EULERIAN POLYNOMIALS AND THE g-INDICES OF YOUNG TABLEAUX

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Abstract. In this paper, we introduce k-Young tableaux and their g-indices. We first present certain expansions of $(c(x)D)^n$ in terms of inversion sequences as well as k-Young tableaux, where $c(x)$ is a smooth function in the indeterminate x and D is the derivative with respect to x. By studying the connections between k-Young tableaux and standard Young tableaux, we then present combinatorial interpretations of Eulerian polynomials, second-order Eulerian polynomials, and André polynomials in terms of standard Young tableaux.

1. Introduction

1.1. Notation and preliminaries.

The *Eulerian polynomials* $A_n(x)$ can be defined by the summation formula

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{1-x} = \sum_{k=0}^{\infty} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}. \tag{1.1}$$

It is well known (see [12, 14] and references therein) that

$$A_n(x) = \sum_{i=0}^{n} \left\langle {n \atop i} \right\rangle x^i = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{des}\,(\pi)},$$

where $\left\langle {n \atop i} \right\rangle$ is called *Eulerian number* and $\mathfrak{S}_n$ is the set of all permutations of the set $[n]=\{1,2,\ldots,n\}$. For $\pi=\pi(1)\pi(2)\cdots\pi(n)\in\mathfrak{S}_n$, the number of *descents* of π is defined by $\mathrm{des}\,(\pi)=\#\{i\in[n]\mid \pi(i)>\pi(i+1)\}$, where we set $\pi(n+1)=0$.

Let $[n]_k=\{1^k,2^k,\ldots,n^k\}$ be a multiset, where each $i\in[n]$ appears k times. A *k-Stirling permutation* of order n is a permutation of the multiset $[n]_k$ such that for each i, $1\leqslant i\leqslant n$, all entries between any two occurrences of i are at least i. Let $\mathcal{Q}_n(k)$ be the set of all k-Stirling permutations of order n. When $k=2$, the set $\mathcal{Q}_n(k)$ reduces to $\mathcal{Q}_n$, which is the set of ordinary Stirling permutations of the multiset $[n]_2$. For $\sigma=\sigma_1\sigma_2\cdots\sigma_{kn}\in\mathcal{Q}_n(k)$, an index $i\in[kn]$ is a *descent* of σ if $\sigma_i>\sigma_{i+1}$ or $i=kn$. Let $\mathrm{des}\,(\sigma)$ be the number of descents of σ. The *k-order Eulerian polynomials* are defined by

$$C_n(x;k)=\sum_{\sigma\in\mathcal{Q}_n(k)} x^{\mathrm{des}\,(\pi)}.$$

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Following [10, Lemma 1], the polynomials $C_n(x;k)$ satisfy the recurrence relation

$$C_{n+1}(x;k)=(kn+1)xC_n(x;k)+x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}C_n(x;k),\ C_0(x;k)=1. \tag{1.2}$$

In particular, $C_n(x;1)=A_n(x)$. When $k=2$, the polynomials $C_n(x;k)$ reduce to the *second-order Eulerian polynomials* $C_n(x)$ (see [6, 14]). In [4], Carlitz found that

$$\sum_{k=0}^{\infty}\left\{{n+k \atop k}\right\}x^k=\frac{C_n(x)}{(1-x)^{2n+1}},$$

where $\left\{{n \atop k}\right\}$ are the *Stirling numbers of the second kind*, i.e., the number of set partitions of $[n]$ with k blocks.

Consider the polynomials $F_n:=F_n(x;\alpha,\beta,a,b,c)$ defined as follows:

$$\left(\frac{a+bx+cx^2}{(1-x)^{\alpha}}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\frac{1}{(1-x)^{\beta}}=\frac{F_n}{(1-x)^{n+n\alpha+\beta}}.$$

It is routine to verify that $F_0=1$ and for $n\geqslant 0$, one has

$$F_{n+1}=(n+n\alpha+\beta)(a+bx+cx^2)F_n+(a+bx+cx^2)(1-x)\frac{\mathrm{d}}{\mathrm{d}x}F_n. \tag{1.3}$$

Comparing (1.2) with (1.3), we get the following result.

Proposition 1.1. *Let k be a nonnegative integer. For $n\geqslant 1$, we have*

$$\left(\frac{x}{(1-x)^k}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\frac{1}{1-x}=\frac{C_n(x;k+1)}{(1-x)^{n+kn+1}}. \tag{1.4}$$

In this paper, we always let $c:=c(x)$ and $f:=f(x)$ be two smooth functions in the indeterminate x, and set $D=\frac{\mathrm{d}}{\mathrm{d}x}$. The expansions of $(cD)^nf$ have been studied as early as 1823 by Scherk [1, Appendix A]. He found that

$$(xD)^n=\sum_{k=0}^{n}\left\{{n \atop k}\right\}x^kD^k.$$

In the sequel, we adopt the convention that $\mathbf{f}_k=D^kf$ and $c_k=D^kc$. Note that $\mathbf{f}_0=f$, $c_0=c$, $(cD)f=(c)\mathbf{f}_1$, $(cD)^2f=(cc_1)\mathbf{f}_1+(c^2)\mathbf{f}_2$. For $n\geqslant 1$, we define

$$(cD)^nf=\sum_{k=1}^{n}A_{n,k}\mathbf{f}_k. \tag{1.5}$$

Note that $A_{n,k}=A_{n,k}(c,c_1,\ldots,c_{n-k})$ is a function of $c,c_1,\ldots,c_{n-k}$. In particular, $A_{1,1}=c$, $A_{2,1}=cc_1$ and $A_{2,2}=c^2$. By induction, it is easy to verify that $A_{n+1,1}=cDA_{n,1}$, $A_{n,n}=c^n$ and for $2\leqslant k\leqslant n$, we have

$$A_{n+1,k}=cA_{n,k-1}+cDA_{n,k}. \tag{1.6}$$

The numbers appearing in $A_{n,k}$ as coefficients can be found in [16, A139605]. We refer the reader to [3, 15] for various examples on the expansions of $(cD)^n$.

1.2. Motivation and the organization of the paper.

In 1973, Comtet obtained the following result by induction.

Proposition 1.2 ([8]). *Let $A_{n,k}$ be defined by* (1.5). *For $1 \leqslant k \leqslant n$, we have*

$$A_{n,k} = \frac{c}{k!} \sum (2-k_1)(3-k_1-k_2)\cdots(n-k_1-k_2-\cdots-k_{n-1}) \frac{c_{k_1}}{k_1!}\cdots\frac{c_{k_{n-1}}}{k_{n-1}!}, \tag{1.7}$$

where the summation is over all sequences $(k_1, k_2, \ldots, k_{n-1})$ of nonnegative integers such that $k_1+k_2+\cdots+k_{n-1} = n-k$ and $k_1+\cdots+k_j \leqslant j$ for any $1 \leqslant j \leqslant n-1$.

The initial motivation of this paper is give a combinatorial proof of (1.7). In order to state some combinatorial interpretations for $A_{n,k}$, we need to introduce several notations on partitions of integers.

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a weakly decreasing sequence of nonnegative integers. Each λ_i is called a *part* of λ. The sum of the parts of a partition λ is denoted by $|\lambda|$. If $|\lambda| = n$, then we say that λ is a *partition* of n, also written as $\lambda \vdash n$. We denote by m_i the number of parts equals i. By using the multiplicities, we also denote λ by $(1^{m_1}2^{m_2}\cdots n^{m_n})$. The partition with all parts equal to 0 is the empty partition. The *length* of λ, denoted $\ell(\lambda)$, is the maximum subscript j such that $\lambda_j > 0$. The *Ferrers diagram* of λ is a graphical representation of λ with λ_i boxes in its ith row and the boxes are left-justified. For a Ferrers diagram $\lambda \vdash n$ (we will often identify a partition with its Ferrers diagram), a *standard Young tableau* (SYT, for short) of shape λ is a filling of the n boxes of λ with the integers $1, 2, \ldots, n$ such that each number is used, and all rows and columns are increasing (from left to right, and from bottom to top, respectively). Given a Young tableau, we number its rows starting from the bottom and going above. Let SYT (n) be the set of SYT of size n. For a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, we define $c_\lambda = \prod_{i=1}^{\ell} c_{\lambda_i},\ c_\emptyset = 1$.

Let $\left[{n \atop k}\right]$ be the Stirling numbers of the first kind, which count permutations in $\mathfrak{S}_n$ with k cycles. We can now recall a recent result.

Proposition 1.3 ([3]). *Let $A_{n,k}$ be defined by* (1.5). *For $n \geqslant 1$, there exist positive integers $a(n,\lambda)$ such that*

$$A_{n,k} = \sum_{\lambda \vdash n-k} a(n,\lambda) c^{n-\ell(\lambda)} c_\lambda,$$

where λ runs over all partitions of $n-k$. The Stirling numbers of the first and second kinds, and the Eulerian numbers can be respectively expressed as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{\lambda \vdash n-k} a(n,\lambda),\ \begin{Bmatrix} n \\ k \end{Bmatrix} = a(n, 1^{n-k}),\ \left\langle {n \atop k} \right\rangle = \sum_{\ell(\lambda)=n-k} a(n,\lambda).$$

Motivated by Propositions 1.2 and 1.3, we first express $(cD)^n f$ in terms of inversion sequences as well as k-Young tableaux in Section 2, and then we present the other main results. In the rest sections, we prove some of the main results.

2. The g-index of Young tableau and main results

2.1. Inversion sequences and a combinatorial proof of Proposition 1.2.

An integer sequence $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ is an *inversion sequence* of length n if $0 \leqslant e_i < i$ for all $1 \leqslant i \leqslant n$. Let I_n be the set of all inversion sequences of length n. There is a natural bijection ψ between I_n and the symmetric group $\mathfrak{S}_n$ defined by $\psi(\pi) = \mathbf{e}$, where $e_i = \#\{j \mid 1 \leqslant j < i \text{ and } \pi(j) > \pi(i)\}$.

Definition 2.1. For any inversion sequence $\mathbf{e} \in \mathrm{I}_n$, let $|\mathbf{e}|_j = \#\{i \in [n] \mid e_i = j\}$. We now define $\phi(\mathbf{e}) = c \cdot c_{|\mathbf{e}|_1} c_{|\mathbf{e}|_2} \cdots c_{|\mathbf{e}|_{n-1}} \cdot \mathbf{f}_{|\mathbf{e}|_0}$.

For example, take $n = 9$ and $\mathbf{e} = (0,0,1,0,4,2,4,0,1)$, then $|\mathbf{e}|_0 = 4, |\mathbf{e}|_1 = 2, |\mathbf{e}|_2 = 1, |\mathbf{e}|_3 = 0, |\mathbf{e}|_4 = 2$ and $|\mathbf{e}|_j = 0$ for $5 \leqslant j \leqslant 8$. So that $\phi(\mathbf{e}) = c \cdot c_2 c_1 c c_2 cccc \cdot f_4 = c^6 c_1 c_2^2 \cdot \mathbf{f}_4$. The following lemma is fundamental.

Lemma 2.2. *For $n \geqslant 1$, we have*

$$(cD)^n f = \sum_{\mathbf{e} \in \mathrm{I}_n} \phi(\mathbf{e}). \tag{2.1}$$

Proof. Note that $\mathrm{I}_1 = \{0\}$ and $\mathrm{I}_2 = \{00, 01\}$. Since $\phi(0) = c\mathbf{f}_1$, $\phi(00) = c \cdot c \cdot \mathbf{f}_2$, $\phi(01) = c \cdot c_1 \cdot \mathbf{f}_1$, then (2.1) is valid for $n = 1, 2$. Assume that (2.1) holds for a given n. Let $\mathrm{I}_{n,k} = \{\mathbf{e} \in \mathrm{I}_n : |\mathbf{e}|_0 = k\}$. Then for any $\mathbf{e} \in \mathrm{I}_{n,k}$, we have $\phi(\mathbf{e}) = c \cdot c_{|\mathbf{e}|_1} \cdot c_{|\mathbf{e}|_2} \cdots c_{|\mathbf{e}|_{n-1}} \cdot \mathbf{f}_k$. Let $\mathbf{e}'$ be obtained from $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ by appending e_{n+1}. We distinguish three cases:

(*i*) If $e_{n+1} = 0$, then $\phi(\mathbf{e}') = c \cdot c_{|\mathbf{e}|_1} \cdot c_{|\mathbf{e}|_2} \cdots c_{|\mathbf{e}|_{n-1}} \cdot c \cdot \mathbf{f}_{k+1}$;
(*ii*) If $e_{n+1} = i$ and $1 \leqslant i \leqslant n-1$, then $\phi(\mathbf{e}') = c \cdot c_{|\mathbf{e}|_1} \cdots c_{|\mathbf{e}|_i + 1} \cdots c_{|\mathbf{e}|_{n-1}} \cdot c \cdot \mathbf{f}_k$;
(*iii*) If $e_{n+1} = n$, then $\phi(\mathbf{e}') = c \cdot c_{|\mathbf{e}|_1} \cdot c_{|\mathbf{e}|_2} \cdots c_{|\mathbf{e}|_{n-1}} \cdot c_1 \cdot \mathbf{f}_k$.

It is routine to check that the first case accounts for the term $cA_{n,k-1}$ and the last two cases account for the term $cDA_{n,k}$. Then $\sum_{\mathbf{e} \in \mathrm{I}_{n+1,k}} \phi(\mathbf{e}) = (cA_{n,k-1} + cDA_{n,k})\mathbf{f}_k = A_{n+1,k}\mathbf{f}_k$, which follows from (1.6). This completes the proof. □

A combinatorial proof of Proposition 1.2:

Proof. For $\mathbf{e} \in \mathrm{I}_n$, let $k = |\mathbf{e}|_0$ and $k_i = |\mathbf{e}|_{n-i}$ for $1 \leqslant i \leqslant n-1$. Note that $\sum_{i=1}^{n-1} k_i = n - k$ and $k_1 + \cdots + k_j \leqslant j$ for each j. The number of such $\mathbf{e}$ is equal to

$$\begin{aligned} &\binom{1}{k_1}\binom{2-k_1}{k_2}\binom{3-k_1-k_2}{k_3}\cdots\binom{n-k_1-k_2-\cdots-k_{n-1}}{k} \\ &= \frac{(2-k_1)(3-k_1-k_2)\cdots(n-k_1-k_2-\cdots-k_{n-1})}{k!k_1!k_2!\cdots k_{n-1}!}. \end{aligned}$$

Then by using Lemma 2.2, we obtain Proposition 1.2. □

2.2. ***k*-Young tableaux and their *g*-indexes.**

Since the $c_{k_1}, c_{k_2}, \ldots, c_{k_{n-1}}$ are commutative, we have to group the terms in (1.7) which produce the same product $c_{k_1} c_{k_2} \cdots c_{k_{n-1}}$. We say that a *type* of n is a pair (k, μ), denoted by $(k, \mu) \vdash n$, where $k \in [n]$ and $\mu = (\mu_1, \ldots, \mu_{n-1})$ is a partition of $n - k$, i.e., μ is written up to $n-1$ terms by appending 0's at the end. Let (k, μ) be a type of n. We define

$$\mathrm{Set}\,(\mu) = \{\mu_j \mid 1 \leqslant j \leqslant n-1\},\ |\mu|_j = \#\{i \mid \mu_i = j, 1 \leqslant i \leqslant n-1\}.$$

Let $(|\mathbf{e}|_0, \mu(\mathbf{e}))$ be the *type* of $\mathbf{e} \in \mathrm{I}_n$, where $\mu(\mathbf{e})$ is the decreasing order of the numbers $|\mathbf{e}|_1, \ldots, |\mathbf{e}|_{n-1}$. For each type (k, μ) of n, let $p_{k,\mu}$ be the number of inversion sequences of type (k, μ). It follows from Lemma 2.2 that

$$(cD)^n f = \sum_{(k,\mu) \vdash n} p_{k,\mu} c c_{\mu_1} c_{\mu_2} \cdots c_{\mu_{n-1}} \mathbf{f}_k, \tag{2.2}$$

where the summation is taken over all types (k, μ) of n.

Lemma 2.3. *By convention, set* $p_{0,\mu}=0$. *When* $(k,\mu)=(1,(1,1,\ldots,1))$, *it is clear that* $p_{k,\mu}=1$. *For the other types* (k,μ) *of* n, *we have*

$$p_{k,\mu}=\sum_{j\in \mathrm{Set}\,(\mu)\setminus\{0\}}(|\mu|_{j-1}+1)p_{k,\mu^{(j)}}+p_{k-1,\mu^{(0)}}, \tag{2.3}$$

where $\mu^{(j)}$ *is obtained from* μ *by replacing the last occurrence of the part* j *by* $j-1$ *and by deleting the last* 0 *and* $\mu^{(0)}$ *is obtained from* μ *by deleting the last* 0. *Thus we have* $(k,\mu^{(j)})\vdash(n-1)$ *and* $(k-1,\mu^{(0)})\vdash(n-1)$.

Proof. Take an element $\mathbf{e}\in \mathrm{I}_n$ of type (k,μ). Let $\mathbf{e}'=(e_1,e_2,\ldots,e_{n-1})\in \mathrm{I}_{n-1}$ be obtained from $\mathbf{e}$ by deleting the last e_n. If $e_n=0$, then the type of $\mathbf{e}'$ is $(k-1,\mu^{(0)})$. This operation is reversible. If $e_n=i$ $(1\leqslant i\leqslant n-1)$ and $|\mathbf{e}|_i=j\in \mathrm{Set}\,(\mu)\setminus\{0\}$, then the type of $\mathbf{e}'$ is $(k,\mu^{(j)})$. In this case, the operation is not reversible. We have exactly $(|\mu|_{j-1}+1)$ ways to generate the inverses. In fact we can append $e_n=i'\neq i$ at the end of $\mathbf{e}'$ with the condition of $|\mathbf{e}|_i-1=j-1=|\mathbf{e}|_{i'}$ to obtain an inversion sequence in I_n of type (k,μ). □

For convenience, we now give an illustration of (2.3), in order to get inversion sequences of type $(k,\mu)=(3,(2,1,1,0,0,0))$, we distinguish three cases:

(i) For each $\mathbf{e}\in \mathrm{I}_6$ that counted by $p_{2,(2,1,1,0,0)}$, we can get one inversion sequence of type (k,μ) by appending $e_7=0$ at the end of $\mathbf{e}$;

(ii) Let $\mathbf{e}\in \mathrm{I}_6$ be an inversion sequence counted by $p_{3,(1,1,1,0,0)}$. If $|\mathbf{e}|_i=1$ then we can append $e_7=i$ at the end of $\mathbf{e}$. As we have three choices for i, we get the term $3p_{3,(1,1,1,0,0)}$;

(iii) Let $\mathbf{e}\in \mathrm{I}_6$ be an inversion sequence counted by $p_{3,(2,1,0,0,0)}$. If $|\mathbf{e}|_i=0$, then we can append $e_7=i$ at the end of $\mathbf{e}$. As we have four choices for i, we get the term $4p_{3,(2,1,0,0,0)}$.

Each type (k,μ) of n can be represented by a picture which contains k boxes in the bottom row, and the Young diagram of μ in the top. Such picture is called a (k,μ)-*diagram*.

Definition 2.4. Let (k,μ) be a type of n. A k-*Young tableau* Z of shape (k,μ) is a filling of the n boxes of the (k,μ)-diagram by the integers $1,2,\ldots,n$ such that (i) each number is used, (ii) all rows and columns in the top Young diagram are increasing (from left to right, and from bottom to top, respectively), (iii) the bottom row is an increasing sequence of length k, starting with 1.

The filling of the top Young diagram of the partition μ is called the *top Young tableau* of the k-Young tableau. There is no condition between the bottom row and the top Young tableau. The rows of the top Young tableau are counted from bottom to top, and the columns are counted from left to right. We always put a special column of n boxes at the left of a k-Young tableau, labelled by the integers $1,2,\ldots,n$ from bottom to top. See Figure 1 (right diagram) for an example.

Definition 2.5. Let Z be a k-Young tableau of shape (k,μ), where $k+|\mu|=n$. For each $v\in[n]$, suppose that v is in the box (i,j) of the top Young diagram, where the first coordinate i means the column index, and the second coordinate j means the row index. The g-*index* of v, denoted by $g_Z(v)$, is the number of boxes $(i-1,j')$ such that $j'\geqslant j$ and the letter in this box is less than or equal to v (see Figures 2 and 3). If v is in the bottom row, then set $g_Z(v)=1$. The g-*index* of Z is given by $G_Z=g_Z(1)g_Z(2)\cdots g_Z(n)$.

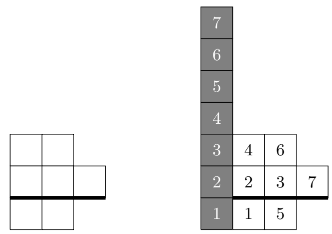

FIGURE 1. ($k = 2, \mu = (3, 2, 0, 0, 0, 0)$)-diagram and a k-Young tableau of shape (k, μ)

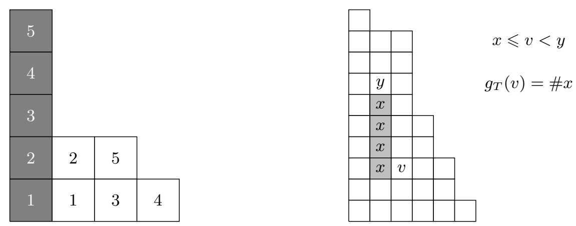

FIGURE 2. Young tableaux and g-index

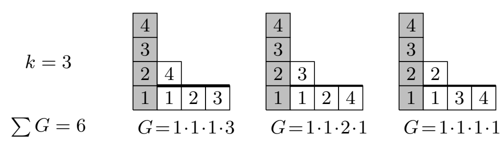

FIGURE 3. All 3-Young tableaux of size 4 and their g-indices

Theorem 2.6. *If $(k, \mu) \vdash n$, then we have*

$$p_{k,\mu} = \sum_Z G_Z, \tag{2.4}$$

where the summation is taken over all k-Young tableaux of shape (k, μ).

Proof. The identity (2.4) is obtained from Lemma 2.3 by induction on n. The maximum letter n in a k-Young tableau Z may be at the end of the bottom row, or a corner in the top Young tableau. In the first case, $g_Z(n) = 1$, and removing the letter n yields a $(k-1)$-Young tableau of shape $(k-1, \mu)$. In the second case, $g_Z(n) = |\mu|_{j-1} + 1$, where j is the length of the row containing n, and removing the letter n yields a k-Young tableaux of shape $(k, \mu^{(j)})$. We recover all terms in (2.3). This completes the proof. $\square$

In the sequel, we give two applications of Theorem 2.6. When $n \geqslant 1$, the Stirling numbers of the first kind $\left[{n \atop k}\right]$ can be defined by $\sum_{k=1}^n \left[{n \atop k}\right] x^k = x(x+1)\cdots(x+n-1)$. According to [1, Proposition A.2], we have $(e^x D)^n f = e^{nx} \sum_{k=1}^n \left[{n \atop k}\right] \mathbf{f}_k$. Setting $c = e^x$ and $c_j = e^x$ in (2.2), it follows from Theorem 2.6 that

$$(e^x D)^n f = e^{nx} \sum_{(k,\mu)\vdash n} \sum_Z G_Z \mathbf{f}_k,$$

So we get the following result.

Proposition 2.7. *For $n \geqslant 1$, we have*

$$\sum_{(k,\mu)\vdash n} \sum_{Z} G_Z x^k = x(x+1)(x+2)\cdots(x+n-1),$$

where the first summation is taken over all type (k,μ) of n, and the second summation is taken over all k-Young tableaux of shape (k,μ).

Proposition 2.8. *Let $\left\{ {n \atop k} \right\}$ be the Stirling numbers of the second kind. We have*

$$\left\{ {n \atop k} \right\} = \sum_{Z} G_Z,$$

where the summation is taken over all k-Young tableaux of shape $\big(k,(1^{n-k}0^{k-1})\big)$.

Proof. Let $c = x$ and $f = 1/(1-x)$. Then $c_1 = 1$ and $c_j = 0$ for $j \geqslant 2$, and $\mathbf{f}_k = k!/(1-x)^{k+1}$. It follows from (2.2) that

$$\begin{aligned}
(xD)^n \frac{1}{1-x} &= \sum_{(k,\mu)\vdash n} p_{k,\mu} c c_{\mu_1} c_{\mu_2} \cdots c_{\mu_{n-1}} \mathbf{f}_k \\
&= \sum_{(k,\mu=(1^{n-k}0^{k-1}))\vdash n} p_{k,\mu} \cdot \frac{k! x^k}{(1-x)^{k+1}} \\
&= \frac{1}{(1-x)^{n+1}} \sum_{(k,\mu=(1^{n-k}0^{k-1}))\vdash n} p_{k,\mu} \cdot k! x^k (1-x)^{n-k}.
\end{aligned}$$

By Theorem 2.6, we have

$$A_n(x) = \sum_{k=0}^{n} p_{k,(1^{n-k}0^{k-1})} \cdot k! x^k (1-x)^{n-k} = \sum_{k=0}^{n} \sum_{Z} G_Z \cdot k! x^k (1-x)^{n-k}, \tag{2.5}$$

where the second summation is taken over all k-Young tableaux of shape $(k,(1^{n-k}0^{k-1}))$. The classical *Frobenius formula* for Eulerian polynomials says that

$$A_n(x) = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\} x^k (1-x)^{n-k}.$$

Comparing the above formula with (2.5), we get the desired result. □

2.3. **The g-indices of Young tableaux.**

Let T be a SYT of shape λ and size n. We define the g-index of T very similarly. We always put a special column of n boxes at the left of T, labelled by $1, 2, 3, \ldots, n$ from bottom to top. The rows of T are counted from bottom to top, and the columns are counted from left to right. For each $v \in [n]$, suppose that v is in the box (i,j). The *g-index* of v, denoted by $g_T(v)$, counts boxes $(i-1,j')$ such that $j' \geqslant j$ and the letter in this box is less than or equal to v. The *g-index* of T is defined by $G_T = g_T(1)g_T(2)\cdots g_T(n)$. For the Young tableau given in Figure 2 (left diagram), we have $g_T(1) = 1,\ g_T(2) = 1,\ g_T(3) = 2,\ g_T(4) = 1,\ g_T(5) = 1$.

Example 2.9. The elements in SYT (4) and their g-indices are listed in Figure 4.

Let $\lambda(T)$ be the partition of T. If $\lambda(T) = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, then let $\lambda(T)! = \prod_{i=1}^{\ell}(\lambda_i!)$.

Theorem 2.10. *Let $C_n(x)$ be the second-order Eulerian polynomials. Then we have*

$$C_n(x) = \sum_{T\in \mathrm{SYT}\,(n)} G_T \ \lambda(T)!\ x^{n+1-\ell(\lambda(T))}. \tag{2.6}$$

Theorem 2.11. *Let $A_n(x)$ be the Eulerian polynomials. Then we have*

$$A_n(x) = \sum_{T\in \mathrm{SYT}\,(n)} G_T \ x^{n+1-\ell(\lambda(T))}. \tag{2.7}$$

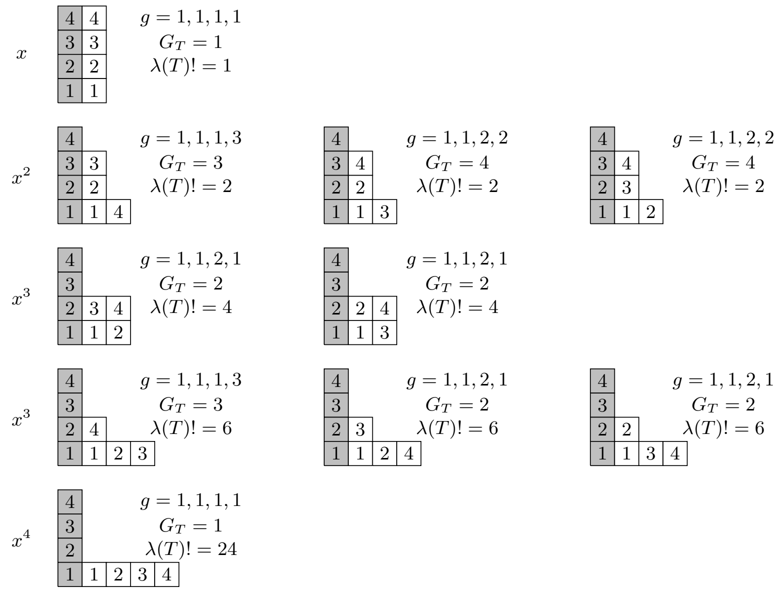

FIGURE 4. The elements in SYT (4) and their g-indices

Let $\pi \in \mathfrak{S}_n$. We say that π has no *double descents* if there is no index $i \in [n-2]$ such that $\pi(i) > \pi(i+1) > \pi(i+2)$. Then π is called a *simsun permutation* if for each $k \in [n]$, the subword of π restricted to $[k]$ (in the order they appear in π) contains no double descents. Simsun permutations are useful in describing the action of the symmetric group on the maximal chains of the partition lattice (see [17]). We define

$$S_n(x) = \sum_{\pi \in \mathcal{RS}_n} x^{\operatorname{des}(\pi)} = \sum_{i=1}^{\lfloor (n+2)/2 \rfloor} S(n,i)x^i,$$

where $\mathcal{RS}_n$ is the set of simsun permutations in $\mathfrak{S}_n$. Set $S_1(x) = x$. It follows from [7, Theorem 1] that $S_n(x) = (n+1)xS_{n-1}(x) + x(1-2x)S'_{n-1}(x)$ for $n \geqslant 2$. The polynomial $S_n(x)$ is known as *André polynomial* (see [6, 11]).

Theorem 2.12. *Let $S_n(x)$ be the André polynomials. For $n \geqslant 1$, we have*

$$S_n(x) = \sum_T G_T \; x^{n+1-\ell(\lambda(T))}, \tag{2.8}$$

where the summation is taken over all tableaux in SYT (n) *with at most two columns.*

We say that $\pi \in \mathfrak{S}_n$ is *alternating* if $\pi(i) < \pi(i+1)$ if i is even and $\pi(i) > \pi(i+1)$ if i is odd. Let E_n denote the number of alternating permutations in $\mathfrak{S}_n$. A remarkable property of simsun permutations is that $\#\mathcal{RS}_n = E_{n+1}$ (see [17, p. 267]).

Corollary 2.13. We have $E_{n+1} = \sum_T G_T$, where the summation is taken over all Young tableaux in SYT (n) with at most two columns.

Combining [2, Corollary 3.2] and [13, Proposition 1], the gamma expansion of Eulerian polynomial is given as follows:

$$A_{n+1}(x) = \sum_{i=1}^{\lfloor (n+2)/2 \rfloor} 2^{i-1} S(n,i) x^i (1+x)^{n+2-2i}.$$

We end this section be giving a characterization for the gamma coefficient polynomial of $A_{n+1}(x)$. For $T \in \mathrm{SYT}\,(n)$, if $\lambda(T) = (1^{n-2i+2}2^{i-1})$, then $n+1-\ell(\lambda(T)) = i$, where $1 \leqslant i \leqslant \lfloor (n+2)/2 \rfloor$. Using (2.8), we immediately get the following corollary.

Corollary 2.14. For $n \geqslant 2$, we have

$$\sum_{i=1}^{\lfloor (n+2)/2 \rfloor} 2^{i-1} S(n,i) x^i = \sum_T G_T \ \lambda(T)! \ x^{n+1-\ell(\lambda(T))},$$

where the summation is taken over all tableaux in $\mathrm{SYT}\,(n)$ with at most two columns.

3. Proof of Theorem 2.10

Setting $k=1$ in (1.4), we obtain that

$$\left(\frac{x}{1-x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{1-x} = \frac{C_n(x)}{(1-x)^{2n+1}}. \tag{3.1}$$

Setting $c = x/(1-x)$ and $f = 1/(1-x)$, then we have

$$c_j = \frac{j!}{(1-x)^{j+1}} \ (j \geqslant 1); \ \mathbf{f}_k = \frac{k!}{(1-x)^{k+1}} \ (k \geqslant 0).$$

By using (2.2), we obtain

$$\begin{aligned}
\left(\frac{x}{1-x}D\right)^n \frac{1}{1-x} &= \sum_{(k,\mu)\vdash n} p_{k,\mu} \cdot \frac{x^{|\mu|_0+1}}{1-x} \frac{\mu_1!}{(1-x)^{\mu_1+1}} \cdots \frac{\mu_{n-1}!}{(1-x)^{\mu_{n-1}+1}} \frac{k!}{(1-x)^{k+1}} \\
&= \frac{1}{(1-x)^{2n+1}} \sum_{(k,\mu)\vdash n} p_{k,\mu} \cdot k!\mu_1! \cdots \mu_{n-1}! x^{|\mu|_0+1}.
\end{aligned}$$

It follows from (3.1) and Theorem 2.6 that

$$\begin{aligned}
C_n(x) &= \sum_{(k,\mu)\vdash n} p_{k,\mu} \cdot k!\mu_1! \cdots \mu_{n-1}! x^{|\mu|_0+1} \\
&= \sum_{(k,\mu)\vdash n} \sum_Z G_Z \cdot k!\mu_1! \cdots \mu_{n-1}! x^{|\mu|_0+1}.
\end{aligned} \tag{3.2}$$

In view of (2.6) and (3.2), we need to establish some relations between k-Young tableaux and standard Young tableaux. Let Z be a k-Young tableau of shape (k,μ). We define $T = \rho(Z)$ to be the unique SYT such that the sets of the letters in the j-th column in Z and T are the same for all j. Let us list some basic facts of this map $Z \mapsto T = \rho(Z)$:

(*i*) We can obtain T from Z by ordering the letters in each column in increasing order. One can easily check that if T is obtained in this way, then T is a SYT;
(*ii*) The partition $\lambda(T)$ is the decreasing ordering of the sequence $(k,\mu_1,\ldots,\mu_{n-1})$, removing the 0's at the end. Hence, $\lambda(T)! = k!\left(\prod_{i=1}^{n-1}\mu_i!\right)$;
(*iii*) We have $n - \ell(\lambda(T)) = |\mu|_0$;
(*iv*) In general $G_Z \neq G_T$.

For example, take the k-Young tableau given in Figure 1, we obtain the SYT given in Figure 5. However ρ is not bijective. Let $\rho^{-1}(T) = \{(k,\mu,Z) \mid \rho(Z) = T\}$. By the above properties of ρ and (3.2), we have

$$\begin{aligned}
C_n(x) &= \sum_{T\in\mathrm{SYT}\,(n)} \sum_{(k,\mu,Z)\in\rho^{-1}(T)} G_Z \cdot k!\mu_1! \cdots \mu_{n-1}! x^{|\mu|_0+1} \\
&= \sum_{T\in\mathrm{SYT}\,(n)} \lambda(T)! x^{n+1-\ell(\lambda(T))} \sum_{(k,\mu,Z)\in\rho^{-1}(T)} G_Z.
\end{aligned} \tag{3.3}$$

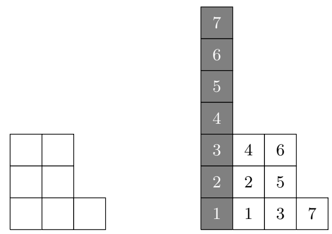

FIGURE 5. $T = \rho(Z)$ for Z given in Figure 1

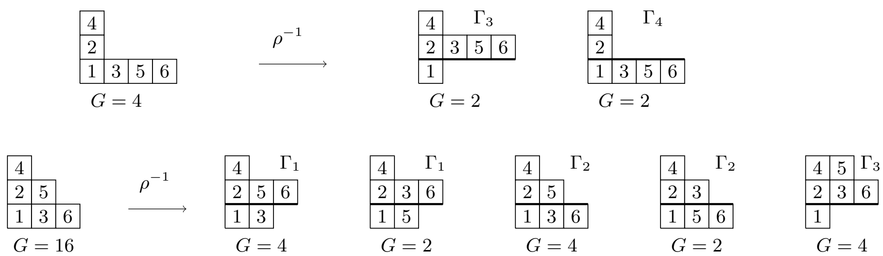

FIGURE 6. Decomposition of $\rho^{-1}(T)$ into $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$

Lemma 3.1. *For each standard Young tableau T, we have*

$$\sum_{Z \in \rho^{-1}(T)} G_Z = G_T, \tag{3.4}$$

where we write $Z \in \rho^{-1}(T)$ instead of $(k, \mu, Z) \in \rho^{-1}(T)$, as we can recover (k, μ) from Z.

Proof. We will prove (3.4) by induction on the size of T. Suppose that (3.4) is true for all standard Young tableaux of size $n-1$. Given a $T \in \mathrm{SYT}\,(n)$. Let T' is a SYT of size $n-1$ obtained from T by removing the letter n. This operation is reversible if $\lambda(T)$ is known. By the hypothesis of induction, we have

$$\sum_{Z' \in \rho^{-1}(T')} G_{Z'} = G_{T'}, \tag{3.5}$$

Clearly, $G_T = G_{T'} \times g_T(n)$. On the other hand, for a k-Young tableau $Z \in \rho^{-1}(T)$, if we remove the letter n, we obtain a k'-Young tableau $Z' \in \rho^{-1}(T')$ of size $n-1$. However, this operation is not always reversible. Let us analyse in detail. Let β be the length of the row containing n in the k-Young tableau $Z \in \rho^{-1}(T)$ with shape (k, μ) if n is in the top Young tableau of Z. The set $\rho^{-1}(T)$ can be divided into four subsets: $\rho^{-1}(T) = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, where $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 are respectively defined as follows:

$$\Gamma_1 = \{Z \in \rho^{-1}(T) : n \text{ is in the top Young tableau and } k = \beta - 1\},$$
$$\Gamma_2 = \{Z \in \rho^{-1}(T) : n \text{ is in the bottom row and } k - 1 \in \mu\},$$
$$\Gamma_3 = \{Z \in \rho^{-1}(T) : n \text{ is in the top Young tableau and } k \neq \beta - 1\},$$
$$\Gamma_4 = \{Z \in \rho^{-1}(T) : n \text{ is in the bottom row and } k - 1 \notin \mu\}.$$

See Figure 6 for two examples. Some of the Γ_i may be empty according to T. We claim that the set Γ_1 and Γ_2 have the same carnality. For each $Z_1 \in \Gamma_1$, there exists $Z_2 \in \Gamma_2$ in a unique manner, such that $Z'_1 = Z'_2 \in \rho^{-1}(T')$, see Figure 7. Moreover, we have the

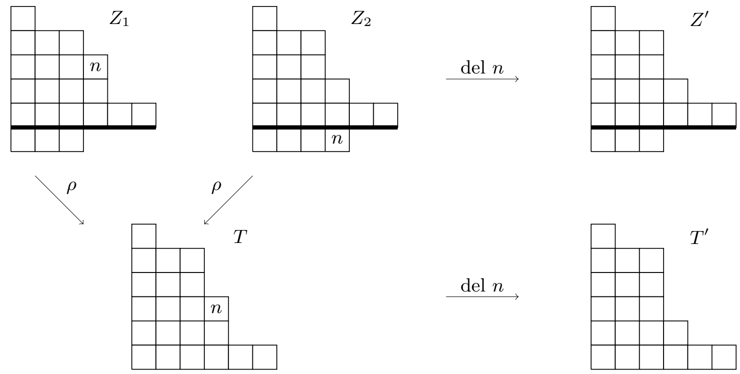

FIGURE 7. $Z_1, Z_2 \in \rho^{-1}(T)$ are mapped to the same $Z' \in \rho^{-1}(T')$ by removing the letter n

relations (see Figure 7): $g_{Z_1}(n) = g_T(n) - 1$ and $g_{Z_2}(n) = 1$. For $Z_3 \in \Gamma_3$ and $Z_4 \in \Gamma_4$, we have $g_{Z_3}(n) = g_T(n)$ and $g_{Z_4}(n) = g_T(n)$. By all these observations, we obtain

$$\begin{aligned}\sum_{Z\in\rho^{-1}(T)} G_Z &= \sum_{Z_1\in\Gamma_1, Z_2\in\Gamma_2} (G_{Z_1} + G_{Z_2}) + \sum_{Z_3\in\Gamma_3} G_{Z_3} + \sum_{Z_4\in\Gamma_4} G_{Z_4}\\ &= \sum_{Z_1\in\Gamma_1, Z_2\in\Gamma_2} (g_{Z_1}(n)G_{Z'} + g_{Z_2}(n)G_{Z'}) + \sum_{Z_3\in\Gamma_3} g_T(n)G_{Z_3'} + \sum_{Z_4\in\Gamma_4} g_T(n)G_{Z_4'}\\ &= g_T(n)\sum_{Z'\in\rho^{-1}(T')} G_{Z'}\\ &= g_T(n)G_{T'}\\ &= G_T.\end{aligned}$$

Hence (3.4) holds. This completes the proof. □

Proof of Theorem 2.10. Combining (3.3) and Lemma 3.1, we get that

$$\begin{aligned}C_n(x) &= \sum_{T\in\mathrm{SYT}\,(n)} \lambda(T)!x^{n+1-\ell(\lambda(T))} \sum_{(k,\mu,Z)\in\rho^{-1}(T)} G_Z\\ &= \sum_{T\in\mathrm{SYT}\,(n)} G_T\ \lambda(T)!x^{n+1-\ell(\lambda(T))}.\end{aligned}$$

This completes the proof. □

4. Proof of Theorem 2.11

For an alphabet V, let $\mathbb{Q}[[V]]$ be the rational commutative ring of formal power series in monomials formed from letters in V. Following Chen [5], a *context-free grammar* over V is a function $G: V \to \mathbb{Q}[[V]]$ that replaces each letter in V by a formal function over V. The *formal derivative* D_G with respect to G satisfies the derivation rules:

$$D_G(u+v) = D_G(u) + D_G(v),\ D_G(uv) = D_G(u)v + uD_G(v).$$

Thus D_G can be understood as a formal differential operator. The reader is referred to [6, 14] for recent progress on this subject.

Setting $u_i = D_G^i(u)$, it follows from (2.2) and (2.4) that

$$(uD_G)^n = \sum_{(k,\mu)\vdash n} \sum_Z G_Z u u_{\mu_1} u_{\mu_2} \cdots u_{\mu_{n-1}} D_G^k, \tag{4.1}$$

where the first summation is taken over all types (k,μ) of n and the second summation is taken over all k-Young tableaux of shape (k,μ). Recall that the Eulerian polynomials $A_n(x)$ are symmetric, i.e.,

$$A_n(x)=\sum_{i=1}^{n}\left\langle {n \atop i}\right\rangle x^i=\sum_{i=1}^{n}\left\langle {n \atop i}\right\rangle x^{n+1-i} \text{ for } n\geqslant 1.$$

In [9], Dumont found a grammatical interpretation of Eulerian numbers.

Proposition 4.1. *If $G=\{x\to y, y\to y\}$, then $(xD_G)^n(y)=\sum_{i\geqslant 1}\left\langle {n \atop i}\right\rangle x^{n+1-i}y^i$.*

Proof of Theorem 2.11. Let $G=\{x\to y, y\to y\}$. From (4.1), we have

$$(xD_G)^n(y)=\sum_{(k,\mu)\vdash n}\sum_{Z}G_Z x x_{\mu_1}x_{\mu_2}\cdots x_{\mu_{n-1}}D_G^k(y),$$

where $x_0=x$ and $x_i=D_G^i(x)=y$ for $i\geqslant 1$ and $D_G^k(y)=y$ for $k\geqslant 0$. Hence

$$(xD_G)^n(y)=\sum_{(k,\mu)\vdash n}\sum_{Z}G_Z y^{n-|\mu|_0}x^{|\mu|_0+1}.$$

Comparing this with Proposition 4.1, we get

$$A_n(x)=\sum_{i=1}^{n}\left\langle {n \atop i}\right\rangle x^{n+1-i}=(xD_G)^n(y)\big|_{y=1}=\sum_{(k,\mu)\vdash n}\sum_{Z}G_Z x^{|\mu|_0+1}, \tag{4.2}$$

where the first summation is taken over all types (k,μ) of n and the second summation is taken over all k-Young tableaux of shape (k,μ). By using Lemma 3.1, along the same lines as in the proof of Theorem 2.10, one can derive (2.7). □

5. Proof of Theorem 2.12

We now recall a grammatical interpretation of André polynomials.

Proposition 5.1 ([9]). *If $G_1=\{x\to xy, y\to x\}$, then*

$$D_{G_1}^n(x)=\sum_{i\geqslant 1}S(n,i)x^iy^{n+2-2i}.$$

Equivalently, we see that if $G_2=\{x\to y, y\to 1\}$, then

$$(xD_{G_2})^n(x)=\sum_{i\geqslant 1}S(n,i)x^iy^{n+2-2i}.$$

Proof of Theorem 2.12. Let $G_2=\{x\to y, y\to 1\}$. From (4.1), we have

$$(xD_{G_2})^n(x)=\sum_{(k,\mu)\vdash n}\sum_{Z}G_Z x x_{\mu_1}x_{\mu_2}\cdots x_{\mu_{n-1}}D_{G_2}^k(x). \tag{5.1}$$

Note that $x_0=D_{G_2}^0(x)=x,\ x_1=D_{G_2}(x)=y,\ x_2=D_{G_2}^2(x)=1$ and $x_i=D_{G_2}^i(x)=0$ for $i\geqslant 3$. Recall that for $(k,\mu)\vdash n$, we have $k\in[n]$. Then $x_{\mu_1}x_{\mu_2}\cdots x_{\mu_{n-1}}D_{G_2}^k(x)\neq 0$ if and only if $0\leqslant \mu_j\leqslant 2$ for all $j\in[n-1]$ and $1\leqslant k\leqslant 2$. Thus

$$\mu=(1^{m_1}2^{m_2}0^{n-1-m_1-m_2}), \tag{5.2}$$

where m_1 and m_2 are nonnegative integers. Assume that Z is a k-Young tableau of shape (k,μ), where μ is given by (5.2). As in the proof of Theorem 2.10, we define $T=\rho(Z)$ to be the unique SYT such that the sets of the letters in the j-th column in Z and T are the same for all j. Then Z has at most two columns. Using Proposition 5.1, we get

$$S_n(x)=(xD_{G_2})^n(x)\big|_{y=1}=\sum_{(k,\mu)\vdash n}\sum_{Z}G_Z x^{|\mu|_0+1}, \tag{5.3}$$

where the first summation is taken over all types (k, μ) of n, the second summation is taken over all k-Young tableaux of shape (k, μ) and the partitions μ have the form (5.2). Along the same lines as in the proof of Theorem 2.10, we get the desired formula (2.8). $\square$

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