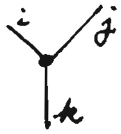
A vector $A = (A_1, A_2, A_3)$ will be indicated by  or  where the arc denotes the index. Multi-indexed objects have multiple arcs. Thus ${}_i\!\frown_j = \delta_{ij}$ and ${}^i_j\!\diagup = \delta^i_j$. Compare [3].

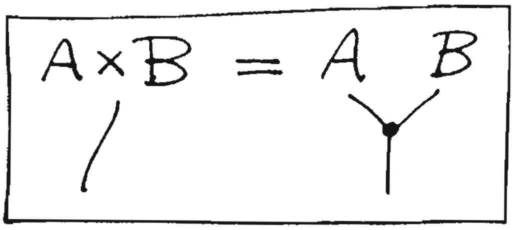
We sum over repeated indices.

Such indices correspond to arcs without free ends.

For example, $\bigcirc = \sum_{i=1}^{3} {}_i\!\frown_i = \sum_{i=1}^{3} \delta_{ii} = 3.$

$$A \cdot B = \sum_{i=1}^{3} A_i B_i = $$ .

Let  $= \varepsilon_{ijk} = \begin{cases} \mathrm{sgn}(ijk) & \text{if } ijk \text{ distinct} \\ 0 & \text{otherwise} \end{cases}$.

Then  since

$$(A \times B)_k = \sum_{i,j} \varepsilon_{ijk} A_i B_j .$$

Note that in a non-commutative context

$$(A \times A)_k = \sum_{ij} \varepsilon_{ijk} A_i A_j$$

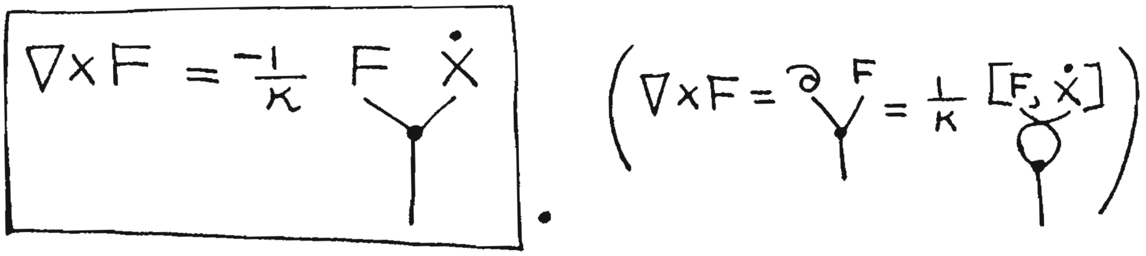
$$(A \times A)_1 = [A_2, A_3]$$
$$(A \times A)_2 = [A_3, A_1]$$
$$(A \times A)_3 = [A_1, A_2].$$

Thus, we can no longer assert $A \times A = 0$, *unless* the coordinates of A commute with one another.

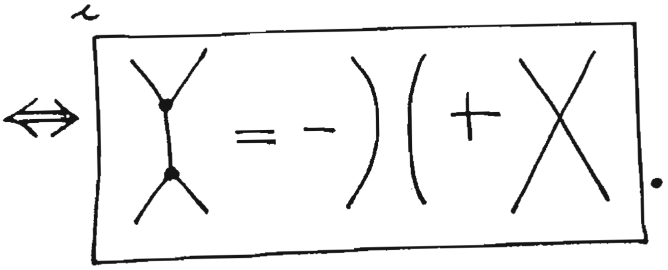
We shall use the axioms stated in the introduction. Thus $X \times X = 0$ is <u>equivalent</u> to the first axiom. The second axiom states that $[X_i, \dot{X}_j] = \kappa \delta_{ij}$. Thus $\frac{1}{\kappa}[X_i, \dot{X}_j] = \partial X_i / \partial X_j$. This means that if F is any function of these non-commuting variables, we can define

$$\boxed{\frac{\partial F}{\partial X_i} = \frac{1}{\kappa}[F, \dot{X}_i]}.$$

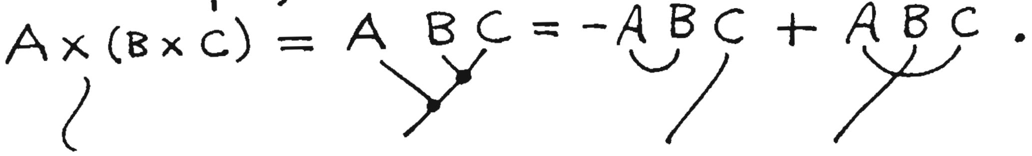
Thus $\nabla \cdot A = \sum_i \frac{\partial A_i}{\partial X_i} = \frac{1}{\kappa}[A, \dot{X}]$ and



With these definitions, we can proceed to work out the details on non-commutative vector calculus. The key to calculations is the basic <u>epsilon identity</u>: $\sum_i \varepsilon_{abi}\varepsilon_{icd} = -\delta^a_d \delta^b_c + \delta^a_c \delta^b_d$

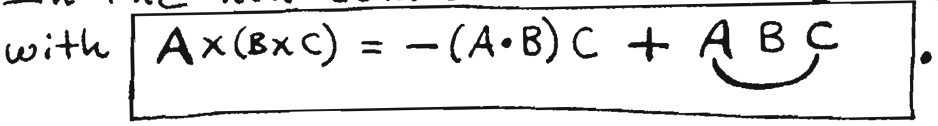


For example,



In the commutative case, this reduces directly to: $A \times (B \times C) = -(A \cdot B)C + (A \cdot C)B$.

In the non-commutative case we are left with



The network formalism articulates the non-commutative vector analysis.

III. Non-Commutative Electromagnetism

As explained in sections I and II, we assume coordinates X such that

$$(1)\quad [X_i, X_j] = 0$$

$$(2)\quad [X_i, \dot{X}_j] = \kappa\,\delta_{ij}.$$

In section II we showed that $(1) \Leftrightarrow X \times X = 0$, and defined differentiation so that (2) implied $\frac{\partial}{\partial X_i} = [\ \ ,\dot{X}_i]/\kappa$. We now define the fields H and E by the formulas

$$\left\{ \begin{array}{l} H = \frac{1}{\kappa}\,\epsilon_{ijk}\dot{X}_i \dot{X}_j = \frac{1}{\kappa}\,\dot{X}\times\dot{X}, \\ \\ E = \ddot{X} - \dot{X}\times H \end{array} \right\}.$$

Our first task is to show that neither E nor H have any dependence on $\dot{X}$. Since (2) (above) holds, this is equivalent to showing that $[E_i, X_j] = 0 = [H_i, X_j]$.

<u>Lemma 1</u>. $\kappa\, H_k\epsilon_{kij} = [X_i, \ddot{X}_j]$

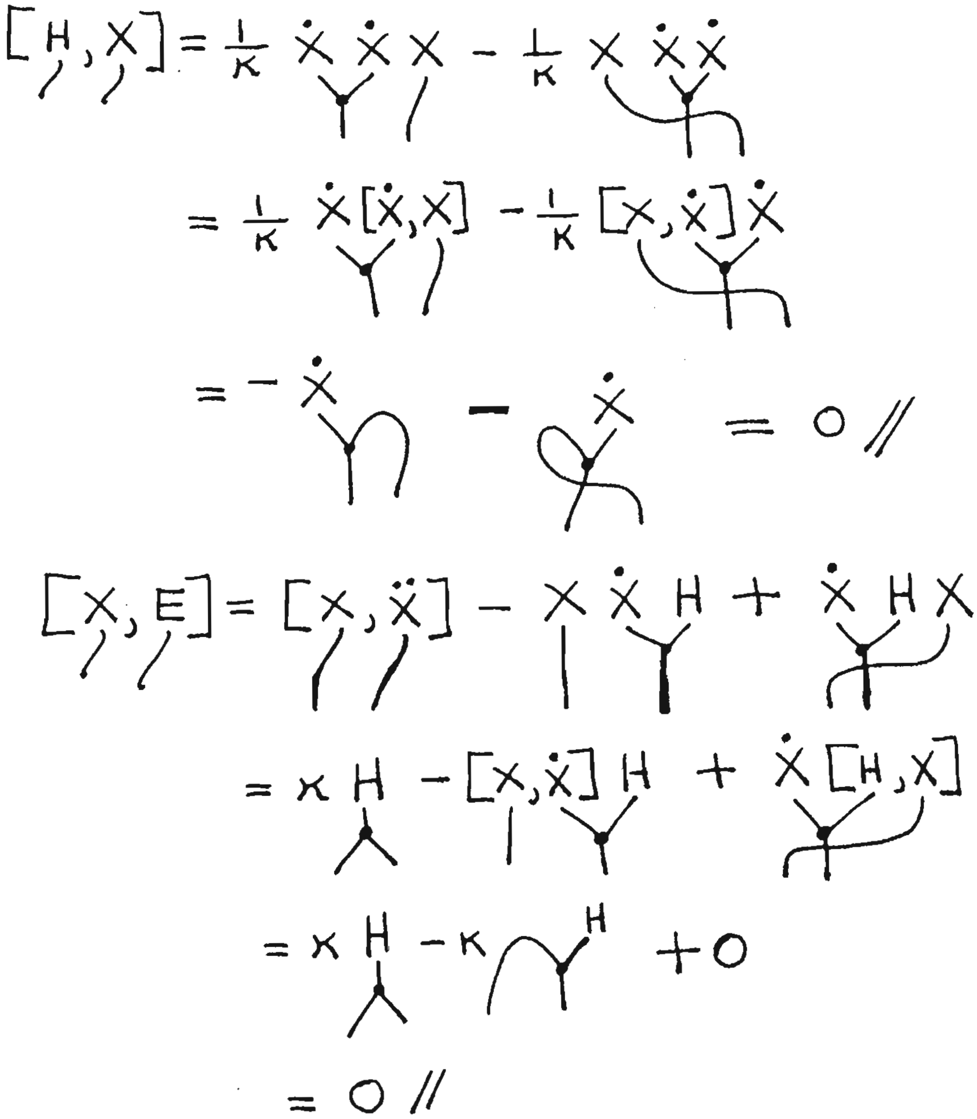
<u>Proof</u>. $H_k\epsilon_{kij} = \frac{1}{\kappa}\,\epsilon_{kab}\dot{X}_a\dot{X}_b\,\epsilon_{kij} = -\frac{1}{\kappa}\,\dot{X}_i\dot{X}_j + \frac{1}{\kappa}\,\dot{X}_j\dot{X}_i$

$$= -\frac{1}{\kappa}[\dot{X}_i, \dot{X}_j]$$

$$= \frac{1}{\kappa}[X_i, \ddot{X}_j] \qquad \left([X_i, \dot{X}_j]^{\bullet} = 0\right) \quad //$$

Lemma 2. $[E, X] = [H, X] = 0.$

Proof.



Remark. Lemma 2 implies that E and H are functions only of X_1, X_2, X_3. Hence $E \times E = H \times H = 0$, since $X \times X = 0$. If F is a function only of X_1, X_2, X_3, then

$$\dot{F} = \frac{\partial F}{\partial t} + \sum_i \dot{X}_i \frac{\partial F}{\partial X_i} = \frac{\partial F}{\partial t} + \frac{1}{\kappa} \dot{X}[F, \dot{X}].$$

Thus
$$\boxed{\partial_t H = \dot{H} + \frac{1}{\kappa} \dot{X}[\dot{X}, H]}.$$

Lemma 3. $\nabla \bullet H = 0$.

Proof. $\nabla \bullet H = \frac{1}{\kappa} [H, \dot{X}]$

$$= \frac{1}{\kappa} H \dot{X} - \frac{1}{\kappa} \dot{X} H$$

$$= \frac{1}{\kappa^2} \dot{X} \dot{X} \dot{X} - \frac{1}{\kappa^2} \dot{X} \dot{X} \dot{X}$$

$$= 0 \quad //$$

Lemma 4. $\partial_t H + \nabla \times E = 0$.

Proof. $\partial_t H = \dot{H} + \frac{1}{\kappa} \dot{X} [\dot{X}, H]$

$$\dot{H} = \frac{1}{2\kappa} [\dot{X}, \dot{X}]^{\bullet} = \frac{1}{\kappa} [\ddot{X}, \dot{X}] = \frac{1}{\kappa} [E, \dot{X}] + \frac{1}{\kappa} [\dot{X} \times H, \dot{X}]$$

$$= -\nabla \times E + \frac{1}{\kappa} [\dot{X} H, \dot{X}]$$

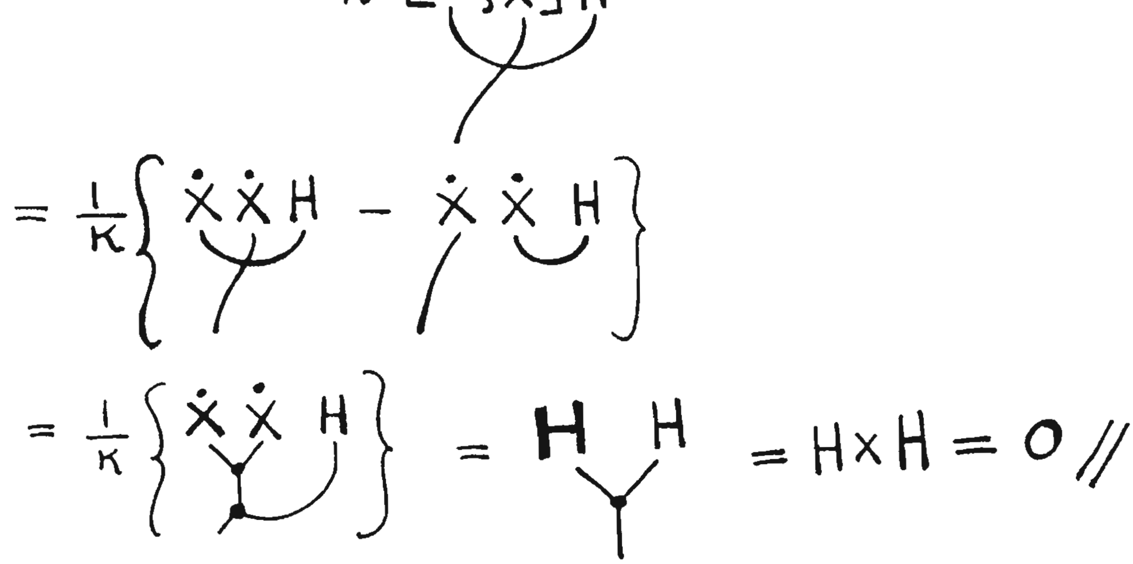
$$= -\nabla \times E - \frac{1}{\kappa} [\dot{X} H, \dot{X}] + \frac{1}{\kappa} [\dot{X} H, \dot{X}]$$

$$= -\nabla \times E - \frac{1}{\kappa} \dot{X} H \dot{X} + \frac{1}{\kappa} \dot{X} \dot{X} H + \frac{1}{\kappa} \dot{X} H \dot{X} - \frac{1}{\kappa} \dot{X} \dot{X} H$$

$$\dot{H} = -\nabla \times E + \frac{1}{\kappa} \dot{X} [H, \dot{X}] - \frac{1}{\kappa} \dot{X} [H, \dot{X}] + \frac{1}{\kappa} [\dot{X}, \dot{X}] H$$

The third term vanishes because $\nabla \bullet H = 0$.

Hence $\partial_t H + \nabla \times E = \frac{1}{K}[\dot{X},\dot{X}]H$



IV. Discussion

In this entire derivation the only appearance of non-algebraic calculus is in the time derivative $\dot{X}$. This can also be made algebraic by choosing a formal discrete time-evolution operator J so that $X' = J^{-1}XJ$ denotes the next (discrete) time value of X. Then we can take $\dot{X} = J(X'-X) = [X,J]$ and proceed as before. This is a capsule summary of the approach taken by the author and Pierre Noyes in [2].

References

1. F. J. Dyson, *Feynman's Proof of the Maxwell Equations*, Amer. J. Phys. (1990) Vol. 58, p. 209–215.
2. L. H. Kauffman and H. P. Noyes, Discrete physics and the derivation of Electromagnetism from the formalism of Quantum Mechanics, Proc. R. Soc. London A (1995) 452, p. 81–95.
3. R. Penrose, Applications of negative dimensional tensors. "Combinatorial Mathematics and Its Applications" edited by D. J. A. Welsh, Academic Press (1971), p. 221–244.

Where are the Twistors in the Null-Surface Formulation of GR?

Simonetta Frittelli[a*], Carlos N. Kozameh[b], Ezra T. Newman[a]
[a] *Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA 15260, USA.*
[b] *FaMAF, Universidad Nacional de Córdoba, 5000 Córdoba, Argentina.*
(July 30, 1996)

We give a brief review of the null-surface approach to general relativity and then speculate on a possible connection between this approach and twistor theory.

We first briefly review a formulation of GR (the Null-Surface Formulation of GR), whose basic variables are families of surfaces on a four-manifold, M, and a scalar function. The metric appears as a derived concept obtained from the assumption that the surfaces are characteristic surfaces of some conformal metric. The scalar function plays the role of a conformal factor converting the conformal metric into a vacuum metric. No proofs are given, as they have already appeared in the literature. We then speculate on the relationship of this reformulation of GR with twistor theory.

The null-surface approach to general relativity rests essentially on the introduction of two real functions, Z and Ω, living on the bundle of null directions over M; i.e., on $M \times S^2$ with x^a in M, and ζ on S^2. These two functions capture the information contained in an Einstein metric: $Z(x^a,\zeta)$ encodes the conformal structure of the Einstein space-time and singles out a preferred member of the conformal class of metrics, while $\Omega(x^a,\zeta)$ represents the appropriate conformal factor that turns the preferred member into an Einstein metric. In the following, we show how these two functions are introduced.

On a manifold M consider an S^2-family of functions $u = Z(x^a,\zeta) = constant$ such that the equation

$$g^{ab}(x^a) Z_{,a}(x^c,\zeta) Z_{,b}(x^d,\zeta) = 0 \tag{1}$$

can be solved for a metric $g^{ab}(x^a)$ for all values of ζ. The surfaces $Z(x^a,\zeta) = const.$ are then null surfaces of the metric g^{ab}. For every fixed value of ζ, the equation $Z(x^a,\zeta) = u$ represents a null foliation of the spacetime (M, g^{ab}). It is an interesting kinematical problem (also studied by L. Mason [1]) to derive the conditions on the function Z that imply the existence of a g^{ab} that satisfies Eq. (1). We approach this problem by resorting to a special set (families) of null coordinate systems $(u, R, \omega, \bar\omega) \equiv \theta^i(x^a)$, $i = 0, 1, +, -$, on M defined by the transformation

$$(u, R, \omega, \bar\omega) = (Z, \bar\eth\eth Z, \eth Z, \bar\eth Z) \tag{2}$$

Eq. (2) should be interpreted as a coordinate transformation $x^a \to \theta^i$ for every fixed value of ζ; i.e., a family of coordinate transformations dependent on ζ. Using the gradient basis $\theta^i{}_{,a}$, the metric g^{ab} can be expressed in the new coordinates as $g^{ij}(\theta^i) = \theta^i{}_{,a}\,\theta^j{}_{,b}\,g^{ab}(x)$.

By repeated differentiation of Eq. (1) with respect to ζ and $\bar\zeta$, and using the independence of the metric on the variable ζ, we obtain our two main results [2]. First, the metric components g^{ij} are all expressible in terms of the two quantities, Z and $\Omega \equiv \sqrt{g^{01}}$ and their derivatives; explicitly we have $g^{ij} = \Omega^2 \tilde g^{ij}(Z)$. Second, we find that Ω and Z are not arbitrary nor independent of each other. They must satisfy two coupled complex differential equations, referred to as the metricity conditions.

The metricity conditions constitute the requirement on Z in order for a metric g^{ab} to exist and not depend on ζ and such that $Z = const.$ are characteristic surfaces of the metric, for every value of ζ. They leave Ω undetermined up to a factor dependent only on x^a.

On this kinematical scheme we impose the (trace-free) vacuum Einstein equations by $\theta^a_1\theta^b_1(R_{ab} - \frac{1}{2}g_{ab}R) = 0$. In terms of the variables Z and Ω this equation takes the following form:

$$D^2\Omega - Q\Omega = 0 \tag{3}$$

where $D \equiv \frac{\partial}{\partial R}$, $\Lambda \equiv \eth^2 Z$ and

---

*e-mail: simo@artemis.phyast.pitt.edu

$$Q \equiv -\frac{1}{4q}D^2\Lambda D^2\bar{\Lambda} - \frac{3}{8q^2}(Dq)^2 + \frac{1}{4q}D^2q \;, \qquad q \equiv 1 - D\Lambda D\bar{\Lambda} \tag{4}$$

The two complex metricity conditions and equation (3) constitute a set of coupled differential equations for the variables Z and Ω that are completely equivalent to the vacuum Einstein equations.

Though any vacuum Einstein space-time can be investigated in this manner we make the specialization, here, to asymptotically flat vacuum space-times. In this case the geometrical meanings to the various quantities become more focused and clearer and the differential equations become easier to handle. We begin with the fact that now null infinity, $\mathcal{I}^+$, exists. It can be coordinatized with a Bondi coordinate system,

$$(u, \zeta, \bar{\zeta}) \tag{5}$$

with $u \in R$, the Bondi retarded time, and $(\zeta, \bar{\zeta}) \in S^2$ labeling the null generators of $\mathcal{I}^+$. With this notation we can give a precise meaning to the null surfaces described by $u = Z(x^a, \zeta)$; they are taken to be the past null cones of the points $(u, \zeta, \bar{\zeta})$ of $\mathcal{I}^+$. In addition to this meaning of Z, there is a dual meaning, namely, if the space-time point x^a is held constant but the $(\zeta, \bar{\zeta})$ is varied over the S^2, we obtain a two-surface on $\mathcal{I}^+$, the so-called light-cone cut of $\mathcal{I}^+$. It consists of all points of $\mathcal{I}^+$ reached by null-geodesics from x^a. Z is then referred to as the light-cone cut function. We now have a geometric interpretation, not only of $Z(x^a, \zeta)$, but also of both, $\omega = \eth Z(x^a, \zeta)$ and $R = \eth\bar{\eth} Z(x^a, \zeta)$. ω is the "stereographic angles" that the light-cone cuts make with the Bondi $u = const.$ cuts (i.e., it labels the backward direction of the null geodesics from $\mathcal{I}^+$ to x^a, and R is a measure of the curvature of the cut and, thus, a measure of the "distance" from $\mathcal{I}^+$ to x^a along the null geodesic. Any null geodesic can be labeled by the five parameters, its intersection point with $\mathcal{I}^+$, i.e., $(u, \zeta, \bar{\zeta})$, and its "null angle", $(\omega, \bar{\omega})$.

In the special case of asymptotic flatness, we obtain a considerable simplification in the two metricity conditions; by differential and algebraic manipulation, they can be expressed as a single equation of the form

$$\eth^2\bar{\eth}^2 Z = \bar{\eth}^2\sigma(Z,\zeta) + \eth^2\bar{\sigma}(Z,\zeta) + \mathcal{D}(\Omega,\Lambda) \tag{6}$$

where σ is a free function of (u, ζ) (the Bondi shear, as characteristic data) on $R \times S^2$. $\mathcal{D}$ is an (explicitly known) non-linear polynomial in Λ and its derivatives and linear in derivatives of $\ln \Omega$. A paper on this is in preparation. Eqs. (3) and (6) are the Einstein equations (including the free data) for asymptotically flat-space-times.

We point out several items of potential interest.

1. There is a very straightforward perturbation scheme (off flat space) for these equations [3].

2. There is a canonical choice of interior space-time coordinates x^a, (essentially the coefficients of the first four spherical harmonics in the expansion of Z) so that there is no gauge freedom [4].

3. All conformal information of the space-time is contained in knowledge of the function $Z(x^a, \zeta, [data])$. The four functions $\theta^i(x^a, \zeta, [data])$, which are defined geometrically on $\mathcal{I}^+$, and describe the interior of the space-time are obtained from derivatives of Z. In turn, they can, in principle, be inverted leading to

$$x^a = x^a(\theta^i, \zeta, [data]), \tag{7}$$

   the location of space-time points in terms of information on $\mathcal{I}^+$. If Eq. (7) is written as

$$x^a = x^a(R, u, \omega, \zeta, [data]), \tag{8}$$

   we have the explicit form of all null geodesics; fixed values of (u, ω, ζ) in the parameter space of null geodesics picks out the geodesic while R is the geodesic parameter, which is affine length for our special metric $\tilde{g}^{ij}$ in the conformal class.

4. The work described here is the (in no way obvious) generalization of the study of self-dual space-times via the good cut equation, $\eth^2 Z = \sigma(Z, \zeta)$ which in turn led to the first insights into asymptotic twistor theory [5].

It is this last remark that raises the issue of what relationship does the present work have with twistor theory? The answer is that we simply do not know, though there are a few suggestions, which we now discuss, of a possible connection.

a. Flat and asymptotically flat twistor theory is clearly intimately concerned with null geodesics and the associated self- (or anti-self-) dual blades; we have, in our generalization, via the function $Z(x^a, \zeta, \bar{\zeta})$, a complete description of all null geodesics with the facility to manipulate them and to add and study any further conformally invariant structures associated with them, e.g., geodesic deviation vectors between different members. In particular, we could have and study deviation vectors at null infinity that make self- (or anti-self-) dual blades with the tangent vectors at null infinity.

b. The light-cone cuts of $\mathcal{I}^+$ for the flat and asymptotically flat self-dual space-times led directly to a definition of asymptotically flat twistors, namely the curves on complexified $\mathcal{I}^+$ defined by $u = Z(x^a, \zeta, \bar{\zeta})$ holding x^a and ζ fixed were the twistor curves. We do not know how this can be related or extended to the general asymptotically flat space-times.

c. In the case of asymptotically flat self-dual space-times, though there was a complete description of the light-cone cuts of interior space-time points (and thus the description of the associated twistor lines) on $\mathcal{I}^+$, for a variety of reasons (e.g., the googly problem) it was of great interest to study the behaviour of the light-cone cuts (and twistor lines) in the limit as the interior points approached $\mathcal{I}^+$. This limit is singular and its study was very difficult and inconclusive. Part of the difficulty was that we did not know how to introduce (interior) Bondi coordinates in place of the x^a so that the limit could be easily achieved. We have, in both the self-dual and the full vacuum case, recently seen how the Bondi coordinates could be introduced and, thus, we have the real hope of being able to study the structure of the light-cone cuts in any of the cases as the interior points x^a approach $\mathcal{I}^+$. This transformation to Bondi coordinates is given in the following fashion. If the $Z(x^a, \zeta, \bar{\zeta})$ is a known function of the local coordinates x^a, then the Bondi coordinates, $y^a = (u_B, r_B, \zeta_B)$ are given implicitly in terms of the x^a by the equations

$$\begin{aligned} u_B &= Z(x^a, \zeta_B, \bar{\zeta}_B) \\ 0 &= \eth Z(x^a, \zeta_B, \bar{\zeta}_B), \qquad \text{and c.c.} \\ r_B &= \eth\bar{\eth} Z(x^a, \zeta_B, \bar{\zeta}_B). \end{aligned} \tag{9}$$

Using these relationships and the known asymptotic form [6,7] of the metric we expect that we will be able to write the Z asymptotically (but explicitly) in terms of inverse powers of the r_B for space-time points near $\mathcal{I}^+$. This, then, would hopefully allow us to see how the limit behaves. Again, though we do not know what relationship this might have with twistor theory, we feel that there is considerable hope from this approach.

d. There is a version of our Eq. (6) that has a chiral appearance, so that it formally resembles the self-dual case. It should be looked at carefully, although we suspect that its resemblance to the self-dual case is purely notational and that it may have no deeper significance.

---

[1] L. J. Mason, J. Math. Phys. **36**, 3704 (1995).
[2] S. Frittelli, C. N. Kozameh, and E. T. Newman, J. Math. Phys. **36**, 4975 (1995).
[3] S. Frittelli, C. N. Kozameh, and E. T. Newman, J. Math. Phys. **36**, 4984 (1995).
[4] S. Frittelli and E. T. Newman, *Pseudo-Minkowskian coordinates from the NSF*, in preparation.
[5] R. O. Hansen, E. T. Newman, R. Penrose, and K. P. Tod, Proc. R. Soc. **A363**, 445 (1978).
[6] E. T. Newman and R. Penrose, J. Math. Phys. **5**, 863 (1966).
[7] E. T. Newman and T. W. J. Unti, J. Math. Phys. **3**, 891 (1962).

26

Heavenly hierarchies and curved twistor spaces

Maciej Dunajski   Lionel J. Mason

The anti-self-dual vacuum equations (ASDVE) share many properties with lower-dimensional integrable systems. It is therefore reasonable to hope that some constructions well known from the theory of such integrable models are also present in the ASDVE. Boyer and Plebański [1] obtained an infinite number of conservation laws for the ASDVE equations and established some connections with the nonlinear graviton construction. Some of their results were later rediscovered and extended in various papers of Strachan and Takasaki. With the appropriate symplectic structure, the sequence of conserved quantities should lead to a hierarchy of evolution equations that are 'hidden symmetries' for the ASDVE equations.

In what follows, the hierarchical structure of the ASDVE in the heavenly form due to Plebański is constructed by looking at ways of generating sequences of solutions to the linearized heavenly equations. This approach is motivated by that in [5] for the treatment of ASDYM hierarchies.

We use the formulation of the ASDVE condition in [4]. All the spinor indices are assumed to be concrete and indices are raised and lowered with $\varepsilon_{AB} = \varepsilon_{[AB]}, \varepsilon_{01} = 1$ according to the usual Penrose and Rindler conventions. We work in the holomorphic category. Let $\mathcal{M}$ be a complex four-manifold equipped with a holomorphic volume form ν. Choose a normalised null tetrad $\nabla_{AA'}$ which consists of four independent and volume preserving vector fields. We will also require that the null tetrad contracted with the volume form yields one. The ASDVE on the metric for which $\nabla_{AA'}$ are a null tetrad arise as a consequence of the integrability of the distribution spanned by the Lax operators $L_A = \pi^{A'}\nabla_{AA'}$

$$[L_A, L_B] = 0 \tag{1}$$

where $\pi^{A'} = (-1, \lambda)$ is a constant spinor; all local solutions of the ASDVE arise in this way. Part of the residual gauge freedom in (1) is fixed by selecting one of Plebański's coordinate systems, $(w, z, \tilde{w}, \tilde{z}) =: (w^A, \tilde{w}^A)$ for the first equation and setting

$$\nabla_{AA'} = \begin{pmatrix} \Omega_{w\tilde{w}}\partial_{\tilde{z}} - \Omega_{w\tilde{z}}\partial_{\tilde{w}} & \partial_w \\ \Omega_{z\tilde{w}}\partial_{\tilde{z}} - \Omega_{z\tilde{z}}\partial_{\tilde{w}} & \partial_z \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\partial^2\Omega}{\partial w^A \partial \tilde{w}^B}\varepsilon^{BC}\frac{\partial}{\partial \tilde{w}^C} & \frac{\partial}{\partial w^A} \end{pmatrix}$$

and coordinate system $(w, z, x, y) =: (w^A, x_A)$ for the second where

$$\nabla_{AA'} = \begin{pmatrix} -\partial_y & \partial_w + \Theta_{yy}\partial_x - \Theta_{xy}\partial_y \\ \partial_x & \partial_z - \Theta_{xy}\partial_x + \Theta_{xx}\partial_y \end{pmatrix}$$
$$= \begin{pmatrix} \varepsilon_{BA}\frac{\partial}{\partial x_B} & \frac{\partial}{\partial w^A} + \varepsilon_{BC}\frac{\partial^2\Theta}{\partial x^A \partial x_B}\frac{\partial}{\partial x_C} \end{pmatrix}.$$

These choices lead to heavenly equations.

$$\Omega_{w\tilde{z}}\Omega_{z\tilde{w}} - \Omega_{w\tilde{w}}\Omega_{z\tilde{z}} = 1 \qquad \text{First} \qquad (2)$$

$$\Theta_{xw} + \Theta_{yz} + \Theta_{xx}\Theta_{yy} - {\Theta_{xy}}^2 = 0 \qquad \text{Second} \qquad (3)$$

Both (2) and (3) were originaly derived from the formulation dual to (1). The one forms $e^{AA'}$ dual to the tetrad $\nabla_{AA'}$ are used to construct the two form $\Sigma(\lambda) = \pi_{A'}\pi_{B'}\varepsilon_{AB}e^{AA'} \wedge e^{BB'} = \varepsilon^{AB}\nu(\cdot,\cdot,L_A,L_B) = \pi_{A'}\pi_{B'}\Sigma^{A'B'}$, which can be used as a set of basic variables in GR. Equation (1) becomes

$$d\Sigma(\lambda) = 0, \qquad \Sigma(\lambda)\wedge\Sigma(\lambda) = 0, \qquad (4)$$

where in the first equation λ is regarded as a parameter and not differentiated. The closure condition is used to introduce ω^A, canonical coordinates on the the spin bundle, holomorphic around $\lambda = 0$ such that $d\Sigma(\lambda) = d\omega^A \wedge d\omega_A$. The various forms of the heavenly equations can be obtained by adapting coordinates etc. to these forms.

Equations (2), (3) admit Lagrangian formulations

$$\mathcal{L}_\Omega = \Omega(\nu - \frac{1}{3}(\partial\tilde{\partial}\Omega)^2),$$

where $\partial = e^{A1'}\otimes\nabla_{A1'} = dz\otimes\partial_z + dw\otimes\partial_w$, $\tilde{\partial} = e^{B0'}\otimes\nabla_{B0'} = d\tilde{z}\otimes\partial_{\tilde{z}} + d\tilde{w}\otimes\partial_{\tilde{w}}$ and forms are multiplied by exterior multiplication and

$$\mathcal{L}_\Theta = \frac{2}{3}\Theta(\partial_2\Theta)^2 - \frac{1}{2}(\partial\Theta)\wedge(\partial_2\Theta)\wedge e^{A0'}\wedge e_A^{0'}$$

where $\partial_2 = e^{B1'}\nabla_{B0'} = dz\otimes\partial_x - dw\otimes\partial_y$. Note that $e^{A0'}\wedge e_A^{0'}$ can be replaced by $dx\wedge dy$ in the second Lagrangian as it is multiplied by $dw\wedge dz$.

A symplectic form on the space of solutions can be derived from the boundary term in the variational principle and is given by

$$\Omega(\delta_1\Omega, \delta_2\Omega) = \frac{2}{3}\int_{\delta\mathcal{M}} e^{1'}{}_B \wedge e^{B0'} \wedge (e^{A0'}\delta_2\Omega\nabla_{A0'}\delta_1\Omega - e^{A1'}\delta_1\Omega\nabla_{A1'}\delta_2\Omega),$$

and similarly for the second equation. They coincide with the symplectic form on the solution space to the wave equation on the ASD background.

Recursion relations

First we observe that the linearized solutions to (2) and (3) satisfy the wave equation on the ASD background given by Ω and Θ respectively.

$$\nabla_{A1'}\nabla^A{}_{0'}\delta\Omega = \nabla_{A1'}\nabla^A{}_{0'}\delta\Theta = 0. \tag{5}$$

From now on we identify tangent spaces to the moduli spaces of solutions to (2, 3) with the space of solutions to the curved background wave equation, $\mathcal{W}_g$. The linearised vacuum metrics corresponding to $\delta\Omega$ and $\delta\Theta$ are

$$h^I{}_{AA'BB'} = \iota_{(A'}o_{B')}\nabla_{(A1'}\nabla_{B)0'}\delta\Omega, \qquad h^{II}{}_{AA'BB'} = o_{A'}o_{B'}\nabla_{A0'}\nabla_{B0'}\delta\Theta.$$

We are now able to generate new linearized solutions from old ones. Given $\phi \in \mathcal{W}_g$ we use the first of these equations to find h^I. If we put the perturbation obtained in this way on the LHS of the second equation and add an appropriate gauge term then we get ϕ' - the new element of $\mathcal{W}_g$ that provides the $\delta\Theta$ which gives rise to $h^{II}_{ab} = h^I_{ab} + \nabla_{(a}V_{b)}$. This reduces to

$$\nabla_{A1'}\nabla_{B1'}\phi = \nabla_{A0'}\nabla_{B0'}\phi'. \tag{6}$$

Define a recursion operator $R : \mathcal{W}_g \longrightarrow \mathcal{W}_g$ by

$$\nabla_{A1'}\phi = \nabla_{A0'}R\phi \tag{7}$$

so formally $R = (\nabla_{A0'})^{-1}\circ\nabla_{A1'}$. From this definition and from (1) it follows that if ϕ belongs to $\mathcal{W}_g$ then so does $R\phi$. Because $[R,\ \nabla_{B1'}] = 0$ on $\mathcal{W}_g$, we have $R^2\delta\Omega = \delta\Theta$.

Define, for $i \in \mathbf{Z}$, a hierarchy of linear fields, $\phi_i \equiv R^i\phi_0$. Put $\Phi = \sum_{-\infty}^{\infty}\phi_i\lambda^i$ and observe that the recursion equations are equivalent to $L_A\Phi = 0$. Thus Φ is a function on $\mathcal{PT}$. Conversely every solution of $L_A\Phi = 0$ defined on a neighbourhood of $|\lambda| = 1$ can be expanded in a Laurent series in λ with coefficients being a series of elements of $\mathcal{W}_g$ related by the recursion operator. The function Φ can be thought of as a Cech representative of the element of $H^1(\mathcal{PT}, \mathcal{O}(-2))$ that corresponds to the solution of the wave equation ϕ (here $\mathcal{PT}$ is the twistor space of $\mathcal{M}$).

It is clear that a series corresponding to $R\phi$ is the function $\lambda^{-1}\Phi$. Note that R is not completely well defined when acting on $\mathcal{W}_g$ because of the ambiguity in the inversion of $\nabla_{A0'}$. This means that if one treats $\Phi(\lambda)$ as a twistor function on $\mathcal{PT}$, pure gauge elements of the first sheaf cohomology group $H^1(\mathcal{PT}, \mathcal{O}(-2))$ of the twistor space corresponding to $\mathcal{M}$ are mapped to nontrivial terms. Note, however, that the action of R is well defined on twistor functions. By iterating R we generate an infinite sequence of elements of $H^1(\mathcal{PT}, \mathcal{O}(-2))$ belonging to different classes.

By a formal application of Stokes' theorem

$$\Omega(R\phi,\ \phi') = \Omega(\phi,\ R\phi'). \tag{8}$$

We can construct an infinite sequence of symplectic forms $\Omega^k(\phi,\ \phi') \equiv \Omega(R^k\phi,\ \phi')$ which play a role in the bihamiltonian formulation.

Twistor surfaces

We can use R to build a family of foliations by twistor surfaces starting from a given one. Put $\omega_0^A = w^A = (w, z)$; the surfaces of constant ω_0^A are twistor surfaces. We have that $\nabla^A{}_{0'}\omega_0^B = 0$ so that in particular $\nabla_{A1'}\nabla^A{}_{0'}\omega_0^B = 0$ and if we define $\omega_i^A = R^i\omega_0^A$ then we can choose $\omega_i^A = 0$ for negative i. We define

$$\omega^A = \omega_0^A + \sum_{i=1}^{\infty}\omega_i^A\lambda^i. \tag{9}$$

We can similarly define $\tilde{\omega}^A$ by $\tilde{\omega}_0^A = \tilde{w}^A$ and choose $\tilde{\omega}_i^A = 0$ for $i > 0$. Note that ω^A and $\tilde{\omega}^A$ are solutions of L_A holomorphic around $\lambda = 0$ and $\lambda = \infty$ respectively and they can be chosen so that they extend to a neighbourhood of the unit disc and a neighbourhood of the complement of the unit disc. We have that $\mathcal{PT}$ can be covered by two sets, U and $\tilde{U}$ with $|\lambda| < 1 + \epsilon$ on U and $|\lambda| > 1 - \epsilon$ on $\tilde{U}$ with (ω^A, λ) coordinates on U and $(\tilde{\omega}^A, \lambda^{-1})$ on $\tilde{U}$. $\mathcal{PT}$ is then determined by the transition function $\tilde{\omega}^B = f^B(\omega^A, \pi_{A'})$ on $U \cap \tilde{U}$.

Newman et. al. [6] make equation (2) λ-dependent and show that ω^A may be found by integrating the hamiltonian system which has Ω as its hamiltonian. In their treatment λ plays the role of time. We give an analogous interpretation of the 2nd equation.

Choose a spinor say $\kappa_{A'} = (0,1)$, in the base space and parametrize a curve by the coordinates

$$x^{AA'} = \frac{\partial \omega^A}{\partial \pi_{A'}}\Big|_{\pi_{A'}=\kappa_{A'}}, \quad x^{A1'} = \omega_0^A = (w,z), \; x^{A0'} = x^A = (-y,x)$$

where $x^{A1'}$ gives the initial point on the curve, while $x^{A0'}$ is a tangent vector to the curve. To procede further, ie to find higher terms in (9) we do one of the following (all give the same answer).

a) Insert the 2nd heavenly tetrad into the recursion relations and solve for ω_3^A

$$\omega^A = x^{A1'} + \lambda x^{A0'} + \lambda^2 \varepsilon^{BA}\frac{\partial\Theta}{\partial x^{B0'}} + \lambda^3\varepsilon^{BA}\frac{\partial\Theta}{\partial x^{B1'}} + \dots \; . \tag{10}$$

Note that (7) is used to find the fourth term in the series, since the third one is the definition of Θ.

b) Use the globality and the degree two homogeneity of $\Sigma(\lambda)$

$$o_{A'_1}o_{A'_2}\dots o_{A'_k}\frac{\partial^k d\omega^A\wedge d\omega_A}{\partial\pi_{A'_1}\partial\pi_{A'_2}\dots\partial\pi_{A'_k}}\Big|_{\pi_{A'_i}=o_{A'_i}}$$

$$= \sum_{i=0}^{k}\varepsilon_{AB}d\omega_i^A\wedge d\omega_{k-i}^B = \begin{cases} 0 & \text{for } k>2 \\ -\Sigma^{0'0'} & \text{for } k=2\;. \end{cases} \tag{11}$$

c) Make the 2nd equation $\pi_{A'}$, i.e. λ-dependent. Define $X^{AA'} = \partial\omega^A/\partial\pi_{A'}$. Continue the curve to another order in λ so that to order λ^2

$$X^{A1'} = x^{A1'} + \lambda x^{A0}, \qquad X^{A0'} = x^{A0'} + \lambda\varepsilon^{BA}\frac{\partial\Theta}{\partial x^{B0'}}.$$

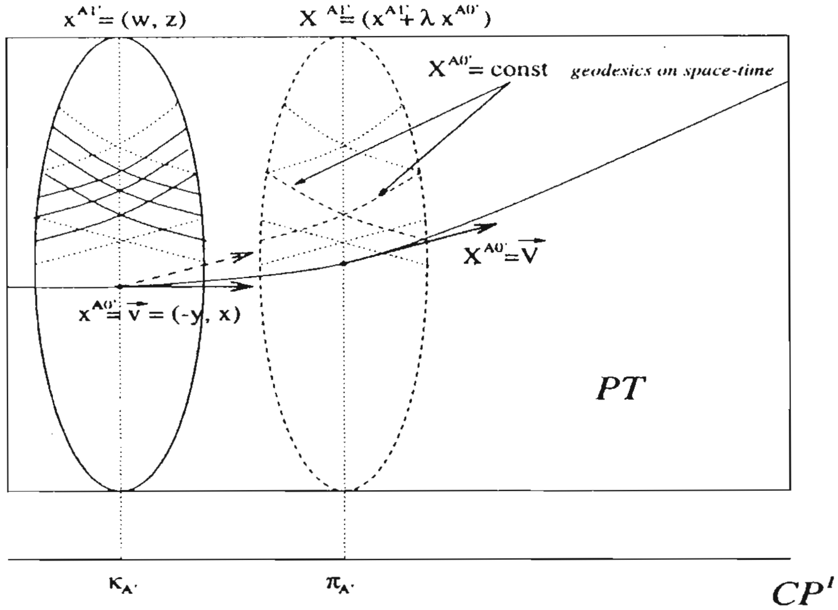
We then put the space-time metric into a standard, 2nd heavenly form with respect to the coordinates $X^{AA'}$

$$ds^2 = \varepsilon_{AB}dX^{A1'}dX^{B0'} + \frac{\partial^2\Theta'}{\partial X^{A0'}\partial X^{B0'}}dX^{A1'}dX^{B1'}$$

which forces us to introduce Θ', differing from Θ by terms of order λ

$$\Theta'(X^{AA'},\pi_{A'}) = \Theta(x^{AA'}) + \lambda\tau(x^{AA'}).$$

We find Θ' and can then iterate the process to obtain the subsequent orders in $X^{AA'}$. The parameter λ plays the role of time and Θ plays the



role of a time dependent hamiltonian. In homogeneous coordinates, Θ is homogeneous of degree -4 in $\pi_{A'}$. The construction may be summarized by the following

$$\dot{X}^{A1'} = X^{A0'}, \quad \dot{X}^{A0'} = \varepsilon^{BA}\frac{\partial\Theta'}{\partial X^{B0'}}, \quad \dot{\Theta}' = \tau, \quad \frac{\partial\tau}{\partial X^{A0'}} = \frac{\partial\Theta'}{\partial X^{A1'}}. \tag{12}$$

Dot means differentiation with respect to λ. The last equation (which gives the recursion relations) is valid up to the addition of $f(X^{A1'})$. The first two equations can (for those familiar with *edth*= $\eth$) be written as

$$\eth^2\omega^A = \pi^{A'}\nabla^A{}_{A'}\Theta, \quad \text{or} \quad \eth X^{AA'} = \{X^{AA'}, \Theta\}_\Pi,$$

where $\Pi = \pi^{A'}\pi^{B'}\nabla_{AA'} \wedge \nabla^A{}_{B'}$ is a (homogeneous) Poisson structure defined on the spin bundle tangent to the α-planes. Note that it projects down to zero by the twistor fibration.

Hierarchies

Finally we embed (1) in an infinite system of overdetermined PDEs. The associated linear system is

$$L_{Ai}s = (\lambda\nabla_{Ai} - D_{Ai-1})s = 0, \tag{13}$$

where

1. $s = s(x^{AA'\ldots A'_k}, \pi_{A'})$ is a function on a spin bundle over $\mathcal{M} \times X$, where $x^{AA'\ldots A'_k} = x^{A(A'\ldots A'_k)}$ are coordinates on the $\mathcal{M} \times X$ and $X \equiv \mathbb{C}^{2(n-1)}$ is a space of parameters ("times"), and

$$x^{Ai} = o_{A'}o_{A'_2}\ldots o_{A'_i}\iota_{A'_{i+1}}\ldots\iota_{A'_n}x^{AA'A'_2\ldots A'_n}$$

for $i = 0, 1$ are coordinates on $\mathcal{M}$ and for $i > 1$ are coordinates on X.

2. $\nabla_{AA'i} = \iota^{A'_1}\ldots\iota^{A'_{i-1}}o^{A'_i}\ldots o^{A'_n}\nabla_{AA'(A'_2\ldots A'_n)}$, $\nabla_{A1'i} = \nabla_{Ai}$, $\nabla_{A0'i} = D_{Ai-1}$.

3. $L_{A(A'_2\ldots A'_n)} = \pi^{A'}\nabla_{AA'(A'_2\ldots A'_n)}$, $L_{Ai} = \pi^{A'}\nabla_{AA'i}$.

Compatibility conditions for (13) yield

$$[\nabla_{Ai}, \nabla_{Bj}] = 0, \tag{14}$$

$$[D_{Ai-1}, D_{Bj-1}] = 0, \tag{15}$$

$$[\nabla_{Ai}, D_{Bj-1}] - [\nabla_{Bj}, D_{Ai-1}] = 0. \tag{16}$$

In what follows we shall give the appropriate generalisation of the second heavenly formulation; the first hierarchy is a consequence of a different gauge choice. (15) is used to fix $D_{Ai-1} = \partial_{Ai-1}$. Equation (16) implies that we can put $\nabla_{Ai} = \partial_{Ai} + [\partial_{Ai-1}, V]$ for some vector field V. In the basic tetrad for the second heavenly equation we see that $V = \varepsilon_{AB}\partial\Theta/\partial x_A \partial/\partial x_B$, i.e. V is the hamiltonian vector field of Θ with respect to the Poisson structure $\varepsilon_{AB}\partial/\partial x_A \wedge \partial/\partial x_B = \partial_x \wedge \partial_y$. The higher flows of the *second heavenly hierarchy* are, after redefinition of Θ, given by reducing (14) to give

$$\partial_{Ai}\partial_{Bj-1}\Theta - \partial_{Bj}\partial_{Ai-1}\Theta + \{\partial_{Ai-1}\Theta, \partial_{Bj-1}\Theta\}_{yx} = 0. \tag{17}$$

The sub-hierarchy $[L_A, L_{B_i}] = 0$ gives the recursion relations (7), which now generalize to

$$\nabla_{Ai}\partial_{Bj-1}\Theta = D_{Ai-1}\partial_{Bj}\Theta = D_{Ai-1}R\partial_{Bj-1}\Theta. \tag{18}$$

Let $e^{AA'_2...A'_n}$ be the set of one forms dual to $\nabla_{AA'A'_2...A'_n}$. In the adopted gauge

$$e^{A1'i} = dx^{Ai}, \quad e^{A0'i} = dx^{Ai-1} + \varepsilon^{AC} dx^{Di} \frac{\partial^2 \Theta}{\partial x^{Di-1} \partial x^{C0'}}.$$

An analogue of the formulation (4) may be achieved by introducing a two form homogeneous of degree $2n$ in $\pi_{A'}$

$$\Sigma_{(n)}(\lambda) = \varepsilon_{AB} \pi_{A'} ... \pi_{A'_n} \pi_{B'} ... \pi_{B'_n} e^{AA'...A'_n} \wedge e^{BB'...B'_n} \tag{19}$$

which satisfies $d\Sigma(\lambda) = 0, \;\; \Sigma(\lambda) \wedge \Sigma(\lambda) = 0$. The second equation follows from the choice of gauge while the closure condition is equivalent to (17).

The corresponding twistor space is obtained by factorizing the spin bundle $\mathcal{M} \times X \times \mathbb{CP}^1$ by the twistor distibution L_{Ai}. The resulting twistor space is still three-dimensional however has it a different topology as the holomorphic curves corresponding to points of $\mathcal{M} \times X$ have normal bundle $\mathcal{O}^A(n)$.

Define a conjugate recursion operator $R^*_{(i)} = D_{Ai-1} R_{(i)} {D_{Ai-1}}^{-1}$ and rewrite (17) in the bihamiltonian form

$$\partial_{Bj}(\partial_{Ai-1}\Theta) = D_{Ai-1}\partial_{Bj}\Theta = \nabla_{Ai}\partial_{Bj-1}\Theta =$$

$$\nabla_{Ai} R^{j-1} \partial_{B0'}\Theta = R^{*j-1} \nabla_{Ai} \partial_{B0'}\Theta.$$

The comparison with the KdV hierarchy

$$u_{t_j} = \mathcal{E} \frac{\delta h_{j-1}}{\delta u} = \mathcal{D} \frac{\delta h_j}{\delta u}$$

suggests that D_{Ai-1} and ∇_{Ai} play the role of first and second Poisson operators, while the solutions of the wave equation correspond to (functional derivatives of) different hamiltonians.

Observe that $\sum_{m=0}^{j} \{\partial_{Ai+j-m-1}\Theta, \partial_{Bm-1}\Theta\}_{yx} = \partial_{Bj}\partial_{Ai-1}\Theta$. This, with the definition $\omega_{Bj}(\lambda) = \sum_{m=0}^{j} \partial_{Bm-1}\Theta\lambda^{m+1}$, gives another form of the 2nd hierarchy

$$\partial_{Bj}\, \omega^A(\lambda) = \{\omega^A(\lambda), \lambda^{-j-1}\omega_{Bj}(\lambda)\}_{yx}. \tag{20}$$

Finally we would like to make a few remarks about where the above ideas could be applied.

- WDVV-the scaling reduction of the equation of associativity in the theory of Frobenius manifolds [2] coincides with Painleve VI for $n = 3$. It seems likely that a general case (which may be viewed as a higher order analogue of the Painleve equations) can be obtained as a reduction of higher flows in (17). Another possibility is that the Lax representation of the N-wave system lifted to the spin bundle will give rise to that of WDVV.

- Closely related to the previous is Krichever's work on Witham hierarchies in the context of equations of hydrodynamic type. In his approach [3], there is a closed two-form of an arbitrary homogeneity associated to each flow. This suggests an analogy with (19).
- In the last **TN** MD showed how $\nabla_{AA'}$ can be constructed from a solution of the Sine-Gordon equation. The same construction applies to KdV and NlS. It should be possible to obtain a recursion operator for these soliton equations from (7). In [5] such a reduction was implemented from the ASDYM recursion operator.

Many thanks to George Sparling for discussions about 2nd equation.

References

[1] Boyer, C.P. & Plebański, J.F.(1985) An infinite hierarchy of conservation laws and nonlinear superposition principles for self-dual Einstein spaces, J. Math. Phys. **26**, 229-234.

[2] Dubrovin, B. (1992) Integrable systems in topological field theory, Nucl. Phys. **B 379**, 627-689.

[3] Krichever, I.M. (1988) The averaging method for two-dimensional integrable equations, Funct. Anal. Appl **3**, 200-213.

[4] Mason, L.J. & Newman, E.T. (1989) A connection between the Einstein and Yang-Mills equations, Comm. Math. Phys., **121**, 659-668.

[5] Mason, L.J. & Woodhouse, N.M.J. (1996) *Integrability, Self-Duality, and Twistor Theory*, L.M.S. Monographs New Series, **15**, OUP.

[6] Newman, E.T., Porter, J.R. & Tod, K.P. (1978) Twistor Surfaces and Right-Flat Spaces, Gen. Rel. Grav. **12** 1129-1142.

Formal Adjoints and a Canonical Form for Linear Operators

Suppose E and F are smooth vector bundles on an oriented smooth manifold M. Let *vol* denote the bundle of volume forms on M. The *formal adjoint* of a linear differential operator $L : E \to F$ is the differential operator $L^* : F^* \otimes vol \to E^* \otimes vol$ characterised by the equation

$$\int_M \langle L^*\sigma, \tau\rangle = \int_M \langle \sigma, L\tau\rangle \quad \text{for } \sigma \in \Gamma(M, F^* \otimes vol) \text{ and } \tau \in \Gamma_*(M, E).$$

If $F = E^* \otimes vol$, then $L^* : E \to F$ and there is a canonical decomposition

$$L = L_+ + L_- = \tfrac{1}{2}[L + L^*] + \tfrac{1}{2}[L - L^*]$$

into self-adjoint and skew-adjoint parts. If E and F are tensor bundles on a Riemannian manifold, then L may be written in terms of the Levi-Civita connection and a formula for its adjoint determined by integration by parts. Suppose, for example, that E and F are both trivial and L is second order. Then we may write

$$L = S^{ab}\nabla_a\nabla_b + T^b\nabla_b + R,$$

where the tensor S^{ab} is symmetric. Adopting the convention that ∇_a acts on everything to its right, we can re-express L in the form

$$L = \nabla_a S^{ab}\nabla_b + (\tilde{T}^b\nabla_b + \nabla_b\tilde{T}^b) + \tilde{R}$$

where $\tilde{T}^b = \frac{1}{2}(T^b - (\nabla_a S^{ab}))$ and $\tilde{R} = R - \nabla_b\tilde{T}^b$. This is congenial since clearly

$$L^* = \nabla_a S^{ab}\nabla_b - (\tilde{T}^b\nabla_b + \nabla_b\tilde{T}^b) + \tilde{R}.$$

In particular,

$$L_+ = \nabla_a S^{ab}\nabla_b + \tilde{R} \quad \text{and} \quad L_- = \tilde{T}^b\nabla_b + \nabla_b\tilde{T}^b.$$

This generalises immediately to give:

Proposition . *A self-adjoint k^{th} order linear differential operator taking functions to functions on an oriented Riemannian manifold has even order and may be canonically written in the form:*

$$\sum_{i=0}^{k/2} \underbrace{\nabla_a\nabla_b\cdots\nabla_c}_{i} S_{(i)}^{ab\cdots cef\cdots g} \underbrace{\nabla_e\nabla_f\cdots\nabla_g}_{i},$$

for suitable symmetric tensors $S_{(i)}^{ab\cdots cef\cdots g}$. A skew-adjoint k^{th} order linear differential operator taking functions to functions on an oriented Riemannian manifold has odd order and may be canonically written in the form:

$$\sum_{i=0}^{(k-1)/2} \underbrace{\nabla_a\nabla_b\cdots\nabla_c}_{i}(\nabla_d A_{(i)}^{ab\cdots cdef\cdots g} + A_{(i)}^{ab\cdots cdef\cdots g}\nabla_d)\underbrace{\nabla_e\nabla_f\cdots\nabla_g}_{i},$$

for suitable symmetric tensors $A_{(i)}^{ab\cdots cdef\cdots g}$.

Now suppose M is an even-dimensional oriented conformal manifold. Graham, Jenne, Mason, and Sparling [2] have shown that there is a conformally invariant operator which, with respect to any Riemannian metric in the conformal class, takes the form

$$L = \Delta^{n/2} + \text{lower order terms}.$$

Since L takes functions to volume forms, so does its adjoint. The self-adjoint part L_+ of L is therefore also conformally invariant. As a conformal analogue of $\Delta^{n/2}$, we may as well replace L by L_+. Write this differential operator in the form given by our Proposition. The scalar part $S_{(0)}$ is a conformal invariant since it may be obtained by applying L to $f \equiv 1$. We may subtract this part and obtain the following result conjectured to us by Tom Branson.

Theorem . *The operator $\Delta^{n/2}$ admits a self-adjoint conformally invariant modification of the form $f \mapsto \nabla_a(Q^{ab}(\nabla_b f))$ for a suitable $(n-2)^{nd}$ order differential operator $Q : \Lambda^1 \to \Lambda^{n-1}$.*

His motivation for this conjecture comes from the case of the sphere where the form of the operator may be verified directly. On the sphere, the operator controls the embedding $L^2_{n/2} \hookrightarrow e^L$ (Orlitz class) as a limiting case of the sharp Sobolev embeddings $L^2_r \hookrightarrow L^{\frac{2n}{n-2r}}$ for $r < n/2$ (equivalently, comparing an L^q norm with the complementary series norm). See [1] for further discussion.

References

[1] T.P. Branson, *Sharp inequalities, the functional determinant, and the complementary series,* Trans. A.M.S. **347** (1995), 3671–3742.

[2] C.R. Graham, R. Jenne, L.J. Mason, and G.A.J. Sparling, *Conformally invariant powers of the Laplacian, I: Existence*, Jour. Lond. Math. Soc. **46** (1992), 557–565.

Michael Eastwood
Department of Pure Mathematics
University of Adelaide
South AUSTRALIA 5005

meastwoo@spam.maths.adelaide.edu.au

Rod Gover
School of Mathematics
Queensland University of Technology
Gardens Point Campus
Brisbane
Queensland 2434
AUSTRALIA

r.gover@fsc.qut.edu.au

Two Puzzles

(1) In ΠN 10, p.22 (July 1980), I proposed, as a puzzle, to find (?) in the following sequence:

$$\ldots, 7, 9, 12, (?), 24, 36, 56, 90, \ldots$$

The point about the answer

$$(?) = 24 \log 2 = 16.6355323\ldots$$

is that it is transcendental, being given (by use of l'Hospital's rule) by $n=0$ in

$$\frac{24}{n}(2^n - 1)$$

all other terms in the sequence being rational numbers.

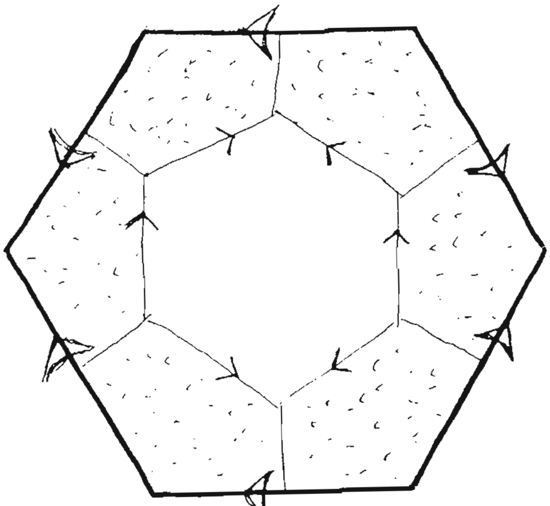
Now try

$$\ldots, 28, 0, 21, 4, 18, 0, (?), 24, 18, 20, 21, 24, 28, \ldots$$

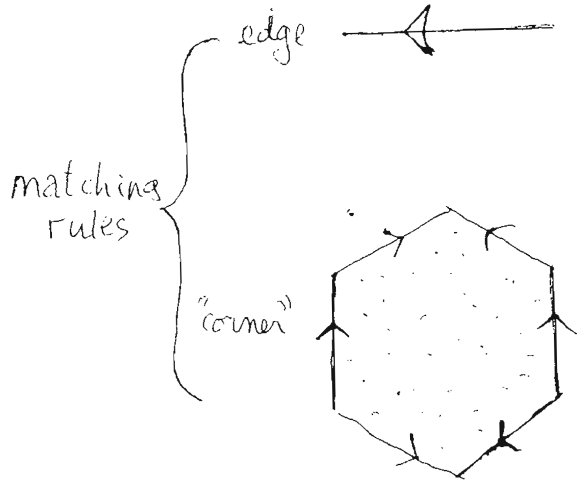
which has a little extra twist about it.

[Hint: first add a certain integer to each term, to make it somewhat more symmetrical]

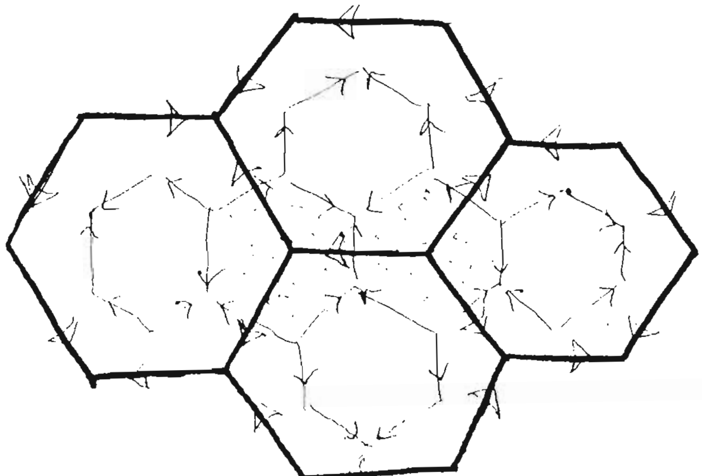
(2) This problem is to tile the plane with a regular hexagon, with edge-matching rules and "corner matching rules":



tile



To start:



Geometric issues in the foundation of science

Oxford 1996

As many readers know, the above conference took place last June and was deemed a success. Roger Penrose, looking impossibly young for his 65 years, managed to attend *all* the talks, including those in the parallel sessions — without apparent recourse to the many-world thesis!

The proceedings of this conference will be published by the Oxford University Press. The intended content is listed below.

Plenary talks

A. Ashtekar: *Geometric issues in quantum gravity*
M.F. Atiyah: *Roger Penrose, a personal reminiscence*
A. Connes: *Gravity coupled with matter and the spectral action principle*
S.K. Donaldson: *Quantum field theory and four manifold topology*
A. Ekert: *From quantum code-making to quantum code-breaking*
H. Friedrich: *The Einstein equations and conformal structures*
S. Hameroff: *Is consciousness a self-organizing process in fundamental space-time geometry? The microtubule connection*
S.W. Hawking: *Loss of information in black holes*
N.J. Hitchin: *Geometry of the space of framings*
C.R. LeBrun: *On 4-dimensional Einstein manifolds*
R. Penrose: *Twistor theory: whence and whither?*
D.W. Sciama: *Decaying neutrinos and the geometry of the universe*
G.B. Segal: *Generalisations of manifolds*
A. Shimony: *Physical implications of objective transiency*
P. Steinhardt: *Penrose tilings and quasicrystals revisited*
G. Veneziano: *Quantum-geometric origin of all forces in string theory*
R.S. Ward: *Integrable systems and twistors*

Parallel sessions

Quantum theory and beyond
A. Hodges, L. Hughston, L. Kauffman, L. Smolin
Geometry and gravity
B. Carter, G. Gibbons, E. Newman, G. Sparling
Fundamental questions in quantum mechanics
J. Anandan, M. Berry, R. Jozsa, L. Vaidman
Mathematical aspects of twistor theory
T. Bailey, S. Merkulov, S. Gindikin, A. Trautman

For further information, please contact us at: *geom96@math.ox.ac.uk* or Geometric Issues 96, Mathematical Institute, 24-29 St. Giles', Oxford OX1 3LB, United Kingdom.

TWISTOR NEWSLETTER No. 41

CONTENTS

Short contributions for **TN 42** should be sent to

Maciej Dunajski
Twistor Newsletter Editor
Mathematical Institute
24–29 St. Giles'
Oxford OX1 3LB
United Kingdom
E-mail: tnews@maths.ox.ac.uk

to arrive before the 15th December 1996.