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MASS, COHOMOLOGY, AND SPIN

I. Consider a field $\phi(x,y)$ which is holomorphic on $M^+ \times M^+$ and satisfies

(1) $$X^a X_a \phi = 0 \quad \text{and} \quad Y^a Y_a \phi = 0 \quad ,$$

where $X^a = \partial/\partial x_a$, etc.. Although each of the two constituents of this state is massless and has zero helicity, the system as a whole will in general possess mass and spin. The total mass-squared operator is $\hat{M}^2 = -(X^a + Y^a)(X_a + Y_a)$. Assuming ϕ to be in a mass eigenstate

(2) $$\hat{M}^2 \phi = m^2 \phi \quad ,$$

the total spin operator is

$$\hat{S}^2 = (x^a - y^a)[X_a - Y_a] - (x^a - y^a)(x^b - y^b)[X_a Y_b + \tfrac{m^2}{4} g_{ab}] \quad ,$$

and if ϕ is in a spin eigenstate then

(3) $$\hat{S}^2 \phi = s(s+1)\phi \qquad (s \text{ integral}) \quad .$$

We shall now describe such 2-point fields cohomologically.

The standard mass-squared and spin-squared operators for the sheaf $\mathcal{O}(-2,-2)$ on $\Xi := \mathbb{P}_3^+ \times \mathbb{P}_3^{*-}$ are $\hat{\mathfrak{M}}^2 = Z^\alpha I_{\alpha\beta}\frac{\partial}{\partial W_\beta} W_\gamma I^{\gamma\delta}\frac{\partial}{\partial Z^\delta}$ and $\hat{\mathfrak{S}}^2 = -Z^\alpha W_\alpha \frac{\partial^2}{\partial Z^\beta \partial W_\beta}$.

Subsheaves $\mathcal{O}_{ms}(-2,-2) \subset \mathcal{O}(-2,-2)$ of mass m and integral spin s are defined by the following exact sequences:

$$0 \longrightarrow \mathcal{O}_m(a,b) \longrightarrow \mathcal{O}(a,b) \xrightarrow{\hat{\mathfrak{M}}^2 - m^2} \mathcal{O}(a,b) \longrightarrow 0 \qquad (a, b \text{ integral})$$

$$0 \longrightarrow \mathcal{O}_{ms}(-2,-2) \longrightarrow \mathcal{O}_m(-2,-2) \xrightarrow{\hat{\mathfrak{S}}^2 - s(s+1)} \mathcal{O}_m(-2,-2) \longrightarrow 0 \qquad (s > 0)$$

$$0 \longrightarrow \mathcal{O}_{m0}(-2,-2) \longrightarrow \mathcal{O}_m(-2,-2) \xrightarrow{\partial^2/\partial Z^\alpha \partial W_\alpha} \mathcal{O}_m(-3,-3) \longrightarrow 0 \qquad (s = 0).$$

By the $\mathcal{P}$-transform there exists a canonical isomorphism

$H^2(\Xi, \mathcal{O}(-2,-2)) \xrightarrow{\cong} \{\text{holomorphic fields on } M^+ \times M^+ \text{ satisfying (1)}\}$.

The sheaf maps $\hat{\mathfrak{M}}^2$ and $\hat{\mathfrak{S}}^2$ induce on $H^2(\Xi, \mathcal{O}(-2,-2))$ the action of the corresponding space-time operators $\hat{M}^2$ and $\hat{S}^2$. Cohomological reasoning, when applied to the three exact sequences above, shows $H^2(\Xi, \mathcal{O}_m(-2,-2)) \cong \ker[\hat{M}^2 - m^2] : H^2(\Xi, \mathcal{O}(-2,-2)) \longrightarrow H^2(\Xi, \mathcal{O}(-2,-2))$, and $H^2(\Xi, \mathcal{O}_{ms}(-2,-2)) \cong \ker[\hat{S}^2 - s(s+1)] : H^2(\Xi, \mathcal{O}_m(-2,-2)) \longrightarrow H^2(\Xi, \mathcal{O}_m(-2,-2))$.

Therefore: <u>THEOREM</u>: There exists a canonical linear isomorphism:

$H^2(\Xi, \mathcal{O}_{ms}(-2,-2)) \xrightarrow{\cong} \{\text{holomorphic fields on } M^+ \times M^+ \text{ satisfying (1), (2), and (3)}\}$.

II. A positive frequency free field of mass m and (integral) spin s is defined to be a symmetric, tracefree, divergence-free field $\phi_{a\ldots d}(r)$ with s Lorentz indices, holomorphic on M^+, satisfying $(R^a R_a + m^2)\phi_{a\ldots d}(r) = 0$.

Given $\phi(x,y)$ holomorphic on $M^+ \times M^+$, make a change of coordinates: $r = \frac{1}{2}(x+y)$, $q = \frac{1}{2}(x-y)$. Then $\Phi(r,q) := \phi(x,y)$ is holomorphic in a neighbourhood of $M^+ \times \{0\} \subset M^+ \times M$, and can be expanded in powers of q^a about $q^a = 0$:

(4) $\Phi(r,q) = \phi^0(r) + \phi^1_a(r)\,q^a + \frac{1}{2!}\phi^2_{ab}(r)\,q^a q^b + \cdots + \frac{1}{n!}\phi^n_{ab\ldots d}\,q^a q^b \cdots q^d + \cdots$

The conditions imposed on $\phi(x,y)$ in Theorem 1 give certain relations among the coefficients $\phi^n_{a\ldots d}(r)$. Using $R_a = X_a + Y_a$, the mass condition (2) is:

(5) $$(R^2 + m^2)\,\phi^n_{a\ldots d}(r) = 0 \qquad \forall n .$$

The ZRM equations (1) are equivalent to $R\cdot Q\,\Phi = 0$ and $(R^2 + Q^2)\,\Phi = 0$, where $Q_a := X_a - Y_a$. From these equations, and (5), one obtains:

$$R^a \phi^{n+1}_{a\ldots d} = 0 \quad \text{and} \quad \phi^{n+2}_{a\ldots bc}{}^{c} = m^2 \phi^n_{a\ldots b} \qquad \forall\, n \geq 0$$

The spin operator, re-expressed in r,q variables, takes the form:

$$\hat{S}^2 = q^a Q_a + q^a Q_a\, q^b Q_b - \mathcal{D} ,$$

where $\mathcal{D} := q^a q^b \mathcal{D}_{ab}$ and $\mathcal{D}_{ab} := m^2 g_{ab} + R_a R_b$. Then (3) becomes:

$$
\begin{aligned}
&(0) \quad s(s+1)\,\phi^0 = 0 \\
&(1) \quad [2 - s(s+1)]\,\phi^1_a = 0 \\
&(2) \quad [4 + 2 - s(s+1)]\,\phi^2_{ab} = 2\,\mathcal{D}_{ab}\,\phi^0 \\
&\quad \vdots \\
&(n) \quad [2n + n(n-1) - s(s+1)]\,\phi^n_{ab\,c\ldots d} = n(n-1)\,\mathcal{D}_{(ab}\,\phi^{n-2}_{c\ldots d)} \quad \text{, etc..}
\end{aligned}
$$

The following conditions are therefore equivalent to (1), (2), and (3):

(6) $\phi^0, \ldots, \phi^{s-1} = 0$

(7) $\phi^s_{a\ldots d}(r)$ is a free field of mass m and spin s

(8) for $n > s$, $n-s$ odd: $\phi^n = 0$

$n-s$ even: $\phi^n_{abc\ldots d} = \dfrac{n(n-1)}{(n-s)(n+s+1)}\,\mathcal{D}_{(ab}\,\phi^{n-2}_{c\ldots d)}$.

THEOREM 2. There exists a canonical linear isomorphism:

$$H^2(\Xi, \mathcal{O}_{ms}(-2,-2)) \xrightarrow{\ \rho\ } \{\text{holomorphic fields on } M^+ \text{ with mass } m \text{ and spin } s\}.$$

Proof: By Theorem 1 an element of $H^2(\Xi, \mathcal{O}_{ms}(-2,-2))$ is a field $\phi(x,y)$

satisfying (1), (2), and (3). The map $\mathcal{P}$ is given explicitly by:

$$\phi(x,y) \xrightarrow{\ \mathcal{P}\ } \phi^{s}_{a\ldots d}(r) .$$

The injectivity of $\mathcal{P}$ follows at once from (6), (7), and (8). Furthermore for any positive frequency field $\phi^{s}_{a\ldots d}(r)$ the series generated by the relations (6), (7), and (8) converges (by elementary analysis) for all $r,q \in M^{+}\times M$ and hence $\mathcal{P}$ is also surjective. □

The map $\mathcal{P}^{-1}$ can be made explicit. We know $\mathcal{P}^{-1}$ is of the form:

$$\mathcal{P}^{-1}: \phi_{a\ldots d}(r) \longmapsto \Phi(r,q) = \frac{1}{s!}\Upsilon_s(\mathcal{D})\, q^a \cdots q^d \phi_{a\ldots d}(r) ,$$

where $\Upsilon_s(\mathcal{D})$ is some operator to be determined, subject to $\Upsilon_s(0)=1$. The requirement that $\Phi(r,q)$ be a spin s eigenstate will be fulfilled provided Υ_s satisfies the following Bessel-type equation:

$$\mathcal{D}^2 \Upsilon_s'' + \left(\tfrac{3}{2}+s\right)\mathcal{D}\Upsilon_s' - \frac{\mathcal{D}}{4}\Upsilon_s = 0 .$$

There is a unique solution with the desired analyticity properties:

$$\Upsilon_s(\mathcal{D}) = \frac{(2s+1)!}{2^s\, s!}\,\frac{J_{s+1/2}(i\mathcal{D}^{1/2})}{\mathcal{D}^{1/4}} ,$$

where $J_{s+1/2}$ is the Bessel function of order $s+1/2$. The series expansion of $\Upsilon_s(\mathcal{D})$ is compatible with (4), (6), (7), and (8).

We expect the analysis above to generalize in a relatively straightforward way for twists other than $(-2,-2)$, and also for systems with more than two twistors (cf. A. Popovich's note in this issue). It should be mentioned that there is an intimate connection between the map $\mathcal{P}^{-1}$ and the so-called "sub-elementary" states introduced by A.P. Hodges in his Ph.D thesis. Cf. also M.G.E. in TN9. Gratitude is expressed to M.G.E. and A.P.H. for useful discussions.

Tim Hurd
Lane Hughston

Moduli for some vector bundles on $\mathbb{P}^3$.

Ross Moore.

According to G. Trautmann (Math. Ann. 237, 167 (1978)), a vector bundle $\mathcal{E}$ on $\mathbb{P}^3$ with $l^1_0 = l^1_{-1} = l^2_{-3} = l^2_{-4} = 2$, where $l^i_d = \dim H^i(\mathbb{P}^3, \mathcal{E}(d))$, fits into an exact sequence of sheaves on $\mathbb{P}^3$:

$$\cdots \longrightarrow \mathcal{L}_2 \xrightarrow{\mathcal{Z}_1} \mathcal{L}_1 \xrightarrow{\mathcal{Z}_0} \mathcal{L}_0 \xrightarrow{\mathcal{Z}_{-1}} \mathcal{L}_{-1} \tag{1}$$

$$\begin{tikzcd}
\mathcal{L}_2 \arrow[r, "\mathcal{Z}_1"] \arrow[d, phantom, "\parallel"] & \mathcal{L}_1 \arrow[r, "\mathcal{Z}_0"] \arrow[d, phantom, "\parallel"] & \mathcal{L}_0 \arrow[r, "\mathcal{Z}_{-1}"] \arrow[d, phantom, "\parallel"] & \mathcal{L}_{-1} \arrow[d, phantom, "\parallel"] \\
{}^{\lambda}\mathcal{O}_{[\alpha\beta\gamma]}(-3) \arrow[r, "Z^\gamma"] & {}^{\lambda}\mathcal{O}_{[\alpha\beta]}(-2) \arrow[r, "Z^\beta"] & {}^{\lambda}\mathcal{O}_{\alpha}(-1) \arrow[r, "Z^\alpha"] & {}^{\lambda}\mathcal{O} \\
\oplus & \oplus & \oplus & \oplus \\
{}^{\mu}\mathcal{O}_{[\alpha\beta\gamma]}(-2) \arrow[r, "Z^\gamma"] \arrow[uur, "{}^{\lambda}_{\mu}A^\gamma"] & {}^{\mu}\mathcal{O}_{[\alpha\beta]}(-1) \arrow[r, "Z^\beta"] \arrow[uur, "{}^{\lambda}_{\mu}A^\beta"] & {}^{\mu}\mathcal{O}_{\alpha} \arrow[r, "Z^\alpha"] \arrow[uur, "{}^{\lambda}_{\mu}A^\alpha"] & {}^{\mu}\mathcal{O}(1) \\
\oplus & \oplus & \oplus & \oplus \\
{}^{\mu'}\mathcal{O}_{[\alpha\beta\gamma\delta]}(-1) \arrow[r, "Z^\delta"] \arrow[uur, "{}^{\mu}_{\mu'}B^{\gamma\delta}"] & {}^{\mu'}\mathcal{O}_{[\alpha\beta\gamma]} \arrow[r, "Z^\gamma"] \arrow[uur, "-{}^{\mu}_{\mu'}B^{\beta\gamma}"] & {}^{\mu'}\mathcal{O}_{[\alpha\beta]}(1) \arrow[r, "Z^\beta"] \arrow[uur, "{}^{\mu}_{\mu'}B^{\alpha\beta}"] & {}^{\mu'}\mathcal{O}_{\alpha}(2) \\
\oplus & \oplus & \oplus & \oplus \\
{}^{\lambda'}\mathcal{O}_{[\alpha\beta\gamma\delta]} \arrow[r, "Z^\delta"] \arrow[uur, "{}^{\mu'}_{\lambda'}C^\delta"] & {}^{\lambda'}\mathcal{O}_{[\alpha\beta\gamma]}(1) \arrow[r, "Z^\gamma"] \arrow[uur, "{}^{\mu'}_{\lambda'}C^\gamma"] & {}^{\lambda'}\mathcal{O}_{[\alpha\beta]}(2) \arrow[r, "Z^\beta"] \arrow[uur, "{}^{\mu'}_{\lambda'}C^\beta"] & {}^{\lambda'}\mathcal{O}_{\alpha}(3)
\end{tikzcd} \tag{2}$$

via $\mathcal{E} \oplus \mathcal{L} \cong \operatorname{im}(\mathcal{Z}_0 : \mathcal{L}_1 \longrightarrow \mathcal{L}_0) = \ker(\mathcal{Z}_{-1} : \mathcal{L}_0 \longrightarrow \mathcal{L}_{-1})$ (3)

where $\mathcal{L} = \bigoplus_j n_j \mathcal{O}(j)$ is a direct sum of line bundles, $n_j \in \mathbb{Z}$, $j \in \mathbb{Z}$. Reading vertically, (2) defines the sheaves $\mathcal{L}_i$ and the maps $\mathcal{Z}_i$. The indices $\lambda, \mu, \mu', \lambda'$ are independent $GL(2,\mathbb{C})$ indices, and are written on the left to avoid confusion with the usual $\mathbb{C}^4$ (twistor) indices $\alpha, \beta \ldots$ A, B, C are (matrices 2×2) of constant (skew-) twistors, considered as maps between the exterior powers $\Lambda^i\mathbb{C}^4 \longrightarrow \Lambda^j\mathbb{C}^4$, and the corresponding sheaves $\mathcal{O}_{[\alpha\ldots\beta]}(j)$ of germs of sections thereof, homogeneous of appropriate degrees. For exactness of (1), we require precisely

$${}^{\mu}_{\mu'}B_{\alpha\beta}\ {}^{\mu'}_{\lambda'}C^{\beta} = 0 = {}^{\mu}_{\mu'}B_{\alpha\beta}\ {}^{\lambda}_{\mu}A^{\beta} \quad , \quad {}^{\mu}_{\mu'}B_{\alpha\beta} = \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\left({}^{\mu}_{\mu'}B^{\gamma\delta}\right) \tag{4}$$

which appears to be 32 conditions on the $\mathbb{C}^{56}$ possible choices of $[A,B,C]$, however not all these conditions need be independent.

PROBLEMS : (i) Describe geometrically the space S of $\{A,B,C\}$ satisfying (4), in particular, what is $\dim S$?

(ii) Given $\varepsilon \in S$, describe the bundle $\mathcal{E}_\varepsilon$, and the line bundles $\mathcal{L}_\varepsilon$, such that $\mathcal{E}_\varepsilon \oplus \mathcal{L}_\varepsilon$ is the bundle in (3);

(6) in particular, what is rank $\mathcal{E}_1$?

Leaving aside the C's temporarily, the condition that a collection $\{{}^{\mu\mu'}B_{\alpha\beta}\}$ admits a solution ${}_\mu A^\beta$ of (4) is a single determinant condition on the clumped indices $\binom{\beta}{\mu}$. There are then ∞^2 solutions ${}^{\lambda}_{\mu}A^\beta = {}^{\lambda}a\,{}_\mu A^\beta$, ${}^\lambda a \in \mathbb{C}^2$. To obtain two independent solutions ${}^{\lambda}_{\mu}A^\beta$ requires $\{{}^{\mu\mu'}B_{\alpha\beta}\}$ to have rank 6 as a matrix in clumped (8-dim^{al}) indices $\binom{\beta}{\mu}\binom{\alpha}{\mu'}$ — 3 further conditions. To determine these

$$(4) \Rightarrow {}^{\mu}_{\mu'}B_{\alpha\beta}\,{}^{\lambda}_{\mu}A^\beta\,{}^{\delta}_{\nu}A^\alpha = 0 \Leftrightarrow {}^{\mu}_{\mu'}B_{\alpha\beta}\,{}^{\delta\lambda}I^{\alpha\beta} = 0 \qquad (5)$$

where ${}^{\delta\lambda}I^{\alpha\beta} = {}^{\delta}_{\mu}A^{[\alpha}\,{}^{\lambda}_{\nu}A^{\beta]}\,\varepsilon^{\mu\nu}$ determines a conic locus I via $\gamma_\lambda \longmapsto \gamma_\delta\,\gamma_\lambda\,{}^{\delta\lambda}I^{\alpha\beta}$ on the Klein quadric $\Omega \subset \mathbb{P}^5 = \mathbb{P}(\Lambda^2\mathbb{C}^4)$, Grassmann variety of lines in $\mathbb{P}^3$. Equation (5) then says that I is apolar to all of the B's, which must therefore lie in a plane $\pi \subset \mathbb{P}^5$ (considered projectively); moreover $I = \tilde{\pi} \cap \Omega$, where $\tilde{\pi}$ is polar to π w.r.t. Ω. There are ∞^4 solutions ${}^{\lambda}_{\mu}A^\beta$ transformable into each other by $GL(2,\mathbb{C})$ acting on the λ index.

Precisely the same conditions on $\{{}^{\mu\mu'}B\}$ guarantee the existence of ${}^{\mu'}_{\lambda}C^\beta$'s, in the same numbers. Hence the greatest dimension of a component of S is $(24-1-3)+4+4 = 28$ when the ${}^{\mu}_{\mu'}B^{\alpha\beta}$'s all lie in a plane π. Otherwise the maximal dimension is $(24-1)+2+2 = 27$, with ${}^{\mu}_{\mu'}B$'s unrestricted. An intermediate case, where I degenerates to a line on Ω, also occurs if the B's span a 3-space tangent to Ω. This component also has dimension $21+4+4-1 = 28$, and will be studied in detail elsewhere.

The line bundles $\mathcal{E} = \oplus_j\, n_j\,\mathcal{O}(j)$ are identified as follows. The image $\mathcal{F} \subset {}^{\mu}\mathcal{O}^2_\alpha$ of ${}^{\mu'}\mathcal{O}_{(\alpha\beta\gamma)}$ under ${}^{\mu}_{\mu'}B^{\alpha\delta}$ is free, B being essentially a map of vector spaces. Define rank B = rank $\mathcal{F}$, and choose rank B independent constant sections ${}^{\mu'}f_{\alpha\beta\gamma}$ generating $\mathcal{F}$ under B. In (3), these generate rank B line subbundles of $\operatorname{im}\mathcal{E}_0$ which are linearly independent everywhere over $\mathbb{P}^3$. Tensoring by $\mathcal{O}(1)$, $\mathcal{O}(-1)$ and repeating this argument for ${}^{\lambda}_{\mu}A^\beta$, ${}^{\mu'}_{\lambda}C^\alpha$ gives n_{-1} = rank A, n_0 = rank B, n_1 = rank C. Hence rank $\mathcal{E} = 24 - \Sigma n_j$, the ranks of A, B, C being in principle computable. $\operatorname{im}\mathcal{E}_0$ is a deformation of $2\Omega^1 \oplus 2\Omega^1(1) \oplus 2\Omega^2(3) \oplus 2\Omega^2(4)$ — Ω^k = differential k-forms on $\mathbb{P}^3$ — so has rank = 24.

<u>Description of E</u>. Consider the general case of 2 independent (7)
${}^{\lambda}_{\mu}A^{\beta}$'s, & ${}^{\mu'}_{\gamma}C^{\beta}$'s satisfying (4) for ${}^{\mu}_{\mu'}B_{\alpha\beta}$'s spanning a plane $\pi \in \mathbb{P}^5$.
Using (i) ${}^{\mu'}_{\gamma}C^{\gamma}$ is injective,

(ii) ${}^{\gamma}\mathcal{O}_{[\alpha\beta\gamma\delta]} \xrightarrow{{}^{\mu'}_{\gamma}C^{\delta}} {}^{\mu'}\mathcal{O}_{[\alpha\beta\gamma]} \xrightarrow{{}^{\mu}_{\mu'}B^{\alpha\gamma}} {}^{\mu}\mathcal{O}_{\alpha}$ is exact,

(iii) $\dim(\ker {}^{\lambda}_{\mu}A^{\beta}) = 4 \Leftrightarrow {}^{\lambda}_{\mu}A^{\beta}$ is surjective (not trivial),

and chasing around the diagram (2), the line bundles generated by A, C can be "peeled off" leaving:

$$\begin{tikzcd}[column sep=small]
 & {}^{\mu}\mathcal{O}_{[\alpha\beta]}(-1) \arrow[dr, "Z^{\beta}"] & & {}^{\lambda}\mathcal{O} \\
{}^{\mu'}\mathcal{O}(-1) \arrow[ur, "{}^{\mu}_{\mu'}B_{\alpha\beta}"] \arrow[dr, "Z^{\delta}"'] & \oplus & {}^{\mu}\mathcal{O}_{\alpha} \arrow[ur, "{}^{\lambda}_{\mu}A^{\alpha}"] \arrow[dr, "Z^{\alpha}"'] & \oplus \\
 & {}^{\mu'}\mathcal{O}_{[\alpha\beta\gamma]} \arrow[ur, "-{}^{\mu}_{\mu'}B^{\alpha\gamma}"'] & & {}^{\mu}\mathcal{O}(1)
\end{tikzcd} \qquad (6)$$

Since $\ker(Z^{\alpha} : {}^{\mu}\mathcal{O}_{\alpha} \longrightarrow \mathcal{O}(1))$ is the (twisted) cotangent bundle $\Omega^1(1)$, this with the result of Barth & Hulek (Manusc. Math. 25, 323 (1978)) that E is the cohomology of the <u>monad</u>:

$$H^2(E(-3)) \otimes \mathcal{O}(-1) \xrightarrow{B \cdot Z} H^1(E(-1)) \otimes \Omega^1(1) \xrightarrow{A} H^1(E) \otimes \mathcal{O}. \qquad (6)'$$

This means that the bundles E corresponding to points in the (Zariski-open) subset S', of maximal dimension 20 in S, are precisely the stable, rank 2 (follows from (6)) bundles on $\mathbb{P}^3$ with $\dim H^1(E(-3)) = 0$ – instanton condition – and Chern classes $c_1 = 0$, $c_2 = 2$.

The moduli space $M(0,2)$ for such stable bundles is described geometrically by Hartshorne (Math. Ann. 238, 229 (1978)), and has dimension 13. Examining (6) more closely, it will be seen that S' is a principal $GL(2,\mathbb{C}) \times [GL(2,\mathbb{C}) \otimes GL(2,\mathbb{C})] \times GL(2,\mathbb{C})$ – bundle over $M(0,2)$ with the group actions being the natural ones on the indices $\lambda, \mu, \mu', \gamma$ respectively. Note that these indices correspond precisely to the cohomology spaces in (6)', and that (6) identifies the maps in (6)' in terms of the extensions of bundles of differentials Ω^1, $\Omega^1(1)$, $\Omega^2(3)$, $\Omega^2(4)$ producing the cohomology of E.

To analyze (6) further, note that $I \in \Omega$ corresponds to a <u>regulus</u> in $\mathbb{P}^3$, i.e. one system of generators on a quadric $Q \in \mathbb{P}^3$. Adapt coordinates / abstract indices to this (non-singular) quadric Q, via $Z^{\alpha} \longleftrightarrow Z^{A\tilde{A}} \in \mathbb{C}^2 \otimes \mathbb{C}^2$, so that $Q(Z) = \frac{1}{2} Z_{A\tilde{A}} Z^{A\tilde{A}}$. The indices $A, \tilde{A}$ projectively refer to the lines of each regulus on Q; the

(8) skew forms ε_{AB}, $\varepsilon_{\tilde A\tilde B}$ can be assumed to be known (at least up to scale). In $\Lambda^2\mathbb{C}^4$, $X^{\alpha\beta} \longleftrightarrow \varphi^{AB}\varepsilon^{\tilde A\tilde B} + \chi^{\tilde A\tilde B}\varepsilon^{AB}$ where $\varphi^{AB} = \varphi^{(AB)}$ and $\chi^{\tilde A\tilde B} = \chi^{(\tilde A\tilde B)}$. The plane $\tilde\pi$ is given by $\varphi^{AB} = 0$, and I is the locus of decomposables $\chi^{\tilde A\tilde B} = \mu^{\tilde A}\mu^{\tilde B}$. Similarly π is the plane $\chi^{\tilde A\tilde B} = 0$, containing the conic $\omega = \pi \cap \Omega$ of decomposables $\varphi^{AB} = \rho^A\rho^B$. Hence

$$^{\mu\mu'}B_{\alpha\beta} \longleftrightarrow {}^{\mu\mu'}B_{AB}\,\varepsilon_{\tilde A\tilde B}$$

$$\text{and} \qquad {}^{\lambda}_{\mu}A^{\alpha} \longleftrightarrow {}_{\mu}a^{A}\cdot{}^{\lambda}\lambda^{\tilde A}\,, \quad {}^{\mu}_{\mu'}B_{AB}\,{}_{\mu}a^{B} = 0 \qquad (7)$$

$$^{\mu'}_{\lambda'}C^{\alpha} \longleftrightarrow {}^{\mu'}c^{A}\cdot{}_{\lambda'}\lambda'^{\tilde A}\,, \quad {}^{\mu}_{\mu'}B_{AB}\,{}^{\mu'}c^{B} = 0$$

is the translation of (4). The $GL(2,\mathbb{C})$ actions on the λ, λ' indices is just the standard basis action, and commutes with the actions on the μ, μ' indices; they need no longer be considered.

The divisor of "jumping lines" of E consists of those lines $X^{\alpha\beta} = Y^{[\alpha}Z^{\beta]}$ in $\mathbb{P}^3$ for which $\mathrm{im}({}^{\mu}_{\mu'}B_{\alpha\beta}Z^{\beta}) \cap \ker(Y^{\alpha}) \neq 0$ in (6). This is the quadratic condition, in $\mathbb{P}^5$

$$q_{ABCD}\,\varphi^{AB}\varphi^{CD} = \tfrac{1}{2}({}_{\mu\mu'}B_{AB})({}^{\mu\mu'}B_{CD})\,\varphi^{AB}\varphi^{CD} = 0 \qquad (8)$$

describing the join of $\tilde\pi$ to a conic $q_{(AB)(CD)} = q_{(CD)(AB)}$ in π giving a quadric cone hypersurface in $\mathbb{P}^5$ with plane vertex $\tilde\pi$. Writing $\langle\varphi,\psi\rangle \equiv \varphi_{AB}\psi^{AB}$, the determinant condition on the $^{\mu\mu'}B_{AB}$'s to admit solutions ${}_{\mu}a^{B}$, ${}^{\mu'}c^{B}$ of (7) is

$$\begin{aligned} 0 &= \langle {}_{\mu\mu'}B, {}_{\nu\nu'}B\rangle\langle {}^{\mu\nu'}B, {}^{\nu\mu'}B\rangle \\ &= \langle {}_{\mu\mu'}B, {}_{\nu\nu'}B\rangle\langle {}^{\mu\mu'}B, {}^{\nu\nu'}B\rangle - \tfrac{1}{4}\langle {}_{\mu\mu'}B, {}^{\mu\mu'}B\rangle^2 \qquad (9) \\ &= 4\,q_{ABCD}\,q^{ABCD} - (q_{AB}{}^{AB})^2 \end{aligned}$$

This is precisely the condition that there exist triangles in π which are inscribed in the conic q, and circumscribed about the conic ω (see Moore [1] p. 40) agreeing with the Hartshorne description. Note that, from (8), q is an invariant of the $GL(2,\mathbb{C}) \otimes GL(2,\mathbb{C})$ action on the indices $\mu\mu'$ of $^{\mu\mu'}B_{AB}$.

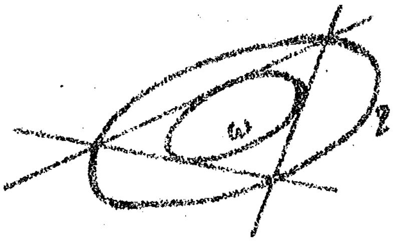

fig. 1

Knowing that an "instanton" bundle is uniquely determined by its divisor of jumping lines (see e.g. Beilinson, Gel'fand, Manin), this is sufficient to deduce the fibre bundle structure of S' over $M(0,2)$. q is also invariant under interchange of indices $\mu \leftrightarrow \mu'$, so that the full symmetry group on the $\mu\mu'$-indices is $\mathbb{C}^* \times O(4,\mathbb{C}) \cong GL(2,\mathbb{C}) \otimes GL(2,\mathbb{C}) \times \mathbb{Z}_2$ preserving $\varepsilon^{\mu\nu}\varepsilon^{\mu'\nu'}$ (up to scale).

Alternatively, the inscribed triangles in q describe a linear system of type g^1_3 on the regulus ω on $Q \subset \mathbb{P}^3$. This is precisely the data of the Hartshorne description of $M(0,2)$ from which the bundle $\mathcal{E}$ can be reconstructed as an extension $\xi \in \mathrm{Ext}^1_{\mathbb{P}^3}(\mathcal{O}_Q(-2,1), 2\mathcal{O}(-1)) \cong H^0(2\,\mathcal{O}_Q(3,0) \otimes \mathcal{O}(-1))$. In Moore (2) such an extension is shown to correspond to a rational morphism

$$f : \mathbb{P}^3 \longrightarrow \tilde{G}(1,3)$$

$$\text{inducing} \quad \mathcal{E}|_{\mathbb{P}^3 - L} \cong f^* \tilde{\mathcal{U}}^\vee \tag{10}$$

where $\tilde{\mathcal{U}}^\vee$ is the universal quotient bundle over the Grassmann variety $\tilde{G}(1,3)$ of lines in the projective space of symmetric 3-spinors $\psi_{\tilde{A}\tilde{B}\tilde{C}}$; L is the subvariety in $\mathbb{P}^3$ where f vanishes, this being the 4 double lines of the linear system g^1_3. This description can also be derived directly from (6).

<u>Analysis of the "monad" (6)</u>: Let $W = \ker({}^{\alpha}_{\mu}A^{\alpha})$. Then $W \cong \mathbb{C}^6$ and $W = \{{}^{\mu}W_{A\tilde{A}} : {}^{\mu}W_{A\tilde{A}}\,{}_{\mu}a^{A} = 0\}$.
Let $W_z = W \cap \ker(Z^\alpha) = \{{}^{\mu}W_{A\tilde{A}} : {}^{\mu}W_{A\tilde{A}}\,Z^{A\tilde{A}} = 0 = {}^{\mu}W_{A\tilde{A}}\,{}_{\mu}a^{A}\}$,
and put $B_z = \mathrm{im}(B\cdot z) = \{\gamma^{\mu'}\,{}^{\mu}_{\mu'}B_{AB}\,Z^{B}{}_{\tilde{A}}\ ,\ \gamma^{\mu'} \in {}^{\mu'}\mathcal{O}(-1)\}$.
The fibre $\mathcal{E}_z$ of $\mathcal{E}$ over $z \in \mathbb{P}^3$ is given by $\mathcal{E}_z = W_z/B_z$. (11)

The following diagram commutes.

$$\begin{tikzcd}[column sep=large]
\mathcal{O}^3 \arrow[d, equal] & W_z \arrow[d, equal] & \tilde{\mathcal{O}}^3 \arrow[d, equal] \\
\{\varphi^{(ABC)}\} \arrow[r, bend right, dashed, "\varepsilon_{A(B}\,{}^{\mu}a_{C}\,Z_{D)\tilde{A}}"'] & \{{}^{\mu}W_{A\tilde{A}}\} \arrow[l, bend right, "Z^{\tilde{A}(B}\,{}_{\mu}a^{C}\,Z^{D)\tilde{A}}"'] \arrow[r, bend left, "{}_{\mu}a^{C}\,Z_{C(\tilde{B}}\,Z^{A}{}_{\tilde{C})}"] & \{\psi_{\tilde{A}\tilde{B}\tilde{C}}\} \arrow[l, bend left, dashed, "{}^{\mu}a_{B}\,Z^{B(\tilde{B}}\,Z_{A}{}^{\tilde{C})}"] \\
\{\gamma_{\mu'}\,{}^{\mu\mu'}B^{(BC}\,{}_{\mu}a^{D)}\cdot Q(z)\} \arrow[u, hook] & \{-\gamma_{\mu'}\,{}^{\mu\mu'}B_{AB}\,Z^{B}{}_{\tilde{A}}\} \arrow[l] \arrow[r] \arrow[u, hook] \arrow[d, equal] & \{-\gamma_{\mu'}\,{}^{\mu\mu'}B^{AB}\,{}_{\mu}a^{C}\,Z_{A\tilde{B}}\,Z_{B\tilde{A}}\,Z_{C\tilde{C}}\} \arrow[u, hook] \\
& B_z &
\end{tikzcd} \tag{12}$$

(10) where $\odot^3$, $\tilde{\odot}^3$ are the spaces of symmetric 3-spinors as shown. For $Z \notin Q$, i.e. $Q(Z) \neq 0$, the maps in the first row are isomorphisms of vector spaces, and the "dashed" maps $\dashrightarrow$ are their inverses, up to multiples of $Q(Z)^k$ for some $k \in \mathbb{Z}^+$.

For constants γ_μ, the $\{ \gamma_\mu{}^{\mu'\mu} B^{(BC}{}_\mu a^{D)} \}$ represents a linear system g_3^1 of divisors of type $(3,0)$ on Q. It is the same g_3^1 as in fig. 1, since $\varepsilon_{\mu\nu}{}^{\mu\mu'} B^{(AB}{}_\mu a^{E)}{}^{\nu\nu'} B^{(CD}{}_\nu a^{F)} \varepsilon_{EF} \propto q^{ABCD}$. Hence projectively, the g_3^1 can be considered as being determined by a line q in $\mathbb{P}\odot^3 \cong \mathbb{P}^3$. The planes (in $\mathbb{P}\odot^3$) through q map to the points of the (projective) fiber $\mathbb{P}E_Z$, giving a trivialization of $\mathbb{P}E$, hence also E, over the (Stein) open set $\mathbb{P}^3 - Q$.

To describe $E|_Q$, look at the maps into $\tilde{\odot}^3$. Again B_Z maps into a linear system g_3^1, but this time of divisors of type $(0,3)$ on Q, and depending non-trivially on $Z \in \mathbb{P}^3$. For each Z there is a line in $\mathbb{P}\tilde{\odot}^3$ the planes through which correspond ~~to~~ points of $\mathbb{P}E$. Explicitly there is a "map" $\mathbb{P}^3 \longrightarrow \tilde{G}(1,3)$ given by

$$Z \longmapsto q^{ABCD} Z_{A\tilde{A}} Z_{B\tilde{B}} Z_{C\tilde{C}} Z_{D\tilde{D}} \qquad (13)$$

which is precisely the map given in Moore [2], denoted f in (10). It is clearly rational, and vanishes along the generators of Q corresponding to the roots of $q^{(ABCD)}$ — these are the double lines of the linear system g_3^1, and correspond to the intersection points $q \cdot W \cong T$. Pulling back the quotient bundle $\tilde{\mathcal{U}}^\vee$ corresponds to taking the planes through the lines $f(Z) \in \mathbb{P}\tilde{\odot}^3$.

Since E is known to be locally free, it can be identified with the bundle, $\mathcal{F}$ say, classified according to Hartshorne by the linear system on Q determined by q. E, $\mathcal{F}$ agree on $\mathbb{P}^3 - L$, where L has codim 2 in $\mathbb{P}^3$, hence they agree everywhere.

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A GENERALIZED DE RHAM SEQUENCE

N. Buchdahl

0. The purpose of this article is to exhibit an exact sequence of sheaves on $\mathbb{M}$ which arose recently in my D.Phil. Qualifying dissertation. This sequence is

$$(0.1)\quad \begin{aligned} 0 &\to \odot^{n-1}\mathbb{T}^* \to \mathcal{O}^{(A'_1\cdots A'_{n-1})} \xrightarrow{\nabla_A^{A'_n}} \mathcal{O}^{(A'_1\cdots A'_n)}_A[-1]' \xrightarrow{\nabla^{AA'_n}} \\ &\xrightarrow{\nabla_{A_1A'_1}\cdots\nabla_{A_{n-1}A'_{n-1}}} \begin{array}{c}\mathcal{O}^{(A'_1\cdots A'_n)}[-1][-2]' \\ \oplus \\ \mathcal{O}_{(A_1\cdots A_n)}[-1]'\end{array} \xrightarrow[\to\nabla^{A A'_n}]{\nabla_{A_1A'_1}\cdots\nabla_{A_{n-1}A'_{n-1}}} \mathcal{O}^{A'}_{(A_1\cdots A_{n-1})}[-1][-2]' \xrightarrow{\nabla^{A_{n-1}}_{A'}} \mathcal{O}_{(A_1\cdots A_{n-2})}[-2][-2]' \to 0 \end{aligned}$$

AA'_n – put n-th ind. at end

A'_n in Kern of Hone

conf -1 -2

For $n = 2$, this is precisely the holomorphic de Rham sequence on $\mathbb{M}$, written in spinor notation.

Using a complicated induction argument, it is possible to show directly that (0.1) is exact; a heuristic proof of this fact will be given in the next section. Note that conformal invariance (in the sense of $g_{ab} \to \Omega^2 g_{ab}$) is broken wherever more than one derivative appears in a map. Nevertheless, all the mappings are defined globally on $\mathbb{M}$.

1. To simplify the notation a little, denote by A_n^{i-1} the i-th non-constant sheaf from the left in (0.1), and by A^2_{n+}, A^2_{n-} the sheaves $\mathcal{O}^{(A'_1\cdots A'_n)}[-1][-2]'$, $\mathcal{O}_{(A_1\cdots A_n)}[-1]'$ respectively; also, let $T_n := \odot^{n-1}\mathbb{T}^*$ ($\approx \mathbb{C}^{\binom{?}{?}}$). (As a mnemonic device, note that when $n = 2$, $A^1_2 \cong \Omega^1_{\mathbb{M}}$ and $A^2_{2\pm} \cong \Omega^2_{\pm}$).

Recall that a holomorphic left-handed massless free field of helicity $-\frac{1}{2}n$ is by definition a section of $F_n := \ker A^2_{n-} \to A^3_n$, whilst a potential for such a field is a section of the sheaf $P_n := \ker A^1_n \to A^2_{n+}$, (where all the maps, including $P_n \to F_n$, are induced from those in (0.1)). With these definitions, the exactness of (0.1) automatically implies that of

$$(1.1)\qquad 0 \to T_n \to A^0_n \to P_n \to F_n \to 0 \qquad ,$$

which simply states that fields are locally isomorphic to potentials/gauge, and that solutions of the dual twistor equation (for arbitrarily many indices) are essentially symmetric dual

(12)

twistors. (Note incidentally that (1.1) is the local version of sequence (5) in Richard Ward's article in Sec. 2.8 of Advances in Twistor Theory.)

Next, observe that the surjectivity of $\Box : \mathcal{O} \to \mathcal{O}$ easily implies that the sequence

$$(1.2) \qquad A^1_n \to A^2_{n+} \to 0$$

is exact. The exactness of (0.1), together with this fact, then implies that of

$$(1.3) \qquad 0 \to L_n \to A^2_{n-} \to A^3_n \to A^4_n \to 0 \quad ,$$

which is just the definition of L_n extended to a resolution.

Conversely, it is elementary to check that the exactness of (1.1), (1.2) and (1.3) together imply that of (0.1) – the exactness of the above three sequences can all be proved seperately, the only difficult part being that of $A^0_n \to P_n \to F_n \to 0$. However, it would be nice to have a simple proof of the exactness of (0.1) since (1.1) and (1.3) follow so easily.

2. Because all the maps in (0.1) are defined globally, one can take sections of it over arbitrary open subsets $U \subset \mathbb{P}^1$; thus, with just a formal change of notation, one can emulate Mike Eastwood's analysis in Sec. 2.10 of ATT to obtain an exact sequence

$$(2.1) \qquad 0 \to \hat{H}^1(U, T_n) \to \frac{\Gamma(U, P_n)}{\mathrm{im}\, \Gamma(U, A^0_n) \to \Gamma(U, P_n)} \to \Gamma(U, E_n) \to \hat{H}^2(U, T_n) \to \frac{\Gamma(U, A^2_{n+})}{\mathrm{im}\, \Gamma(U, A^1_n) \to \Gamma(U, A^2_{n+})}$$

where

$$\hat{H}^p(U, T_n) := \frac{\ker \Gamma(U, A^p_n) \to \Gamma(U, A^{p+1}_n)}{\mathrm{im}\, \Gamma(U, A^{p-1}_n) \to \Gamma(U, A^p_n)} \quad ,$$

The reason for this notation is that when U is Stein, (0.1) is an acyclic resolution of $\mathcal{O}^{-2}\,\mathbb{T}^* = T_n$, so that $\hat{H}^p(U, T_n) = H^p(U, T_n)$ for all p by the abstract de Rham theorem.

From (2.1) one sees that a potential/gauge description of helicity $-\frac{1}{2}n$ fields on U is equivalent to the classical description if $\hat{H}^1(U,T_n) = \hat{H}^2(U,T_n) = 0$. Alternatively, one can break (1.1) into a pair of short exact sequences and consider the resulting long exact cohomology sequences. This has the effect of splitting the conditions $\hat{H}^p(U,T_n) = 0$ into $H^p(U,T_n) = 0$ plus a holomorphic condition which is trivially satisfied when U is Stein and which is vacuous for $n = 1$. In my dissertation there is a proof of the existence of contractible domains $U \subset \mathbb{C}^4$ which do not satisfy this holomorphic condition for any $n > 1$, thereby demonstrating its necessity in general.

skip

3. One obtains an interesting result if one applies the $\mathcal{O}$-transform of Eastwood, Penrose & Wells' recent article on cohomology and massless fields: the open set $U \subset \mathbb{M}$ is said to be $\mathcal{O}_N$ ($N = 0,1,\ldots$) if the intersection of U with every α-plane is connected and has vanishing j-th Betti numbers for $j = 1,2,\ldots,N$. Similarly, $\mathcal{O}^*_N$ is defined with 'α-plane' replaced by 'β-plane'. Denoting by $U'', U^{*''}$ the subsets of $\mathbb{PT}, \mathbb{PT}^*$ respectively associated to U by the twistor correspondence, it can be shown that

(a) U is $\mathcal{O}_1$ $\Rightarrow$ $\Gamma(U,F_n)/\mathrm{im}\,\Gamma(U,A^0_n)\to\Gamma(U,F_n) \simeq H^1(U'',\mathcal{O}(n-2))$

(b) U is $\mathcal{O}^*_1$ $\Rightarrow$ $\Gamma(U,F_n) \simeq H^1(U^{*''},\mathcal{O}(-n-2))$

(c) U is $\mathcal{O}_2$ $\Rightarrow$ $\Gamma(U,A^1_{n+})/\mathrm{im}\,\Gamma(U,A^1_n)\to\Gamma(U,A^0_{n+}) \simeq H^2(U'',\mathcal{O}(n-2))$

(The proof of (a) and (b) is in the paper mentioned above, and a more complicated proof of all three is in my dissertation.)

The last group on the right of (c) above can be shown to vanish always because of the form of U''; thus if, for example, U is an open Stein subset of $\mathbb{M}$ which is both $\mathcal{O}^*_1$ and $\mathcal{O}_2$, then inserting the above into (2.1) gives the exact sequence

$$0 \to H^1(U,T_n) \to H^1(U'',\mathcal{O}(n-2)) \to H^1(U^{*''},\mathcal{O}(-n-2)) \to H^2(U,T_n) \to 0, \quad (3.1)$$

a generalization of the twistor transform isomorphism $H^1(\mathbb{PT}_+,\mathcal{O}(n-2)) \simeq H^1(\mathbb{PT}^*_-,\mathcal{O}(-n-2))$. In particular, every point in $\mathbb{M}$ has arbitrarily small neighbourhoods U for which $H^1(U'',\mathcal{O}(n-2)) \to H^1(U^{*''},\mathcal{O}(-n-2))$ is an isomorphism for all n.

The End.

(14) On the Evaluation of Twistor Cohomology Classes

In the programme for evaluating twistor diagrams initiated by M.L.G and S.A.H much use is made of the evaluation map

$$H^n(M, \Omega^n) \to \mathbb{C} \qquad (A)$$

where M is a compact n-dimensional complex manifold and Ω^n the sheaf of holomorphic n-forms on M. (This is the map used in Serre duality, but is, by itself, a more elementary thing.) The idea is to piece together various cohomology group elements corresponding to different portions of the diagram, making much use of the "multiply map" operation

$$\bullet : H^p(\mathcal{U}, \mathcal{S}) \times H^q(\mathcal{V}, \mathcal{T}) \to H^{p+q+1}(\mathcal{U}\cup\mathcal{V}, \mathcal{S}\times\mathcal{T}) \qquad (B)$$

the aim being, in each case, to fit together the entire compact manifold (generally $\mathbb{CP}^3 \times \mathbb{CP}^3 \times \dots \times \mathbb{CP}^3$) out of $\mathcal{U}, \mathcal{V}, \dots$ and finally to make use of (A) to evaluate the completed diagram. For the simple (scalar product) diagram (f)—•—(g), where $f \in H^1(\mathbb{PT}^+, \mathcal{O}(n-2))$, $g \in H^1(\mathbb{PT}^-, \mathcal{O}(-n-2))$, M.G.E. has suggested the alternative procedure of forming the cup product

$$f \cup g \in H^2(\mathbb{PN}, \Omega^3) \qquad (C)$$

where M.L.G. – S.A.H would have

$$f \bullet g \in H^3(\mathbb{PT}, \Omega^3). \qquad (D)$$

The evaluation of (C), to obtain the complex number equal to the amplitude (f)—•—(g) can be achieved either by Dolbeault means (which, however, does not make its essentially holomorphic character obvious (to me)) or by a Čech (or abstract) procedure. In fact, the evaluation of (C) and (D) yield the same (correct) answer.

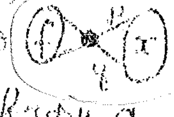
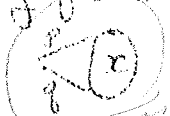
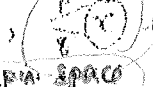
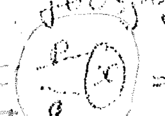
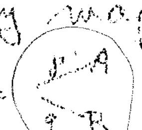
This alternative method suggests the use of an obvious generalization, called V

$$H^p(M, \Omega^n) \xrightarrow{V} \mathbb{C}, \qquad (E)$$

where M is an n-dimensional complex manifold, now not necessarily compact, and where V is a compact $(n+p)$-real-dimensional submanifold of M (or, more generally, an $(n+p)$-cycle in M) $(0 \le p \le n)$. Again this has an obvious Dolbeault interpretation (not immediately obviously holomorphic) or a Čech (or abstract nonsense) interpretation obtained by using $0 \to \mathbb{C} \to \Omega^0 \to \Omega^1 \to \ldots \to \Omega^n \to 0$ to derive a sequence (non-exact) of maps

$$H^p(M,\Omega^n) = H^p(M, d\Omega^{n-1}) \to H^{p+1}(M, d\Omega^{n-2}) \to \ldots \qquad (F)$$
$$\ldots \to H^{p+n-1}(M, d\Omega^0) \to H^{p+n}(M,\mathbb{C}) \xrightarrow{V} \mathbb{C}$$

the last map being ordinary (Čech) cohomology evaluation on the cycle V. This incorporates two special cases: (i) good old-fashioned contour integration (given when $p=0$ and $V=$ the contour); (ii) the original evaluation map (A) (given when $p=n$ and $V = M$ (compact)).

One powerful advantage of this is that it enables us to treat cases when M is non-compact. (Previously I had had to resort to using compact cohomology for (A) when M is non-compact, but this is not so transparent.) The disadvantage of course, is that we need to specify V. Two examples of the use of (E) spring to mind: (I) non-projective twistor integrals, e.g. the $\frac{1}{(2\pi i)^{2n}}\oint \ldots f(Z) \ldots d^{4}\Pi \ldots d^{2}\pi$ that fit in with the twistor particle programme. (I've checked this explicitly for the normal 1-twistor function evaluation at a field point; here $V \cong S^3$.) (II) evaluation of a twistor function (H^1) at a googly field point — which amounts effectively to 
where  is the cohomology class (H^1) which arises from a googly map $x : \mathcal{T} \to \mathcal{S}$ (ω-spin space),  being $x^{-1}\left(\frac{d^2\omega}{(\omega^0)^{n+1}(\omega^1)^{\ldots}}\right)$. We have
 =  in ordinary twistor diagrams, the line x in $\mathbb{PT}''$ being the intersection of planes A and B. Here $\mathcal{T}$ is asymptotic twistor space, which need not be compact. Deforming slightly away from the flat case, we find that a suitable V exists with $V \cong S^3 \times S^3$.

Finally, it is not hard to see that any evaluation using a multiply map can also be achieved using a cup product, the specific V for "∪" arising as the intersection of the V for "•" with $\partial\mathcal{U}$, or, homologously, with $\partial\mathcal{V}$. "∪" is more flexible, though "•" is more unique.

Roger Penrose

References: Ginsberg–Huggett in Advances in Twistor Theory
Penrose in Quantum Gravity & Googlies in A.in T.T.

(16)

The Spin-Squared Operators and $SU(n)$ Casimir Operators for an n-point system

A.S. Popovich

It is possible to define an n-point field $\phi(x_1,\dots,x_n)$, holomorphic on $M^+\times\dots\times M^+$ by means of the contour integral expression:

(1) $$\phi(x_1,\dots,x_n) = \oint \rho_{x_1\cdots x_n} f(Z^\alpha_i)\Delta\pi \quad ,$$

where $\rho_{x_1\cdots x_n}$ denotes evaluation of Z^α_i at x^a_i $(i=1,\dots,n)$, and $f(Z^\alpha_i)$ is homogeneous of degree $(-2,-2,\dots,-2)$. Assume $n \geq 2$. Let $\hat{\mathcal{D}}$ denote a holomorphic differential operator which is polynomial in the quantized generators of the Lie-algebra of the n-twistor internal symmetry group. In certain cases the action of $\hat{\mathcal{D}}$ on $f(Z^\alpha_i)$ in (1) can be interpreted as the corresponding action of an operator $\hat{D}$ on ϕ: $\hat{D}\phi(x_1\cdots x_n) = \oint \rho_{x_1\cdots x_n}\hat{\mathcal{D}} f(Z^\alpha_i)\Delta\pi$. In this note we shall record expressions for $\hat{D}$ corresponding to $\hat{\mathcal{D}} = \hat{\mathcal{S}}^2$, $\hat{\mathcal{C}}_2 := \hat{E}^i_j\hat{E}^j_i$, and $\hat{\mathcal{C}}_3 := \hat{E}^i_j\hat{E}^j_k\hat{E}^k_i$, where $\hat{\mathcal{S}}^2$ is the total spin-squared operator, and $\hat{\mathcal{C}}_2$, $\hat{\mathcal{C}}_3$ denote the quadratic and cubic Casimir operators for internal $SU(n)$. In deriving the following expressions use is made of the identity $\rho_{x_r}\hat{\omega}^{rA'} = \hat{\omega}^{rA'}\rho_{x_r} - i x^{AA'}_r \rho_{x_r}\hat{\pi}^r_A$, and the notation: $q^a_{ij} = x^a_i - x^a_j$, $R_a = \sum'_i \nabla_{ia}$, $Q^a_{ij} = \nabla^a_i - \nabla^a_j$.

Then we have:

$$\hat{S}^2 = -6\sum_{i=1}^{n}{}'\sum_{j=1}^{n}{}' x^{[a}_i \nabla^b_i R^{c]} x_{ja}\nabla_{jb}R_c \quad ,$$

$$\hat{C}_2 = -\sum_{i=1}^{n}{}'\sum_{j=1}^{n}{}'\left(q^{AA'}_{ij}q^{BB'}_{ij}\nabla_{iAB'}\nabla_{jA'B} - q^a_{ij}Q_{ija}\right) \quad ,$$

and $\hat{C}_3 =$

$$\sum_{i,j,k}\left[q^{AA'}_{b\,ij}\, q^{BB'}_{b\,jk}\, q^{CC'}_{b\,ki}\, \nabla_{iCA'}\nabla_{jA'B}\nabla_{kB'C} + q^{AA'}_{b\,ji}\, q^{BB'}_{b\,ik}\left(\nabla_{iAB'}\nabla_{kA'B} + \nabla_{jAB'}\nabla_{iA'B} - \nabla_{kAB'}\nabla_{iA'B}\right)\right]$$
$$- \tfrac{1}{2}n \sum_{i,j,a} q^{a}_{b\,ij}\, Q_{ija} \quad .$$

It is not too difficult to check that for the case $n=2$ $\hat{S}^2$ and $\hat{C}_2$ both reduce to the same expression as that given for the spin-squared operator by L.P.H. and T.R.H. in this issue (and which had been derived by M.G.E. also). The $\hat{C}_2$ operator for $n=3$ has been worked out also by Z. Perjés. Expressions for $\hat{C}_r$ $r\geq 4$ can likewise be computed, but are rather complicated and so will not be given here. It should be noted that the dimensionality of twistor space requires that $\hat{C}_r$ be constructable from $\hat{C}_2$, $\hat{C}_3$, and $\hat{C}_4$ when $r\geq 5$.

On Non-Projective Twistor Cohomology

Let $\mathcal{H}(n)$ denote the sheaf of germs of holomorphic functions on $\mathbb{T}-\{0\}$ homogeneous of degree n. Equivalently, $\mathcal{H}(n)$ may be defined by the exact sequence

$$0 \longrightarrow \mathcal{H}(n) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O} \longrightarrow 0 \qquad *$$
$$f \longmapsto Z.\frac{\partial f}{\partial Z} - nf$$

Let $\pi : \mathbb{T}-\{0\}$ $\quad \mathbb{PT}$ be the usual projection and let D be an open subset of $\mathbb{PT}$. We ask: How is $H^p(D, \mathcal{O}(n))$ related to $H^p(\tilde{D}, \mathcal{H}(n))$ where $\tilde{D} = \pi^{-1}(D)$? This is relevant to twistor quantization etc., i.e. how do we interpret $-\frac{1}{2}Z.\partial/\partial Z - 1$ as the helicity operator? To calculate $H^p(\tilde{D}, \mathcal{H}(n))$ we proceed as follows.

The short exact sequence $*$ gives rise to the long exact sequence on direct images

$$0 \longrightarrow \pi_*\mathcal{H}(n) \longrightarrow \pi_*\mathcal{O} \longrightarrow \pi_*\mathcal{O} \longrightarrow \pi^1_*\mathcal{H}(n) \longrightarrow 0$$
$$\pi_*\mathcal{O} = \hat{\bigoplus}\, \mathcal{O}(j) \quad \longleftarrow \text{using Laurent series expansion along } \pi \longrightarrow \quad \pi_*\mathcal{O} = \hat{\bigoplus}\, \mathcal{O}(j)$$
$$\textstyle\sum a_j \longmapsto \sum (j-n)\, a_j$$

$\hat{\bigoplus}$: completed direct sum.

So $\pi_*\mathcal{H}(n) = \mathcal{O}(n) = \pi^1_*\mathcal{H}(n)$ and all higher direct images vanish. The Leray spectral sequence now gives the long exact sequence

$$0 \to H^1(D,\mathcal{O}(n)) \to H^1(\tilde{D}, \mathcal{H}(n)) \to \Gamma(D, \mathcal{O}(n)) \to H^2(D,\mathcal{O}(n)) \to H^2(\tilde{D}, \mathcal{H}(n)) \to H^1(D,\mathcal{O}(n)) \to \cdots$$

For example: $H^1(\mathbb{PT}, \mathcal{O}) = 0$ but $H^1(\mathbb{T}-\{0\}, \mathcal{H}(0)) = \mathbb{C}$.

Michael Eastwood.

(18) Three Channels for the Box Diagram

Introduction.

In this article we show that all three channels for the ϕ^4 integral do in fact survive translation into twistors. (We do not use any cohomological techniques, and nor do we blow the box diagramup.) This result refutes the ancient belief that the box diagram only has one of these channels and vindicates the hope that twistor diagrams incorporate crossing symmetry [1].

Translation Procedure.

The massless scalar ϕ^4 integral is $\int_S \phi_1 \phi_2 \phi_3 \phi_4 \, d^4x$ (1)
where S is a 4 real dimensional contour in $\mathbb{C}M$. Suppose the z.r.m. fields $\phi_1, \dots, \phi_4$ are elementary states. The first step in the translation is to choose the following twistor functions to generate these elementary states:

$$\frac{1}{\begin{smallmatrix} W & W \\ | & | \\ A & B \end{smallmatrix}} \Longleftrightarrow \phi_1, \quad \frac{1}{\begin{smallmatrix} C & D \\ | & | \\ X & X \end{smallmatrix}} \Longleftrightarrow \phi_2, \quad \frac{1}{\begin{smallmatrix} Y & Y \\ | & | \\ E & F \end{smallmatrix}} \Longleftrightarrow \phi_3, \quad \frac{1}{\begin{smallmatrix} G & H \\ | & | \\ Z & Z \end{smallmatrix}} \Longleftrightarrow \phi_4 .$$

Given $A, B, \dots, H$ in a particular position move them until one of the following conditions holds:

channel $\langle 12|34 \rangle$ ϕ_1, ϕ_2 negative frequency and ϕ_3, ϕ_4 positive
channel $\langle 13|24 \rangle$: ϕ_1, ϕ_3 negative frequency and ϕ_2, ϕ_4 positive
channel $\langle 14|23 \rangle$: ϕ_1, ϕ_4 negative frequency and ϕ_2, ϕ_3 positive

Now (i) draw in the contour $S = \mathbb{M}$ and (ii) continuously move $A, B, \dots, H$ and the contour until $A, B, \dots, H$ are in their original positions. We now have three spacetime contours, one for each channel. We use the notation $W_\alpha = (W_A, W^{A'})$, so that (1) becomes

$$\int \frac{W^A dW_A \wedge X^{A'} dX_{A'} \wedge Y^B dY_B \wedge Z^{B'} dZ_{B'} \wedge d^4x}{\rho_x \begin{pmatrix} W W & C D & Y Y & G H \\ | \; | & | \; | & | \; | & | \; | \\ A B & X X & E F & Z Z \end{pmatrix}} \qquad (2)$$

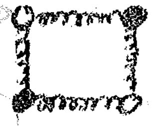
where ρ_x restricts all the twistors to go through x^a. This integral is over a contour in the 8 complex dimensional space $\mathbb{B}$. $\mathbb{B}$ is a bundle over $\mathbb{C}M$ with fibre $\{(W_A, X_{A'}, Y_A, Z_{A'}) \in (\mathbb{CP}^1)^4\}$.

We define two subbundles of $\mathbb{B}$.

$\mathbb{I} =: \{(W_A, X_{A'}, Y_A, Z_{A'}, x^a) \in \mathbb{B} : W_A \propto Y_A \text{ and } X_{A'} \propto Z_{A'}\}$

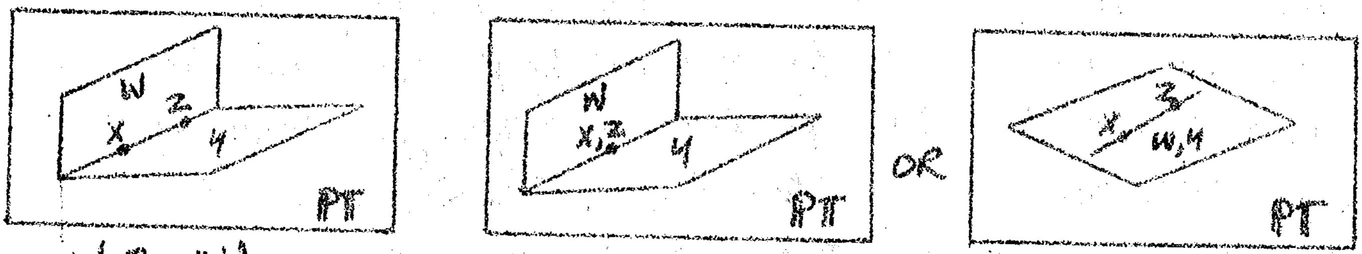
$\mathbb{U} =: \{(W_A, X_{A'}, Y_A, Z_{A'}, x^a) \in \mathbb{B} : W_A \propto Y_A \text{ or } X_{A'} \propto Z_{A'}\}$

The next step in the translation is the projection p from $\mathbb{B}$ to

 $= \left\{ \genfrac{}{}{0pt}{}{W}{1} ; \genfrac{}{}{0pt}{}{1}{X} ; \genfrac{}{}{0pt}{}{Y}{1} ; \genfrac{}{}{0pt}{}{1}{Z} : \genfrac{}{}{0pt}{}{W}{X} = \genfrac{}{}{0pt}{}{W}{Z} = \genfrac{}{}{0pt}{}{Y}{X} = \genfrac{}{}{0pt}{}{Y}{Z} = 0 \right\}$

given by $p(W_A, X_{A'}, Y_A, Z_{A'}, x^a)$

$= (W_A, -ix^{AA'}W_A ; ix^{AA'}X_{A'}, X_{A'} ; Y_A, -ix^{AA'}Y_A ; ix^{AA'}Z_{A'}, Z_{A'})$

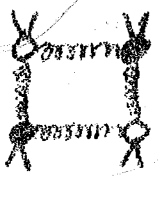


$p(\mathbb{B}-\mathbb{U})$ $\qquad$ $p(\mathbb{U}-\mathbb{I})$

This projection is a biholomorphism on $\mathbb{B}-\mathbb{I}$, but consider

$p(\mathbb{I}) = \left\{ \genfrac{}{}{0pt}{}{W}{1} ; \genfrac{}{}{0pt}{}{1}{X} ; \genfrac{}{}{0pt}{}{Y}{1} ; \genfrac{}{}{0pt}{}{1}{Z} : \genfrac{}{}{0pt}{}{W}{1} \propto \genfrac{}{}{0pt}{}{Y}{1}, \genfrac{}{}{0pt}{}{1}{X} \propto \genfrac{}{}{0pt}{}{1}{Z}, \text{ and } \genfrac{}{}{0pt}{}{W}{X} = 0 \right\}$ which (as was realised after a question from Mike Eastwood) is singular in 

$\mathbb{I}$ has 6 complex dimensions and is a blowup of $p(\mathbb{I})$ which only has 5. Therefore $\mathbb{B}$ is a blowup of . The differential form in (2) is non-zero on $\mathbb{I}$ so when we blow $\mathbb{I}$ down this form will be singular on $p(\mathbb{I})$. Therefore if a contour in $\mathbb{B}$ is to survive the translation it must avoid $\mathbb{I}$, so that the restriction of the form to this contour remains nonsingular. The last step in obtaining the box diagram is to go from the space  $-p(\mathbb{I})$ to the space  using Cauchy's theorem $f(0) = \frac{1}{2\pi i}\oint \frac{f(z)\,dz}{z}$ once for each subspace of the form $\left\{\genfrac{}{}{0pt}{}{W}{X} = 0\right\}$.

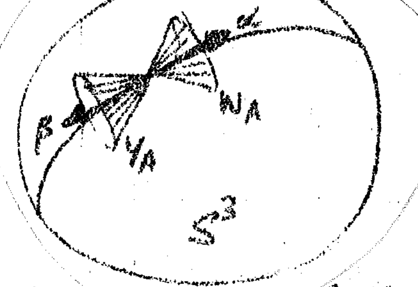
Suppose γ is a contour in  $-p(\mathbb{I})$. Then this last stage maps γ to the contour $84(\gamma)$ in

, where δ is Leray's cobord map. If γ is to survive this last step we have to show that $\delta^4(\gamma)$ is not homologous to zero. This can be done properly using Leray sequences. They get a little involved, however, so it is much clearer to actually construct $\delta^4(\gamma)$. We do this in the last section.

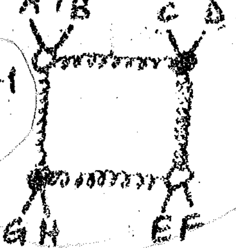
Proof that the three contours avoid $\mathbb{I}$.

Easy case – channel $\langle 13|24\rangle$:

We can start with a particular location for ABC...H and then move them away from that location (keeping the lines AB and EF in $\mathbb{PT}^+$ for negative frequency, and the lines C,D and G,H in $\mathbb{PT}^-$, for positive frequency). Think of M as the Einstein universe identified: $S^3 \times S^1$. Choose $A = E$, $B = F \in \mathbb{PT}^+$ in such a way that the Robinson congruence α defined by A points (when projected spatially) along a right rotation of S^3 and that β defined by B along the opposite rotation of S^3. Similarly, choose $\bar{C} = \bar{G}$, $\bar{D} = \bar{H} \in \mathbb{PT}^+$ so that C and D define γ and δ whose spatial projections are also opposite right rotations on S^3.



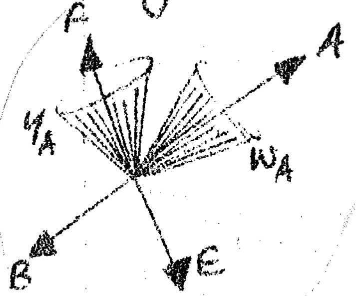
We need a contour for the space $\to p^{-1}$ avoiding $\mathbb{I}$. Actually, in this case we can even avoid $\mathbb{U}$.



For each x^a we need $(W_A, X_{A'}, Y_A, Z_{A'})$ such that the flagpoles of $W_A, X_{A'}, Y_A$, and $Z_{A'}$ avoid the Robinson congruences α, β, γ, and δ respectively, and to miss $\mathbb{U}$ we also need $W_A \nparallel Y_A$ and $X_{A'} \nparallel Z_{A'}$. Easy. Let the W_A flagpole (when projected into S^3) execute a narrow cone about the α-direction and let the Y_A flagpole do the same (independently) about the β-direction (see picture). We get $S^1 \times S^1$'s worth of contour at each point of the S^3. Treat $X_{A'}$ and $Z_{A'}$ similarly (and independently). Since these Robinson congruences are constant in Einstein (S^1) time we have $S^3 \times S^1$'s worth of $S^1 \times S^1 \times S^1 \times S^1$ contours avoiding $\mathcal{U}$. This is the required 8 real dimensional contour.

Harder case—channel $\langle 12|34 \rangle$:

Here $A, B, \bar{G}$, and $\bar{H}$ are in $\mathbb{PT}^+$ and $E, F, \bar{C}$, and $\bar{D}$ are in $\mathbb{PT}^-$. Now we cannot chose $A = E$ because they correspond to a right rotation and a left rotation respectively. So although

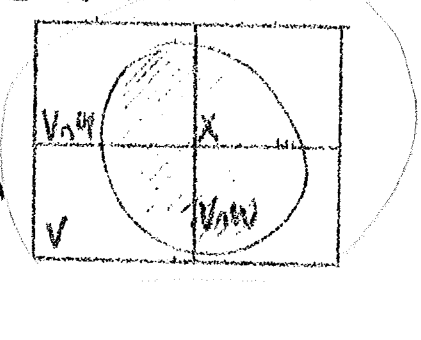


we can make W_A and Y_A execute narrow cones about the directions on S^3 corresponding to A and F respectively (as before, remember F was equal to B) Y_A and W_A will encounter one another when these directions get close. This will happen near a great circle on the S^3. Similarly for $X_{A'}$ and $Z_{A'}$ – they will encounter one another near another great circle on the S^3. So all we have to do is make sure that these great circles do not intersect. Then at no stage do we have $W_A \propto Y_A$ and $X_{A'} \propto Z_{A'}$, so that $\mathcal{I}$ is avoided for all x^a on the S^3, as required.

Notice that the channels $\langle 12|34 \rangle$ and $\langle 14|23 \rangle$ do intersect $\mathcal{U}$.

Getting from  to .

Suppose γ is a contour in  $- p(\mathcal{I})$ and suppose for now that it also avoids $p(\mathcal{U})$ (as does the $\langle 13|24 \rangle$ contour). A point on γ is given by two points and two planes as drawn in the picture of $p(B - \mathcal{U})$. Fix the planes W and Y and consider the neighbourhood of X in any plane V through X not containing Y (see picture). The planes W and Y never coincide on γ so we can take X off them and impose the conditions



$\overset{W}{\underset{X}{|}} = \overset{Y}{\underset{X}{|}} = 0$ using two Cauchy integrals. Similarly for the point Z we use two more Cauchy integrals, so that we integrate over four S's in all before using γ.
Now suppose γ intersects $\rho(V)$. The above procedure still works when $X \propto Z$, but if $W \propto Y$ we have to dualise (draw the whole thing in dual twistor space and fix the planes X and Z ...). In the previous section we saw that γ never intersects $\rho(I)$, so we can always use one of these procedures.

Reference.

Stephen Huggett

1. "Crossing and Twistor Diagrams" Andrew Hodges in Advances in twistor theory

Roger Penrose

More on the Evaluation of Twistor Cohomology Classes

(?) As TN 10 was just going to press, the following obvious(!) generalization of (E) occurred to me:

$$H^p(M, d\Omega^{q-1}) \xrightarrow{V} \mathbb{C}, \qquad (G)$$

where M is complex n-dimensional and V is real (p+q)-dimensional. ($d\Omega^{q-1}$ = sheaf of holomorphic closed q-forms.) Case q = n gives (E). Proof the same.
"Application": ordinary twistor integral $\frac{1}{2\pi i}\oint f(Z)\pi_{A'}d\pi^{A'}$ viewed as $S^1 (= V)$ in $\mathbb{C}^2$, $p=0, q=1, n=2$. Roger Penrose

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A Sequence for Twistorians

Fill in the missing number:

..., 7, 9, 12, (?), 24, 36, 56, 90, ...

Roger Penrose

Some Expository Notes on Affine Bundles and Googly Photons

P.S. Green.

An affine space is, roughly speaking, a vector space without a choice of origin. Similarly an affine bundle is a vector bundle without a zero section. Since the translations of an affine space forma vector space of the same dimension, one may associate to every affine bundle its vector bundle of fibrewise translations. Two local choices of origin for an affine bundle differ by a unique local translation. It follows that the obstruction to a global choice of origin lies in H^1 with coefficients in the sheaf of germs of local translations. In as much as a choice of origin defines an isomorphism (as affine bundles) between an affine bundle and its translation bundle, an affine bundle is classified by its translation bundle and the obstruction to a zero section.

Let D be a domain in PT. The total space of an affine line bundle over D admits a volume form for which the divergence operator annihilates infinitesimal fibrewise translations if and only if the associated translation bundle is the fourth power of the universal bundle (not coincidentally the bundle of volume forms on D). Such affine line bundles are parametrized by $H^1(D, \mathcal{O}(-4))$. The volume form in question is unique up to a constant factor provided D contains a projective line.

All of the foregoing, of course, was written with the holomorphic case in mind. However, it all makes perfect sense in the topological category and then the obstruction to a zero section always vanishes. Therefore one may choose a continuous (even smooth) origin in any holomorphic affine line bundle. The complement is still a holomorphically foliated space, each of whose leaves is a punctured complex line. Under the conditions described in the previous paragraph, this complement admits a four-fold cover which is connected over each leaf. The pull-back of the special volume form to the covering space gives a googly photon.

The zero section in the foregoing may be replaced by an subset of the total space which meets each fibre in a non-empty compact contractible set. Any two such subsets are contained in a third; hence the particular choice is immaterial and only the original affine bundle is significant.

This construction also works in reverse; i.e. it may be verified that any googly photon arises from the construction described above.

Historical comment concerning affine bundles in twistor theory: M.F.A hinted at the idea as long ago as 1976, but we failed to appreciate it properly. But R.S.W. was quite explicit and clear about it in Spring 1977. That the "googly photon" (G.A.J.S.; R.S.W. TN1; R.P. TN 3/5; L.P.H. TN3) was a 4-fold cover after removing a topological "zero section" was fully appreciated then. R.P.

SPACES OF TORSION-FREE NULL GEODESICS I: DIMENSIONS ABOVE THREE

IN THE LAST TN, WE EXAMINED SPACES OF TORSION NULL GEODESICS, MEANING THE NULL GEODESICS OF ANY OLD (HOLOMORPHIC) AFFINE CONNEXION PRESERVING SOME (HOLOMORPHIC) METRIC; IT WAS ONLY BY THE INCLUSION OF TORSION THAT ONE COULD PROVE A THEOREM SAYING, ROUGHLY, THAT A SMALL DEFORMATION OF A SPACE OF NULL GEODESICS IS AGAIN A SPACE OF NULL GEODESICS. BUT IF ONE IS INTERESTED IN CONFORMAL STRUCTURES, SURELY ONE WILL ONLY WISH TO CONSIDER SPACES OF TORSION-FREE NULL GEODESICS — THAT IS, THE NULL GEODESICS OF SYMMETRIC METRIC-PRESERVING CONNEXIONS. WHAT EXTRA STRUCTURE DISTINGUISHES SUCH SPACES FROM THE OTHERS? ANSWER: A CONTACT STRUCTURE, IF WE'RE WORKING WITH "SPACETIMES" OF DIMENSION EXCEEDING 3.

A CONTACT STRUCTURE ON A COMPLEX MANIFOLD X IS A HOLOMORPHIC 1-FORM θ WITH VALUES IN A HOLOMORPHIC LINE-BUNDLE E, $\theta \in H^0(X, \mathcal{O}(T^*X \otimes E)) =: \Gamma(\Omega^1(E))$, HAVING THE PROPERTY THAT

$$\theta \wedge \underbrace{d\theta \wedge \cdots \wedge d\theta}_{\frac{n-1}{2}\ \text{FACTORS}}$$

IS A NON-VANISHING VOLUME ELEMENT (WITH VALUES IN THE LINE-BUNDLE $E^{(n+1)/2}$). IN PARTICULAR, n IS ODD AND E IS THE $[(n+1)/2]$-TH ROOT OF THE DUAL CANONICAL BUNDLE $\Lambda^n TX$. (NOTICE THAT ALTHOUGH WE CAN'T REALLY DEFINE $d\theta$, WE CAN DEFINE $d\theta$ mod θ, AND SO THE ABOVE EXPRESSION MAKES SENSE; IF ONE CHANGES THE LOCAL TRIVIALIZATION OF E BY MULTIPLYING BY A NON-VANISHING HOLOMORPHIC FUNCTION f, OUR IDEA OF WHAT $d\theta$ (AS A 2-FORM WITH VALUES IN E) MIGHT MEAN JUST GOES OVER TO

$$f^{-1}d(f\theta) = d\theta + (f^{-1}df)\wedge\theta.$$

IT IS IMPORTANT THAT θ IS ALLOWED TO BE "TWISTED" BECAUSE OTHERWISE WE'D BE LOOKING ONLY AT X'S WITH TRIVIAL CANONICAL BUNDLES, WHICH ARE RATHER USELESS FOR OUR PURPOSES.)

FIRST LET'S SEE THAT A SPACE OF TORSION-FREE NULL GEODESICS HAS A CONTACT STRUCTURE (REGARDLESS OF DIMENSIONS). IF γ IS A NULL GEODESIC OF THE TORSION-FREE AFFINE CONNECTION PRESERVING A METRIC g ON A COMPLEX MANIFOLD M, THEN A TANGENT VECTOR TO THE SPACE-OF-NULL-GEODESICS $N(M)$ IS A (JACOBI) VECTOR FIELD J ALONG γ SATISFYING

$$\begin{cases} D^2 J = R(V, J)V \\ D g(J, V) = 0 \end{cases}$$

WHERE V IS AN AUTOPARALLEL TANGENT FIELD ALONG γ AND WHERE D IS COVARIANT DIFFERENTIATION WITH RESPECT TO V; THE SECOND EQUATION STATES THAT THE "GEODESIC TO WHICH J POINTS" IS NULL. (STRICTLY SPEAKING, A TANGENT VECTOR ON $N(M)$ IS THE EQUIVALENCE CLASS OF SUCH A J, CONSIDERED MODULO FIELDS OF THE FORM $\alpha V + \beta \tau V$, WHERE τ IS AN AFFINE PARAMETER ($D\tau = 1$) AND WHERE α AND β ARE CONSTANTS.) THEN THE ASSIGNMENT $J \longmapsto g(J, V)$ IS INDEPENDENT OF POSITION ALONG γ, AND SO CAN BE THOUGHT OF AS A (TWISTED) 1-FORM ON $N(M)$; THIS FORM IS HOLOMORPHIC AND TAKES VALUES IN THE DUAL OF THE LINE BUNDLE OVER $N(M)$ WHOSE FIBRE OVER SOME POINT CONSISTS OF AUTOPARALLEL TANGENT FIELDS ALONG THE CORRESPONDING NULL GEODESIC. THE NONDEGENERACY CONDITION "$\theta \wedge d\theta \wedge \cdots \wedge d\theta \neq 0$" FOLLOWS FROM THE RELATION OF THE ABOVE "TIME-DELAY" FORM TO THE CANONICAL FORM ON THE COTANGENT BUNDLE OF M, WHICH IS, AS WE ALL KNOW, A POTENTIAL FOR A SYMPLECTIC FORM.

THE FOLLOWING CONVERSE IS TRUE IF $\dim M \geq 4$: $N(M)$ ADMITS A CONTACT STRUCTURE ONLY IF THE GEODESICS ONE IS CONSIDERING ARE THOSE OF TORSION-FREE METRIC-PRESERVING CONNECTIONS. FOR IF ONE RESTRICTS THE APPROPRIATE ROOT OF THE CANONICAL BUNDLE, IN WHICH A CONTACT FORM IS SUPPOSED TO TAKE ITS VALUES, TO THE QUADRIC OF NULL GEODESICS PASSING THROUGH ANY POINT OF M, ONE FINDS THAT THE RESULTING LINE BUNDLE IS THE INVERSE HOPF BUNDLE; AND CALCULATION SHOWS THAT THE ONLY HOLOMORPHIC 1-FORM ON A QUADRIC (WHOSE DIMENSION EXCEEDS ONE) WITH VALUES IN THIS BUNDLE IS NOUGHT, SO THE ANNIHILATOR DISTRIBUTION OF THE CONTACT FORM IS TANGENT TO EACH "CELESTIAL QUADRIC", AND THIS IMPLIES THAT THE NULL CONE OF ANY POINT IS A NULL SURFACE. THIS IN TURN CAN BE SHOWN TO IMPLY THAT THE NULL GEODESICS ARE TORSION FREE. FOR FULL DETAILS, SEE THE AUTHOR'S THESIS.

C. R. LeBrun

26

SPACES OF TORSION-FREE NULL GEODESICS II

HEAVEN WITH A COSMOLOGICAL CONSTANT

WE NOTED IN THE LAST TN THAT THE ANALYTIC FAMILY OF $\mathbb{P}_1$'S IN THE SPACE OF NULL GEODESICS FOR A COMPLEX 3-MANIFOLD WITH "TORSION CONFORMAL STRUCTURE" WHICH CONSISTS OF THE "CELESTIAL QUADRICS" IS INCOMPLETE, AND COMPLETING THIS FAMILY GIVES RISE TO AN EMBEDDING OF THE 3-FOLD IN A 4-FOLD WITH HALF-FLAT CONFORMAL STRUCTURE IN SUCH A WAY AS TO REALISE THE CONFORMAL METRIC AS THE INDUCED ONE AND SUCH THAT THE TORSION OF THE ORIGINAL STRUCTURE IS CANONICALLY RELATED (IN A LINEAR FASHION) TO THE "EXTRINSIC CURVATURE" (SECOND FUNDAMENTAL FORM) OF THE REALISED 3-FOLD; IT TURNS OUT, IN PARTICULAR, THAT THE TORSION VANISHES JUST WHEN THE HYPERSURFACE IS UMBILIC (CONFORMALLY TOTALLY GEODESIC). NOW WHEN THE TORSION VANISHES, WE SAW IN ST-ENG(I) THAT THE SPACE OF NULL GEODESICS CARRIES A "TIME DELAY" CONTACT FORM, AND THIS FORM HAS THE PROPERTY THAT ITS ANNIHILATOR DISTRIBUTION IS TANGENT TO EACH "CELESTIAL QUADRIC" — AND THESE ARE PRECISELY ALL OF THE $\mathbb{P}_1$'S IN THE COMPLETE 4-PARAMETER FAMILY HAVING THIS PROPERTY, AT LEAST LOCALLY. HOWEVER, A GENERALIZATION OF THE NON-LINEAR GRAVITON CONSTRUCTION DISCOVERED INDEPENDENTLY BY RICHARD WARD & NIGEL HITCHIN (AND, UNFORTUNATELY, PUBLISHED BY NEITHER; VARIOUS DETAILS REMAINED FOR W.T. TO WORK OUT) SHOWS THAT A CONTACT FORM ON A TWISTOR SPACE CORRESPONDS PRECISELY TO AN EINSTEIN METRIC WITH NON-ZERO COSMOLOGICAL CONSTANT; THE DEFINITION OF THE METRIC IS SUCH THAT IT HAS A SIMPLE POLE AT POINTS WHOSE CELESTIAL SPHERES LIE TANGENT TO THE CONTACT FORM'S ANNIHILATOR. HENCE:

EVERY 3-DIMENSIONAL HOLOMORPHIC CONFORMAL STRUCTURE IS THE UMBILIC CONFORMAL INFINITY FOR EXACTLY ONE CONFORMALLY HALF-FLAT EINSTEIN 4-FOLD WITH COSMOLOGICAL CONSTANT Λ.

THIS CAN BE FIXED UP TO MAKE THE ANALOGOUS STATEMENT FOR REAL-ANALYTIC POSITIVE DEFINITE METRICS (ON BOTH 3 & 4-FOLD), BUT THE COSMOLOGICAL CONSTANT MUST BE NEGATIVE.

Claude LeBrun

(THE INTERESTED READER IS REFERRED TO THE AUTHOR'S THESIS, OR JUST LOOK WITH YOUR EYES CLOSED)

Introduction: The blowing up technique [1] can be used to generate explicit deformations of subspaces of PT obtained by removing a line from it. There are two different approaches to generating infinitesimal deformations of such subspaces of PT. The first one consists in calculating the group $H^1(PT-P^1,\Theta)$, elements of which are infinitesimal deformations of $PT-P^1$. While the second approach involve a homomorphism:

$$H^1(PT-P^1,\mathcal{O}(2)) \longrightarrow H^1(PT-P^1,\Theta)$$

arising from the non-linear graviton construction.

The blowing up of a line of PT consists in replacing the submanifold P^1 by $P^1\times P^1$ to obtain a new manifold $B = (PT-P^1)\cup(P^1\times P^1)$ equipped with a projection map $\pi: B \longrightarrow PT$ such that:

1 $$\pi: B-\pi^{-1}(P^1) \longrightarrow PT-P^1$$ is

an isomorphism. The blown up space B is better defined as a hypersurface of $PT\times P^1$ by:

2 $$B = \left\{ (Z^\alpha,\lambda^A) \in PT\times P^1 \;\middle|\; f = \lambda^A L_{A\alpha} Z^\alpha = 0 \right\}$$

where Z^α and λ^A are homogeneous coordinates on PT and P^1; and where P^1 is defined by $L_{A\alpha}Z^\alpha = 0$ (see [1]).

The essential point here is the isomorphism (1), which induces injective maps of sheaves:

3 $$S_B(a,b)\Big|_{B-\pi^{-1}(P^1)} \longrightarrow S_{PT}(a+b)\Big|_{PT-P^1}$$

Where a and b denote the twists of elements of S in Z^α and λ^A, while the twist (a+b) arises from the fact that $\lambda^A \sim L^A{}_\alpha Z^\alpha$ on $B-\pi^{-1}(P^1)$; S is for the moment essentially an arbitrary analytic sheaf. Therefore we have homomorphisms:

4 $$H^1(B,S_B(a,b)) \xrightarrow{\rho_{B-\pi^{-1}(P^1)}} H^1(B-\pi^{-1}(P^1),S_B(a,b)) \longrightarrow H^1(PT-P^1,S_{PT}(a+b))$$

where ρ is the restriction map.

<u>The group $H^1(PT-P^1,\Theta)$</u>: The study of the group $H^1(PT-P^1,\Theta)$ can be achieved by first calculating the groups $H^1(B,\Theta_B(n,-n))$, where $\Theta_B(n,-n)$ are sheaves, of holomorphic cross-sections of certain twisted vector bundles on B, defined by the following exact sequences:

5
$$\begin{tikzcd}[column sep=small]
 & & & 0 \arrow[d] & & & & 0 \arrow[d] & & \\
 & & 0 \arrow[r] & S(n-1,-n-1) \arrow[rrrr, "f"] \arrow[dd] & & & & S(n,-n) \arrow[r, "\rho_{f=0}"] \arrow[dd] & \Theta_B(n,-n) \arrow[r] & 0 \\
 & & & & 0 & & & & 0 & \\
 & & & \Theta(n-1,-n-1) \arrow[ur] \arrow[dddd] & & & & \Theta(n,-n) \arrow[ur] \arrow[dddd] & & \\
 & & \Theta^\alpha(n,-n-1)\oplus\Theta^A(n-1,-n) \arrow[ur] & & & & \Theta^\alpha(n+1,-n)\oplus\Theta^A(n,1-n) \arrow[ur] & & & \\
 & \Theta(n-1,-n-1)\oplus\Theta(n-1,-n-1) \arrow[ur] & & & & \Theta(n,-n)\oplus\Theta(n,-n) \arrow[ur] & & & & \\
 0 \arrow[ur] & & & & 0 \arrow[ur] & & & & & \\
 & & & \Theta(n,-n) \arrow[d] & & & & \Theta(n+1,1-n) \arrow[d] & & \\
 & & & 0 & & & & 0 & &
\end{tikzcd}$$

(28)

All the other sheaves of the above diagram are defined on $PT\times P^1$. The sheaf $\Theta(n,-n)$ is obtained by tensoring the sheaf Θ, of germs of holomorphic vector fields on $PT\times P^1$, by $\mathcal{O}(n,-n)$; while $S(n,-n)$ is similarly obtained from the sheaf S of germs of holomorphic vector fields, on $PT\times P^1$, parallel to the hypersurface $f=0$. i.e. $S\subset\Theta$, and elements of S satisfy $\langle v,df\rangle=0$. (see [2] and [3]) Applying the map (3) to the sheaves Θ we obtain the following isomorphism:

6 $$\Theta_{PT}\big|_{PT-P^1}\simeq\int_{B-\pi^{-1}(P^1)}\sum_n\Theta_B(n,-n)$$

Therefore we have an isomorphism:

7 $$H^1(PT-P^1,\Theta)\simeq\int_{B-\pi^{-1}(P^1)}\sum_n H^1(B,\Theta_B(n,-n))$$

Calculation of the groups $H^1(B,\Theta_B(n,-n))$, using the above diagram is rather lengtly; details can be found in [3]. The main results are that H^1 is non-vanishing only for $n\geq 1$. (In fact for $n=0$ we get back to a result obtained in [2]; that B is rigid) These results don't however guarantee that the restriction to $B-\pi^{-1}(P^1)$ is non-vanishing, because this restriction can map a cocycle of $H^1(B,\Theta_B(n,-n))$ in a coboundary of $H^1(B-\pi^{-1}(P^1),\Theta_B(n,-n))$, which happens in fact in the case $n=1$. For $n>1$ the groups $H^1(B-\pi^{-1}(P^1),\Theta_B(n,-n))$ don't generaly vanish and this prove that $H^1(PT-P^1,\Theta)\neq 0$.

In the case $n=2$ a representative is easily obtained for $H^1(B,\Theta_B(2,-2))$. The long exact cohomology sequences associated with diagram (5) lead to a surjective map: $H^1(PT\times P^1,S(2,-2))\longrightarrow H^1(B,\Theta_B(2,-2))$ where the first group is isomorphic to $H^1(PT\times P^1,\Theta(2,-2))$. Using these facts we find the following representative: $\dfrac{Z^\alpha Z^\beta Z^\gamma \breve{E}^\delta{}_{\alpha\beta\gamma}}{(P_A\lambda^A)(Q_A\lambda^A)}\dfrac{\partial}{\partial Z^\delta}$, where $\breve{E}^\delta{}_{\alpha\beta\gamma}$ is trace free and satisfy $L_{A\alpha}\breve{E}^\alpha{}_{\beta\gamma\delta}=0$. Then the restriction to $B-\pi^{-1}(P^1)$ gives, according to (7), a representative:

8 $$f^\delta(Z^\alpha)\frac{\partial}{\partial Z^\delta}=\frac{Z^\alpha Z^\beta Z^\gamma E^\delta{}_{\alpha\beta\gamma}}{(P_\alpha Z^\alpha)(Q_\alpha Z^\alpha)}\frac{\partial}{\partial Z^\delta}$$

for a subgroup, that we denote by $H^1_1(PT-P^1,\Theta)$, of elements of $H^1(PT-P^1,\Theta)$, which are singular of order 1 over P^1. In fact we can write: $H^1_{n-1}(PT-P^1,\Theta)\simeq\int_{B-\pi^{-1}(P^1)}H^1(B,\Theta_B(n,-n))$.

<u>The non-linear graviton map</u>: The second approach arises from the non-linear graviton conditions [4]:

$$\hat{\omega}^A=\omega^A+f^A(\omega^B,\pi_{B'})\quad,\qquad\hat{\pi}_{A'}=\pi_{A'}$$

with $\det\left(\epsilon^B_A+\dfrac{\partial f^B}{\partial\omega^A}\right)=1$

Elements of $H^1(PT-P^1,\Theta)$ which satisfy these conditions are those for which $f^A=\partial F/\partial\omega_A$ and $f_{A'}=0$; where F is a homogeneous function of degree 2. So $\partial/\partial\omega_A$ define a homomorphism:

9 $$H^1(PT-P^1,\mathcal{O}(2))\xrightarrow{\partial/\partial\omega^A}H^1(PT-P^1,\Theta)$$

According to (3) and (4) we must have the following isomorphisms:

10 $$\Theta_{PT}(2)\big|_{PT-P^1}\simeq\int_{B-\pi^{-1}(P^1)}\sum_{n\geq 2}\Theta_B(n+2,-n)$$

and

11 $$H^1(PT-P^1,\Theta(2))\simeq\int_{B-\pi^{-1}(P^1)}\sum_{n\geq 2}H^1(B,\Theta_B(n+2,-n))$$

On B the sheaf $\Theta_B(n+2,-n)$ is defined by the following exact sequence:

12 $$0\longrightarrow\Theta_{PT-P^1}(n+1,-n-1)\xrightarrow{\lambda^A L_{A\alpha}Z^\alpha}\Theta_{PT-P^1}(n+2,-n)\xrightarrow{\rho_B}\Theta_B(n+2,-n)\longrightarrow 0$$

where n refers to the order of singularity $n-1$ and the number 2 to the helecity -2 of the field [1].

From the long exact cohomology sequence associated to (12) we obtain that $H^1(B, \mathcal{O}_B(n+2,-n))$ is non-vanishing only for $n \geq 2$ and then given by the exact sequence:

13 $$0 \to H^1(B, \mathcal{O}_B(n+2,-n)) \to H^1(PT \times P^1, \mathcal{O}(n+1,-n-1)) \to H^1(PT \times P^1, \mathcal{O}(n+2,-n)) \xrightarrow{\rho_B} H^1(B, \mathcal{O}_B(n+2,-n)) \to 0$$

So $\dim_{\mathbb{C}} H^1(B, \mathcal{O}_B(n+2,-n)) = \frac{(n+3)(n+4)(2n-5)}{6} + 10$

Let us take a representative for $H^1(PT \times P^1, \mathcal{O}(n+2,-n))$ as follows:

$$\frac{\overbrace{Z^{\alpha} \cdots Z^{\gamma}}^{n+2} E_{\alpha \cdots \gamma B \cdots D} \lambda^{B} \cdots \lambda^{D}}{(P_A \lambda^A)^{n-1} (Q_A \lambda^A)^{n-1}}$$

which becomes when the numerator is developed in terms of $(P_A \lambda^A)$ and $(Q_A \lambda^A)$ the following expression:

14 $$\sum_{m=1}^{n-1} \frac{\overbrace{Z^{\alpha} \cdots Z^{\gamma}}^{n+2} E_{m\alpha \cdots \gamma}}{(P_A \lambda^A)^m (Q_A \lambda^A)^{n-m}}$$

The last map of the sequence (13) is surjective, then elements of $H^1(B, \mathcal{O}_B(n+2,-n))$ arise from the restriction of (14) to B. Then the homomorphism (11) provides an appropriate representative for the subgroup $H^1_{n-1}(PT-P^1, \mathcal{O}(2))$ of $H^1(PT-P^1, \mathcal{O}(2))$, elements of which are singular on P^1 at the order n-1. This representative is obtained from (14) by replacing λ^A by $L^A{}_\alpha Z^\alpha$ in it. According to (11), $H^1(PT-P^1, \mathcal{O}(2)) \simeq \sum_{n \geq 2} H^1_{n-1}(PT-P^1, \mathcal{O}(2))$ and a general representative will be given by:

15 $$F = \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \frac{\overbrace{Z^{\alpha} \cdots Z^{\gamma}}^{n+2} E_{nm\alpha \cdots \gamma}}{(P_\alpha Z^\alpha)^m (Q_\alpha Z^\alpha)^{n-m}}$$

Now we can apply the homomorphism (9) and obtain a representative for a subgroup of $H^1(PT-P^1, \mathcal{O})$, corresponding to left-handed gravitons.

<u>Some finite deformations of $PT-P^1$</u>: Finite deformations can be obtained from (15) as solution of:

16 $$\frac{d\omega^A}{d\lambda} = \frac{\partial F(\omega^B, \pi_{B'})}{\partial \omega_A}$$

where λ is a parameter on the integral curves of the vector field $\frac{\partial F}{\partial \omega_A} \frac{\partial}{\partial \omega^A}$

Solutions of (16) can be found only for some specific forms of F, e.g. when it depends on ω^A only through a global homogeneous polynomial of degree 2, preserved under the deformation. (see [4]) The case where this polynomial is of the form $q = (o_A \omega^A)(o^{A'} \pi_{A'})$ is easy to handle, the solution of (16) being:

$$\omega^A = \omega^A_0 + \lambda\, \partial F / \partial \omega_A$$

A general technique, to obtain the associated metric, is available in the appendix 2 of [4]. The metric takes the general form:

$$g_{ab} = \eta_{ab} - K_a A_b - K_b A_a$$

which is the familiar form of p.p. waves. $K_a = o_A o_{A'}$ is a killing vector, while A_a is a potential for an A.S.D. Maxwell field arising from:

$$\pi^{A'} \nabla_{AA'} h = - A_{AA'}(x^b) \pi^{A'}, \quad \text{where } \frac{\partial F}{\partial \omega_A} = o^A (o^{A'} \pi_{A'}) \left(h(x^a, \pi_{A'}) - \hat{h}(x^a, \pi_{A'}) \right)$$

Let us consider examples, corresponding to deformations of PT-I, which generate self-dual solutions of Einstein's equations singular* on the null cone at infinity. A case corresponding to the first order of singularity

(* in the linear approximation)

is already known, having been obtained by George Sparling.
For this case:

$$F = \frac{(\omega^A o_A)^4}{(o^{A'}\pi_{A'})(\iota^{A'}\pi_{A'})}$$

Then $\dfrac{\partial F}{\partial \omega_A} = \dfrac{4(\omega^A o_A)^3 o^A}{(o^{A'}\pi_{A'})(\iota^{A'}\pi_{A'})}$, and $h = \dfrac{-4i(o^{A'}\pi_{A'})(ix^{AA'}o_A\iota_{A'})^3}{(\iota^{A'}\pi_{A'})}$

$$A_{AA'}(x^b) = -12(ix^{AA'}o_A\iota_{A'})^2 o_A o_{A'}$$

And if we define $\begin{pmatrix} v & X \\ Y & u \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix}$

The metric is: $ds^2 = 2(du\,dv - dX\,dY - \mu X^2 dv^2)$ where μ is a constant.

At the second order of singularity we can take: $F = \dfrac{(\omega^A o_A)^5}{(o^{A'}\pi_{A'})(\iota^{A'}\pi_{A'})}\left\{\dfrac{1}{(\iota^{A'}\pi_{A'})} + \dfrac{1}{(o^{A'}\pi_{A'})}\right\}$

Then $A_{AA'}(x^b) = -20i(x^{AA'}o_A\iota_{A'})^2\left\{3(x^{AA'}o_A o_{A'}) - (x^{AA'}o_A\iota_{A'})\right\}$

And the metric is: $ds^2 = 2(du\,dv - dX\,dY - \nu X^2(3v - X)dv^2)$, where ν is a constant

As a different and last example let us consider a deformation of PT minus a P^1, intersecting I, and take: $F = \dfrac{(\pi_{A'}\iota^{A'})^4}{(o_A\omega^A)(o^{A'}\pi_{A'})}$

Then $h = -i\left\{\dfrac{(\iota^{A'}\pi_{A'})^2}{(ix^{AA'}o_A o_{A'})^2(o^{A'}\pi_{A'})^2} + \dfrac{2(ix^{AA'}o_A\iota_{A'})}{(ix^{AA'}o_A o_{A'})^3(o^{A'}\pi_{A'})}\right\}$

$$A_{AA'}(x^b) = \frac{6i(x^{AA'}o_A\iota_{A'})o_A\iota_{A'}}{(x^{AA'}o_A o_{A'})^4}$$

And the metric is: $ds^2 = 2(du\,dv - dX\,dY - \kappa\frac{X}{v^4}\,dv\,dX)$

Finally, there is a conjecture (see P. Tod ... in person) according to which the Sparling-Tod metric and Eguchi-Hansen metric, given in [5], should be obtainable from:

$$F = \frac{(\pi_{A'}E^{A'})^4}{(P_\alpha Z^\alpha)(Q_\alpha Z^\alpha)} \quad \text{and} \quad F = \frac{(\pi_{A'}E^{A'})^2(\pi_{A'}F^{A'})}{(P_\alpha Z^\alpha)(Q_\alpha Z^\alpha)}$$

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