ISSN: 0025-5742

THE MATHEMATICS STUDENT

Volume 93, Nos. 3-4, July - December (2024)
(Issued: November, 2024)

PUBLISHED BY
THE INDIAN MATHEMATICAL SOCIETY
Website: https://indianmathsoc.org

THE MATHEMATICS STUDENT

Edited by G. P. Youvaraj

In keeping with the current periodical policy, THE MATHEMATICS STUDENT seeks to publish material of interest not just to mathematicians with specialized interest but to the postgraduate students and teachers of mathematics in India and abroad. With this in view, it will ordinarily publish material of the following type:

1. research papers,
2. the texts (written in a way accessible to students) of the Presidential Addresses, the Plenary talks and the Award Lectures delivered at the Annual Conferences.
3. general survey articles, popular articles, expository papers and Book-Reviews.
4. problems and solutions of the problems,
5. new, clever proofs of theorems that graduate / undergraduate students might see in their course work, and
6. articles that arouse curiosity and interest for learning mathematics among readers and motivate them for doing mathematics.

Articles of the above type are invited for publication in THE MATHEMATICS STUDENT. Manuscripts intended for publication should be submitted online in the LaTeX and .pdf file including figures and tables to the Editor-in-Chief on the E-mail: msindianmathsociety@gmail.com.

Manuscripts (including bibliographies, tables, *etc.*) should be typed double spaced on A4 size paper with 1 inch (2.5 cm.) margins on all sides with font size 11 pt. in LaTeX. Sections should appear in the following order: Title Page, Abstract, Text, Notes and References. Comments or replies to previously published articles should also follow this format. In LaTeX the following preamble be used as is required by the Press:

```
\ documentclass[11 pt,a4paper,twoside,reqno]{amsart}
\ usepackage {amsfonts, amssymb, amscd, amsmath, enumerate, verbatim, calc}
\ renewcommand{\ baselinestretch}{1.2}
\ textwidth=12.5 cm
\ textheight=20 cm
\ topmargin=0.5 cm
\ oddsidemargin=1 cm
\ evensidemargin=1 cm
\ pagestyle{plain}
```

The details are available on Society's website: https://indianmathsoc.org

Authors of articles / research papers printed in the Mathematics Student as well as in the Journal shall be entitled to receive a *soft copy (PDF file)* of the paper published. There are no page charges for publication of articles in these journals.

All business correspondence should be addressed to Prof. M. M. Pawar, Treasurer, Indian Mathematical Society, Formerly at K.B.C. North Maharastra University, Jalgon, Maharashtra - 425001, India, on the E-mail: treasurerindianmathsociety@gmail.com.

In case of any query, one may contact the Editor through the e-mail.

ISSN: 0025-5742

THE MATHEMATICS STUDENT

Volume 93, Nos. 3-4, July - December (2024)
(Issued: November, 2024)

PUBLISHED BY
THE INDIAN MATHEMATICAL SOCIETY
Website: https://indianmathsoc.org

ISSN: 0025-5742

Published by Prof. M. M. Shikare for the Indian Mathematical Society, type set by Prof. G. P. Youvaraj, Former Director and Head, Ramanujan Institute for Advanced Study in Mathematics, Chepauk, Chennai - 600005, India and printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No. 129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune-411041 (India).

Printed in India.

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July - December (2024)

CONTENTS

****

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July - December (2024), 01–15

ON APPROXIMATE SOLUTIONS OF FRACTIONAL RICCATI DIFFERENTIAL EQUATIONS VIA SUMUDU DECOMPOSITION METHOD

N. B. MANJARE, H. T. DINDE, S. D. JADHAV

(Received : 13 - 04 - 2022; Revised : 22 - 09- 2023)

Abstract. In this paper, the Sumudu decomposition method is used to solve nonlinear fractional Riccati differential equations, which is an innovative coupling of two powerful techniques specifically Sumudu transform and Adomian decomposition method. In science and engineering, the progression of set of mathematical models for any experimental data is developed by using this innovative mixture. An approximate analytical solution is founded in the form of rapidly convergent Taylor series about the function $u_0(x)$. The comparison of existing results with previous results are presented to show efficiency of proposed method and it is plotted graphically.

1. Introduction

Fractional calculus is the field of mathematical analysis which is stated by S. G. Samko et al. [22]. This book described the investigation and applications of derivatives and integrals of arbitrary (real or complex) order. The progress of fractional calculus has been continuously discussed in various fields of mathematical analysis and it is stimulated by various ideas and results. Plenty of research articles were published by many mathematicians. K. B. Oldham et al. [14] addressed that the order q of the operator $\frac{d^q}{dx^q}$ becomes an arbitrary parameter.
In 1999, I. Podlubny [16] focused on the methods of solution of arbitrary real order of fractional differential equations. In the invention of fractional integro-differential equations, S. Behera et al. [4] developed Euler wavelets method for solving fractional-order linear Voltera-Fredholm

2010 Mathematics Subject Classification: 34A08, 26A33, 49M27, 34A45
Key words and phrases: Fractional Derivatives, Sumudu transform method, Adomian Polynomials, Decomposition method, Fractional Riccati Differential Equations

integro-differential equations with weakly singular kernels. S. S. Ray [19] presented numerical solution of fractional differential equations by using new wavelet operational matrix of general order. Further, S. S. Ray [20] introduced a new approach by two-dimensional wavelets operational matrix method for solving variable-order fractional partial integro-differential equations. S. Behera et al. [5] discussed a wavelet-based novel technique for linear and nonlinear fractional Volterra-Fredholm integro-differential equations. S. Behera et al. [6] applied fractional integral operational matrix for the reduction of pantograph Volterra delay integro-differential equations into algebraic equations. Moreover, S. Behera et al. [7] proposed wavelet-based numerical method for linear and nonlinear fractional Volterra integro-differential equations with weakly singular kernels.

In applications, Riccati differential equations and its generalizations appear in the classical problems of the calculus of variations and also used in optimal control and dynamic programming [18]. Particularly, different type of IVP and BVP on fractional differential equations cannot be solved by any unique method because they don't have exact solutions. Therefore, various methods have been investigated to solve fractional order Riccati differential equations for approximate solutions. M. G. Sakar et al. [21] used iterative reproducing kernel Hilbert spaces method (IRKHSM) to achieve the solutions of fractional Riccati differential equations. B. Agheli [3] found a numerical solution for the fractional Riccati differential equations of non-integer order (FRDEs) via trigonometric basic functions, where they successfully applied trigonometric transform method (TTM).

In 1993, G. K. Watugala [23] introduced Sumudu integral transform to solve differential equations from control engineering, which easily converts t-parameter function $f(t)$ into u-parameter function $F(u)$. In 1994, G. Adomian [2] introduced the book on Solving Frontier Problems of Physics: The Decomposition Method. This book was purposefully designed for quantitative solutions of mathematical models of physics, applied mathematics, engineering, biomathematics and astrophysics. Motivated by above cited work, new method called Sumudu decomposition method is used by N. B. Manjare et al. [12] for solving nonlinear Riccati differential equations. In Sumudu decomposition method, the limiting value of n^{th} term of infinite power series provides finite value which satisfies the definition of convergence of real analysis: $u = \lim_{n\to\infty} \varphi_n = \sum_{i=0}^{\infty} u_i$ where, $\varphi_n = \sum_{i=0}^{n-1} u_i$.

The present research work is compared with the solution of FRDEs with an iterative reproducing kernel Hilbert spaces method (IRKHSM) [21] and trigonometric transform method (TTM) [3]. The tabular representation shows that approximate solutions and exact solution for different values of α are very close to each other.
The present paper is organised as follows. The first section takes literature review. The second section presents basic definitions of fractional calculus, Sumudu transform and a few properties of Sumudu transform. The third section depicts analysis of Sumudu Decomposition Method (SDM). The fourth section describes convergence of Adomian decomposition method. The fifth section illustrates comparative study of approximate and exact solution of the FRDEs and finally the sixth section briefly concludes summary of the paper.

2. PRELIMINARIES OF FRACTIONAL CALCULUS

In preliminary part, the basic definitions and properties of the fractional calculus theory and Sumudu transform have been given for understanding the theme of this research paper. The definitions of Riemann-Liouville and Caputo fractional derivatives are used to complete present research work.

Definition 2.1. [17] The Riemann-Liouville fractional integral and differential operator of order $\alpha > 0$; for $t > 0$ is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, \tag{2.1}$$

$$J^0 f(t) = f(t) and \tag{2.2}$$

$$D^{\alpha} f(t) = \frac{d^m}{dt^m} I^{m-\alpha} f(t), m-1 < \alpha < m,\ m \in \mathbb{N}. \tag{2.3}$$

Definition 2.2. [13] The modified Riemann-Liouville derivative is defined as

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} (f(\xi) - f(0)) d\xi, \quad where\ x \in [0,1], 0 < \alpha < 1. \tag{2.4}$$

The Riemann-Liouville derivative does not help for formation of real-world phenomenon model with fractional differential equations. M. Caputo overcame this problem. He developed a novel fractional differential operator D^α and applied it in his theory of viscoelasticity [9]. This Caputo derivative made an impact on historical development of fractional calculus.

Definition 2.3. [15] The Caputo fractional derivative of $f(t)$ of order $\alpha > 0$ with $t > 0$ is defined as

$$D^\alpha f(t) = J^{m-\alpha} D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\xi)^{m-\alpha-1} f^{(m)}(\xi) d\xi,$$
$$m-1 < \alpha \leq m,\ m \in \mathbb{N}, t > 0. \quad (2.5)$$

Definition 2.4. [8] The Sumudu transform is defined over the set of functions

$$A = \{f(t) : \exists M, \tau_1, \tau_2 > 0, f(t) < Me^{\frac{t}{\tau_j}}, if t \in (-1)^j \times [0, \infty)\}, \quad (2.6)$$

which is defined through definite integral by using the following formula:

$$F(u) = S[f(t)] = \frac{1}{u} \int_0^\infty e^{\frac{-t}{u}} f(t) dt, u \in (-\tau_1, \tau_2) \quad (2.7)$$

where S is a Sumudu transform operator and M is a positive constant. The Sumudu transform of elementary functions and its various properties are mentioned and tabulated in [8]. We introduce some selected properties of Sumudu transform of elementary functions as follows:
1.$S\{1\} = 1$,
2.$S\{t^n\} = u^n \Gamma(n+1), n > 0$,
3.$S\{f(t) \pm g(t)\} = S\{f(t)\} \pm S\{g(t)\}$.

Definition 2.5. [11] The Sumudu transform of Caputo fractional derivative is defined as follows

$$S\{D^\alpha f(t)\} = u^{-\alpha} S\{f(t)\} - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0), m-1 < \alpha \leq m. \quad (2.8)$$

3. ANALYSIS OF THE METHOD [SDM]

In this paper, we will consider a class of Riccati differential equation of the form

$$D^{\alpha}y(t)+R(y)+N(y)=f(t), n-1<\alpha\leq n, \tag{3.1}$$

with intial condition

$$y^{k}(0)=y_{0}^{k}, \tag{3.2}$$

where R is a linear bounded operator and N is a nonlinear bounded operator, $f(t)$ is a given continuous function and $D^{\alpha}y(t)$ is the term of the fractional order derivative.
The Sumudu transform and Adomian polynomials consist Sumudu decomposition method.
First, we apply Sumudu transform on both sides of Eq. (3.1) to obtain
$S\{D^{\alpha}y(t)\}+S\{R(y)\}+S\{N(y)\}=S\{f(t)\}$
By applying definition (2.5) and initial condition (3.2), we have
$\frac{S\{y(t)\}}{u^{\alpha}}-\frac{C}{u^{\alpha-k}}+S\{R(y)\}+S\{N(y)\}=S\{f(t)\}$, where $C=\sum_{k=0}^{n-1}f^{(k)}(0)$

$$S\{y(t)\}=u^{k}C+u^{\alpha}S\{f(t)\}-u^{\alpha}S\{R(y)\}-u^{\alpha}S\{N(y)\} \tag{3.3}$$

The standard Sumudu decomposition method defines the solution $y(t)$ by the series

$$y(t)=\sum_{n=0}^{\infty}y_{n}(t), \tag{3.4}$$

and the non-linear term is decomposed as

$$N(y)=\sum_{n=0}^{\infty}A_{n}, \tag{3.5}$$

where A_n i.e. Adomian polynomials of $y_0, y_1, y_2, \ldots\ldots., y_n$ that are given by using the relation
$A_{n}=\frac{1}{n!}\frac{d^{n}}{d\lambda^{n}}\left[N\left(\sum_{n=0}^{\infty}\lambda^{n}y_{n}\right)\right]_{\lambda=0}, n=0,1,2,\ldots$
The first few Adomian Polynomials are defined by

$$A_{0}=N(y_{0}), \tag{3.6}$$

$$A_{1}=y_{1}N^{'}(y_{0}), \tag{3.7}$$

$$A_{2}=y_{2}N^{'}(y_{0})+\frac{1}{2!}y_{1}^{2}N^{''}(y_{0}), \tag{3.8}$$

$$A_{3}=y_{3}N^{'}(y_{0})+y_{1}y_{2}N^{''}(y_{0})+\frac{1}{3!}y_{1}^{3}N^{'''}(y_{0}), \tag{3.9}$$

and so on. We apply Eq. (3.4) to Eq. (3.5) in Eq. (3.3), we obtain

$$S\left\{\sum_{n=0}^{\infty} y_n\right\} = u^k C + u^\alpha S\{f(t)\} - u^\alpha S\left\{R\sum_{n=0}^{\infty} y_n\right\} - u^\alpha S\left\{\sum_{n=0}^{\infty} A_n\right\} \quad (3.10)$$

Comparing both side of Eq. (3.10)

$$S\{y_0\} = u^k C + u^\alpha S\{f(t)\}, \quad (3.11)$$

$$S\{y_1\} = -u^\alpha S\{Ry_0\} - u^\alpha S\{A_0\}, \quad (3.12)$$

$$S\{y_2\} = -u^\alpha S\{Ry_1\} - u^\alpha S\{A_1\}, \quad (3.13)$$

In general, the recursive relation is derived by

$$S\{y_n\} = -u^\alpha S\{Ry_{n-1}\} - u^\alpha S\{A_{n-1}\}, n \geq 1, \quad (3.14)$$

Further, we apply inverse Sumudu transform to Eq. (3.11) - Eq. (3.14) then

$$y_0 = F(t), \quad (3.15)$$

$$y_n = -S^{-1}[u^\alpha S\{Ry_{n-1}\} + u^\alpha S\{A_{n-1}\}], n \geq 1. \quad (3.16)$$

Where $F(t)$ is a function that arises from the source term and prescribed initial conditions. We treat y_0 as an initial approximation which helps to calculate further approximation.

4. CONVERGENCE OF ADOMIAN DECOMPOSITION METHOD

K. Abbaoui and Y. Cherruault [1] suggested new ideas for proving the convergence of decomposition method. G. Adomian [2] investigated a new technique for solving exactly nonlinear functional equations of various kinds (algebraic, differential, partial differential, integral...). In this section, we prove convergence of the series solution with the help of a new formula giving the Adomian polynomials. This formula produces the series solution as a function of the first term of the series. We have a simple formula for calculation of A_n

$$A_n = \sum_{\alpha_1+\alpha_2+\cdots+\alpha_n} N^{(\alpha_1)}(u_0)\frac{u_1^{\alpha_1-\alpha_2}}{(\alpha_1-\alpha_2)!}\cdots\cdots\frac{u_{n-1}^{\alpha_{n-1}-\alpha_n}}{(\alpha_{n-1}-\alpha_n)!}\frac{u_n^{\alpha_n}}{(\alpha_n)!}, n \neq 0.$$

Theorem 4.1. *With the following hypothesis,*

(1) *N is C^∞ in a neghbourhood of u_0 and $||N^{(n)}(u_0)|| \leq M^{'}$, for any n (the derivatives of N at u_0 are bounded in norm) where N is a nonlinear operator from a Hilbert space H into H;*

(2) *$||u_i|| \leq M < 1, i = 1, 2, \ldots,$ where $||\cdot||$ is the norm in the Hilbert space H; the series $\sum_{n=0}^{\infty} A_n$ is absolutely convergent and furthermore, $||A_n|| \leq \left(exp\left(\pi\sqrt{\frac{2}{3}n}\right)\right) M^{'}M^n, n \geq M, n \geq M^{'}$ where $M > 0$ and $M^{'} > 0$ both are finite numbers.*

Theorem 4.2. *If N is C^∞ and satisfies $||N^{(n)}(u_0)|| \leq M < 1$, for any $n \in N$, then the decompositional series $\sum_{n=0}^{\infty} u_n$ is absolutely convergent and we have $||u_{n+1}|| = ||A_n|| \leq M^{n+1} n^{\sqrt{n}} \left(exp\left(\pi\sqrt{\frac{2}{3}n}\right)\right)$.*

Previously Y. Cherruault [10] discussed that the Adomian technique is equivalent to determining the sequence: $S_n = y_1 + y_2 + \cdots + y_n, S_{n+1} = N(y_0 + S_n), S_0 = 0$.

Theorem 4.3. *N being a contradiction ($\delta < 1$), if we assume that $||N_n - N|| = \in_{n(n\to\infty)} \to 0$, (satisfied in our case), then the sequence S_n is given by $S_{n+1} = N_n(y_0 + S_n), S_0 = 0$ converges towards the S solution of $N(y_0 + S) = S$.*

Theorem 4.4. (1) *For every $f \in V^{'}$, there exists $y \in V$ such that: $y - N(y) = f$ where V is a Hilbert space and $V^{'}$ its dual.*

(2) *The sequence y_n defined by $y_{n+1} = y_n - \varrho[N(y_0 + y_n)]$, $\varrho > 0$ is strongly convergent in V and its limit y is the solution of $y = N(y_0 + y)$ for $\varrho > 0$ well chosen. A first consequence is that $u = y_0 + y$ with $y_0 = f$ is a solution of $u = f + N(u)$.*

These convergence theorems are powerful and easy to handle Linear (L) and Non-linear (N) terms. The series solution is convergent with remarkable rapidity and successive terms y_i are easily computed. In Adomian research papers, we can see that a very large number of difficult problems have been successively solved.

5. NUMERICAL EXAMPLES

***Example* 5.1**

We consider the Fractional Riccati differential equation [21]

$$D^{\alpha}y(t) - 2y(t) + y^2(t) = 1, 0 < \alpha \leq 1, 0 < t < 1, \tag{5.1}$$

subject to initial condition

$$y(0) = 0, \tag{5.2}$$

The exact solution when $\alpha = 1$ is

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right).$$

By using Eq. (3.15) to Eq. (3.16), we can obtain initial approximation and general iteration formula for the Eq. (5.1) to Eq. (5.2) as

$$y_0(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} \tag{5.3}$$

$$y_{n+1}(t) = S^{-1}[u^{\alpha}S\{2y_n(t) - A_n\}] \tag{5.4}$$

Further, we use Adomian polynomials Eq. (3.6) to Eq. (3.8), initial approximation Eq. (5.3) and $(n+1)^{th}$ order approximation Eq. (5.4). We can derive the following successive approximations.

$$y_1(t) = \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}$$

$$y_2(t) = \frac{4t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{[2(\Gamma(2\alpha+1))^2 + 4\Gamma(\alpha+1)\Gamma(3\alpha+1)]t^{4\alpha}}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)\Gamma(4\alpha+1)} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{5\alpha}}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}$$

$$= \int_1^{\frac{1+\sqrt{5}}{2}} \frac{\ln\left(v^2-1\right)}{v}dv - \int_0^{\ln\left(\frac{1+\sqrt{5}}{2}\right)} u du - \ln 2 \ln\left(\frac{1+\sqrt{5}}{2}\right)$$

$$y_3(t) = \frac{8t^{4\alpha}}{\Gamma(4\alpha+1)} - 2\frac{[2(\Gamma(2\alpha+1))^2 + 4\Gamma(\alpha+1)\Gamma(3\alpha+1)]t^{5\alpha}}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)\Gamma(5\alpha+1)}$$
$$+4\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{6\alpha}}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(6\alpha+1)} - 8\frac{\Gamma(4\alpha+1)t^{5\alpha}}{\Gamma(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}$$
$$+2\frac{[2(\Gamma(2\alpha+1))^2\Gamma(5\alpha+1) + 4\Gamma(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)]t^{6\alpha}}{(\Gamma(\alpha+1))^3\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)}$$
$$-4\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)t^{7\alpha}}{(\Gamma(\alpha+1))^4\Gamma(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)} - 4\frac{\Gamma(4\alpha+1)t^{5\alpha}}{(\Gamma(2\alpha+1))^2\Gamma(5\alpha+1)}$$

$$+4\frac{\Gamma(2\alpha+1)\Gamma(5\alpha+1)t^{6\alpha}}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)\Gamma(3\alpha+1)\Gamma(6\alpha+1)}-\frac{(\Gamma(2\alpha+1))^2\Gamma(6\alpha+1)t^{7\alpha}}{(\Gamma(\alpha+1))^4(\Gamma(3\alpha+1))^2\Gamma(7\alpha+1)}.$$

The convergent series solution is given by

$$y(t)=\sum_{n=0}^{\infty}y_n(t)$$

$$\begin{aligned}y(t)&=\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{[4(\Gamma(\alpha+1))^2-\Gamma(2\alpha+1)]t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}\\
&+\frac{[8(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)-2(\Gamma(2\alpha+1))^2-4\Gamma(\alpha+1)\Gamma(3\alpha+1)]t^{4\alpha}}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}\\
&+\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{5\alpha}}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}-2\frac{[2(\Gamma(2\alpha+1))^2+4\Gamma(\alpha+1)\Gamma(3\alpha+1)]t^{5\alpha}}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)\Gamma(5\alpha+1)}\\
&+4\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{6\alpha}}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(6\alpha+1)}-8\frac{\Gamma(4\alpha+1)t^{5\alpha}}{\Gamma(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}\\
&+2\frac{[2(\Gamma(2\alpha+1))^2\Gamma(5\alpha+1)+4\Gamma(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)]t^{6\alpha}}{(\Gamma(\alpha+1))^3\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)}\\
&-4\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)t^{7\alpha}}{(\Gamma(\alpha+1))^4\Gamma(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)}-4\frac{\Gamma(4\alpha+1)t^{5\alpha}}{(\Gamma(2\alpha+1))^2\Gamma(5\alpha+1)}\\
&+4\frac{\Gamma(2\alpha+1)\Gamma(5\alpha+1)t^{6\alpha}}{(\Gamma(\alpha+1))^2\Gamma(2\alpha+1)\Gamma(3\alpha+1)\Gamma(6\alpha+1)}-\frac{(\Gamma(2\alpha+1))^2\Gamma(6\alpha+1)t^{7\alpha}}{(\Gamma(\alpha+1))^4(\Gamma(3\alpha+1))^2\Gamma(7\alpha+1)}\\
&+\dots\end{aligned}$$

In particular case $\alpha=1$, then we get

$$y(t)=t+t^2+\frac{t^3}{3}-\frac{t^4}{3}-\frac{7t^5}{15}+\dots$$

The exact solution when $\alpha=1$ is given by

$$y(t)=1+\sqrt{2}\tanh\left(\sqrt{2}t+\tfrac{1}{2}\log\left(\tfrac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right).$$

Table 1 shows that approximate solutions $y(t)$ for Eq. (5.1) which are obtained for different values of α using the Sumudu decomposition method [SDM]. From the numerical results, it is clear that the approximate solutions of SDM are in best agreement with approximate solutions of IRKHSM [21]. According to convergence of decomposition method, the obtained infinite series is rapidly convergent. Table 2 shows the achieved absolute errors of SDM are minor in range as compared to IRKHSM.

Figure 1 compare the efficiency and accuracy of approximate solutions and exact solution for distinct values of $\alpha=0.75, 0.8, 0.9\&1$. We can see that at $\alpha=1$, two non-linear curves are coincident with each other. It

TABLE 1. Comparison of numerical results of Sumudu Decomposition Method (SDM) with IRKHSM for various values of t. (for $\alpha = 0.75, 0.9, 1, N = 6, n = 5$)

t	Exact solution	SDM		
		$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.75$
0.0	0.0	0.0	0.0	0.0
0.2	0.241976	0.241984	0.316894	0.490154
0.4	0.567812	0.568021	0.729250	1.071474
0.6	0.953566	0.952512	1.264254	1.798569
0.8	1.346363	1.321216	1.936047	2.680022
1	1.689498	1.533333	2.757228	3.721149

t	Exact solution	IRKHSM [21]		
		$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.75$
0.0	0.0	0.0	0.0	0.0
0.2	0.241976	0.241884	0.314571	0.473076
0.4	0.567812	0.567738	0.697246	0.936880
0.6	0.953566	0.953490	1.107569	1.333068
0.8	1.346363	1.346324	1.477434	1.622033
1	1.689498	1.689427	1.765103	1.817550

TABLE 2. Comparison of absolute errors of SDM with IRKHSM for $\alpha = 1$.

t	Absolute Errors	
	SDM	IRKHSM [21]
0.0	0.0	0.0
0.2	8.00E-6	9.23E–5
0.4	2.09E-4	7.35E–5
0.6	1.05E-3	7.56E–5
0.8	2.51E-2	3.94E–5
1	1.56E-1	7.12E–5

means that the obtained solutions are very close to the analytical solution. Also, other three branches of curves at $\alpha = 0.75, 0.8\&0.9$ shows the closeness between approximate and analytical solution.

***Example*5.2**

We consider the Fractional Riccati differential equation [3]

$$D^{\alpha}y(t) + y(t) - y^2(t) = 0, 0 < \alpha \leq 1, 0 < t \leq 1, \tag{5.5}$$

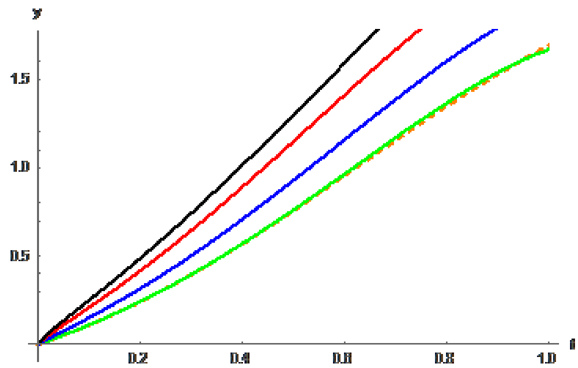


FIGURE 1. The behavior of approximate solutions and exact solution for distinct values of α.

subject to initial condition

$$y(0) = 0.5, \tag{5.6}$$

The exact solution when $\alpha = 1$ is $y(t) = \frac{e^{-t}}{e^{-t}+1}$.
By using Eq. (3.15) to Eq. (3.16), we can obtain initial approximation and general iteration formula for the Eq. (5.5) to Eq. (5.6) as

$$y_0(t) = 0.5 \tag{5.7}$$

$$y_{n+1}(t) = S^{-1}[u^{\alpha} S\{-y_n(t) + A_n\}] \tag{5.8}$$

Further, we use Adomian polynomials Eq. (3.6) to Eq. (3.9), initial approximation Eq. (5.7) and $(n+1)^{th}$ order approximation Eq. (5.8). We can derive the following successive approximations.

$$y_1(t) = -0.25\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \quad y_2(t) = 0,$$

$$y_3(t) = 0.0625\frac{\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}, \quad \text{and} \quad y_4(t) = 0.$$

The convergent series solution is given by

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

$$y(t) = 0.5 - 0.25\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 0.0625\frac{\Gamma(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)} + \dots$$

TABLE 3. Comparison of numerical results of SDM with TTM for various values of t. (for $\alpha = 1, N = 5, n = 3$)

t	SDM	TTM [3]	Exact Solution
0.0	0.5	0.5	0.5
0.2	0.450167	0.450065	0.450166
0.4	0.401333	0.401178	0.401312
0.6	0.354499	0.354203	0.354344
0.8	0.310667	0.309897	0.310026
1	0.270833	0.268837	0.268941

TABLE 4. Comparison of absolute errors of SDM with TTM for $\alpha = 1$.

t	Absolute Errors	
	SDM	TTM [3]
0.0	0.0	0.0
0.2	1.00E-6	1.01334E-4
0.4	2.10E-5	1.34719E-4
0.6	1.55E-4	1.40666E-4
0.8	6.41E-4	1.28611E-4
1	1.89E-3	1.04154E-4

In particular case $\alpha = 1$, then we obtain

$$y(t) = 0.5 - 0.25t + 0.0625\frac{t^3}{3} + \dots$$

The exact solution when $\alpha = 1$ is given by $y(t) = \dfrac{e^{-t}}{e^{-t} + 1}$.

Table 3 shows that approximate solutions $y(t)$ for Eq. (5.5) which are obtained for different values of α using the Sumudu decomposition method [SDM]. From the computed results, it is clear that the approximate solutions of SDM are in best agreement with approximate solutions of TTM [3]. According to convergence of decomposition method, the obtained infinite series is rapidly convergent. Table 4 shows the achieved absolute errors of SDM which are minor in range as compared to TTM.

Figure 2 shows the comparison of the efficiency and accuracy of approximate solution and exact solution for distinct values of $\alpha = 0.75, 0.8, 0.9\&1$. We can see that at $\alpha = 1$, two non-linear curves are coincident with each

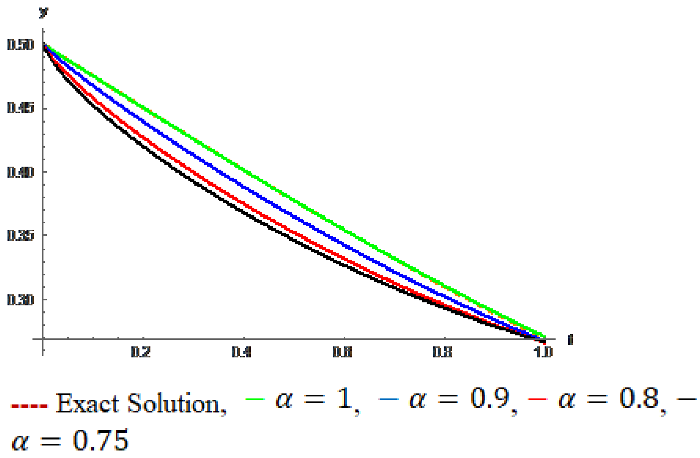


FIGURE 2. The behavior of approximate solutions and exact solution for distinct values of α.

other. It means that the obtained solutions are very close to the analytical solution. Also, other three branches of curves at $\alpha = 0.75, 0.8\&0.9$ shows the closeness between approximate and analytical solution.

CONCLUDING COMMENTS

The Sumudu decomposition method is useful to find convergent series solution of linear and non-linear fractional differential equations. The first example shows that the results of SDM are identical with IRKHSM with negligible absolute errors. The second example demonstrates approximate solutions which are similar with approximate solutions of TTM. The results of SDM show its efficiency and effectiveness because of its ability to solve FDEs without calculation of arbitrary constants. The important fact is that the present mixture is suitable for linear and nonlinear problems without taking help of He's polynomial, without recognizing Lagrange's multiplier and without applying quasilinearization procedure. We can clarify that the derived numerical outcomes reach to an excellent level of approximate results which are obtained by existing methods such as IRKHSM and TTM.

Acknowledgement: The authors would like to express sincere thanks to the referee for the helpful comments and suggestions. Also, authors are very much thankful to Nagesh Karajagi ÖRCHID College of engineering and Technology, Solapur (India) for providing all type of support and computational resources to complete the present research work.

References

[1] Abbaoui, K. A., Cherruault, Y., New Ideas for Proving Convergence of Decomposition Method, *Comput Math Applic.* **29(7)** (1995), 103–108.

[2] Adomian, G., Solving Frontier Problems of Physics: The Decomposition Method, *Springer-Science Business Media* **60** (1994), 1–352.

[3] Agheli, B., Approximate solution for solving fractional Riccati differential equations via trigonometric basic functions, *Trans A Razmadze Math Inst* **172(3)** (2018), 299–308.

[4] Behera, S., Saha Ray, S., Euler wavelets method for solving fractional-order linear Volterra–Fredholm integro-differential equations with weakly singular kernels,*Comput Appl Math.* **40(6)** (2021), 1–30.

[5] Behera, S., Ray, S. S., A wavelet-based novel technique for linear and nonlinear fractional Volterra–Fredholm integro-differential equations,*Comp. Appl. Math.* **41** (2022).

[6] Behera, S., Ray, S. S., An efficient numerical method based on Euler wavelets for solving fractional order pantograph Volterra delay-integro-differential equations,*Journal of Computational and applied Mathematics* **41** (2022), 406.

[7] Behera, S., Saha Ray, S., On a wavelet-based numerical method for linear and nonlinear fractional Volterra integro-differential equations with weakly singular kernels,*Comp. Appl. Math.* **41** (2022), 406.

[8] Belgacem, F. B. M., Karaballi, A. A., Sumudu transform fundamental properties investigations and applications,*J Appl Math Stoch Anal.* **2006** (2006), 1–23.

[9] Caputo, M., Linear Models of Dissipation whose Q is almost,*Geophys J R astr Soc.* **13** (1967), 529–539.

[10] Cherruault, Y., Convergence of Adomian's Method,*emerald insight* **13** (2005), 31–38.

[11] Mahdy, A. M. S., Marai, G. M. A., Sumudu decomposition method for solving fractional Riccati equation,*J. Abstr. Comput. Math.* **50** (2018), 42–50.

[12] Manjare, N. B., Dinde H. T., Sumudu Decomposition Method for Solving Fractional Bratu-Type Differential Equations,*J Sci Res.* **12(4)** (2020), 585–605.

[13] Merdan, M., On the solutions fractional Riccati differential equation with modified Riemann-Liouville derivative,*Int J Differ Equations* (2012), 1-17.

[14] Keith, B., Oldham, J. S., The Fractional Calculus,*Academic Press* (1974), 1–225.

[15] Ordokhani, Y., Rahimkhani, P., Babolian, E., Application of fractional-order Bernoulli functions for solving fractional Riccati differential equation,*Int J Nonlinear Anal Appl.* **8(2)** (2017), 277–292.

[16] Podlubny, I., FRACTIONAL DIFFERENTIAL EQUATIONS,*Academic Press* (1999), 1–337.

[17] Rawashdeh, M. S., Al-Jammal H., New approximate solutions to fractional nonlinear systems of partial differential equations using the FNDM,*Adv Differ Equations* **2016(1)** (2016), 277–292.

[18] Reid, W. T., Riccati Differential Equations,*In: ACADEMIC PRESS New York and London* (1972), 1–213.

[19] Saha Ray, S., Numerical Solution of Fractional Differential Equations by Using New Wavelet Operational Matrix of General Order, *In: Nonlinear Differential Equations in Physics. Springer, Singapore* (2020).

[20] Saha Ray, S., A new approach by two-dimensional wavelets operational matrix method for solving variable-order fractional partial integro-differential equations,*Numer Methods Partial Differ Equ.* **37(1)** (2021), 341–359.

[21] Sakar, M. G., Akgül, A., Baleanu, D., On solutions of fractional Riccati differential equations.,*Adv Differ Equations* **1** (2017), 1-10.

[22] Stefan, G., Samko, A., Natol, A. K., and O. I. M., Fractional integrals and derivatives Theory and Applications,*Gordon and Breach Science Publishers* **1** (1993), 1–973.

[23] Watugala, G. K., Sumudu transform: A new integral transform to solve differential equations and control engineering problems,*Int J Math Educ Sci Technol.* **24(1)** (1993), 35–43.

Manjare Ñagesh Balasaheb
Department of Mathematics
General Science & Engineering
N. K.Öchid College of Engineering,
413002, Solapur/Maharashtra, INDIA.
E-mail: nageshmanjare@gmail.com

Dinde Ḧambirrao T.
Department of Mathematics
Faculty of Sciences
Shivaji University, Kolhapur
416004, Kolhapur/Maharashtra, INDIA.
E-mail: drhtdmaths@gmail.com

Jadhav S̈antosh D.
Department of Mathematics
General Science & Engineering
N. K.Öchid College of Engineering,
413002, Solapur/Maharashtra, INDIA.
E-mail: sdjmaths23@gmail.com

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July - December (2024), 16–27

ON SOME IDENTITIES OF PSEUDO FIBONACCI POLYNOMIALS

SUCHITA AROLKAR, C. N. PHADTE AND Y. S. VALAULIKAR
(Received : 19 - 05 - 2023 ; Revised : 17 - 05 - 2024)

Abstract. In this article, we present identities involving pseudo Fibonacci polynomials and their derivatives. We also obtain the representation of r^{th} order derivative of pseudo Fibonacci polynomials in terms of derivatives of Fibonacci polynomials and pseudo Fibonacci polynomials. Finally, we show that the n^{th} pseudo-Fibonacci polynomial is a solution of a non-homogeneous second-order linear hypergeometric differential equation.

1. Introduction

Like Fibonacci sequence, Fibonacci polynomials also play a very important role in the development of combinatorial related fields in mathematics. Various identities of Fibonacci polynomials and their extensions are studied in [1, 9, 12]. Two sequences of polynomials $J_n(x)$ and $j_n(x)$, Jacobsthal and Jacobsthal-Lucas polynomials, respectively, and their properties are studied in [7]. In [15], the author has obtained results concerning the diagonal functions associated with generalized Fibonacci and Lucas polynomials. The author also derives a number of interesting new results concerning the derivatives of these polynomials. In [4], authors have defined Fibonacci polynomials by

$$F_n(x) = \begin{cases} 0, \text{if } n = 0 \\ 1, \text{if } n = 1 \\ xF_{n-1} + F_{n-2}, \text{if } n > 1, \end{cases} \tag{1.1}$$

and following identities are obtained.

2010 Mathematics Subject Classification: 11B37, 11B39, 11B83
Key words and phrases: Pseudo Fibonacci polynomials, Fibonacci polynomials, Generating function

(1) The combinatorial form of (1.1) is given by

$$F_n(x) = \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n-1-k}{k} x^{n-1-2k}. \tag{1.2}$$

(2) The generating function of (1.1) is given by

$$G(x,u) = \frac{u}{1-xu-u^2}. \tag{1.3}$$

Equation (1.3) can also be written as

$$\sum_{n=0}^{\infty} F_n(x)\ u^n = \frac{u}{1-xu-u^2}. \tag{1.4}$$

Note that, if the equation (1.4) is differentiated both sides r times, with respect to x, then we get

$$\sum_{n=0}^{\infty} \frac{d^r F_n}{dx^r} u^n = \frac{r!\ u^{r+1}}{(1-xu-u^2)^{r+1}}.$$

Further, they have presented derivatives of these polynomials in the form of convolution of k-Fibonacci polynomials. The identities showing the relation of Fibonacci polynomials and their derivatives are also proved. One such identity which we shall be using is

$$nF_n(x) = \frac{dF_{n+1}}{dx} + \frac{dF_{n-1}}{dx}. \tag{1.5}$$

The study of properties of derivatives of the Morgan-Voyce polynomials can be seen in [8]. In [5] and [6], identities on first and second order derivatives of Fibonacci and Lucas polynomials, respectively are obtained. Identities on the higher order derivatives of Fibonacci and Lucas polynomials are introduced in [16].

In [11], a new type of Fibonacci polynomials, called pseudo Fibonacci polynomials, denoted by $g_n(x,t)$, are defined by

$$g_n(x,t) = xg_{n-1}(x,t) + g_{n-2}(x,t) + At^{n-2}, \text{ for all } n \geq 2, \tag{1.6}$$

with $g_0(x,t) = 0$ and $g_1(x,t) = 1$, where A is constant and t is non-zero real number such that $t \neq \frac{x \pm \sqrt{x^2+4}}{2}$.

We list below first few pseudo Fibonacci polynomials.

$g_2(x,t) = x + A$, $g_3(x,t) = x^2 + 1 + Ax + At$,

$g_4(x,t) = x^3 + 2x + A(x^2+1) + Atx + At^2$,

$g_5(x,t) = x^4 + 3x^2 + 1 + A(x^3+2x) + At(x^2+1) + At^2x + At^3$.

Note that

$$g_n(x,t) = F_n(x) + A\sum_{i=0}^{n-1} F_{n-1-i}(x)\ t^i \tag{1.7}$$

and

$$\frac{\partial g_n}{\partial t} = A\sum_{i=0}^{n-2}(i+1)F_{n-2-i}(x)\ t^i. \tag{1.8}$$

Various identities for these polynomials are also proved. We list a few of these below.

(1) Binet type formula

$$g_n(x,t) = c_1\alpha^n + c_2\beta^n + zt^n, \tag{1.9}$$

where $c_1 = \frac{1+z(\beta-t)}{\alpha-\beta}$, $c_2 = -\frac{1+z(\alpha-t)}{\alpha-\beta}$, $\alpha = \frac{x+\sqrt{x^2+4}}{2}$, $\beta = \frac{x-\sqrt{x^2+4}}{2}$ and $z = \frac{A}{t^2-xt-1}$.

(2) Generating function

$$G(x,t,u) = \frac{u(1+(A-t)u)}{(1-xu-u^2)(1-tu)}. \tag{1.10}$$

Thus, we have

$$\sum_{n=0}^{\infty} g_n(x,t)\ u^n = \frac{u\big(1+(A-t)u\big)}{(1-xu-u^2)(1-tu)}. \tag{1.11}$$

The pseudo Fibonacci polynomial $g_n(x,t)$ can be written in the combinatorial form using (1.2) and (1.7).

(3) Combinatorial form

$$g_n(x,t) = \sum_{k=0}^{[\frac{n-1}{2}]}\binom{n-1-k}{k}x^{n-1-2k} + A\sum_{i=0}^{n-2}\sum_{k=0}^{[\frac{n-2-i}{2}]}\binom{n-2-i-k}{k}x^{n-2-i-2k}\ t^i. \tag{1.12}$$

In this paper, we present identities of pseudo Fibonacci polynomials involving derivatives of these polynomials. We also present the r^{th} order derivative of these polynomials in the form of convolution of Fibonacci polynomials and pseudo Fibonacci polynomials. In the last section we prove that n^{th} pseudo Fibonacci polynomial is a solution of second order linear hypergeometric differential equation.

2. Identities of pseudo Fibonacci polynomials

In this section, we first obtain some identities involving derivatives of pseudo Fibonacci polynomials. Then we prove convolution property for rth derivatives of $g_n(x,t)$.

Theorem 2.1. *For all $n, r \in \mathbb{Z}^+$,*

$$\frac{\partial^r g_{n+1}}{\partial x^r} = n\,\frac{\partial^{r-1} g_n}{\partial x^{r-1}} - \frac{\partial^r g_{n-1}}{\partial x^r} - \frac{\partial^r g_{n+1}}{\partial t \partial x^{r-1}}. \tag{2.1}$$

Proof. Differentiating the equation (1.7) with respect to x, we get

$$\frac{\partial g_n}{\partial x} = \frac{dF_n}{dx} + A\sum_{i=0}^{n-1} \frac{dF_{n-1-i}}{dx}\, t^i. \tag{2.2}$$

Thus, we have

$$\begin{aligned}
\frac{\partial g_{n+1}}{\partial x} + \frac{\partial g_{n-1}}{\partial x} &= \frac{dF_{n+1}}{dx} + \frac{dF_{n-1}}{dx} + A\sum_{i=0}^{n} \frac{dF_{n-i}}{dx} t^i + A\sum_{i=0}^{n-2} \frac{dF_{n-2-i}}{dx} t^i \\
&= nF_n + A\sum_{i=0}^{n-1} (n-1-i) F_{n-1-i}\, t^i, \text{ using (1.5)} \\
&= nF_n + nA\sum_{i=0}^{n-1} F_{n-1-i}\, t^i - A\sum_{i=0}^{n-1} (i+1) F_{n-1-i}\, t^i \\
&= ng_n - \frac{\partial g_{n+1}}{\partial t}.
\end{aligned}$$

Therefore

$$\frac{\partial g_{n+1}}{\partial x} = ng_n - \frac{\partial g_{n-1}}{\partial x} - \frac{\partial g_{n+1}}{\partial t}. \tag{2.3}$$

Differentiating equation (2.3) both sides $(r-1)$ times, with respect to x, we obtain (2.1). □

Theorem 2.2. *For $n, r \in \mathbb{Z}^+$,*

$$(x^2+4)\frac{\partial g_n}{\partial x} = n\big(g_{n+1} + g_{n-1}\big) - xg_n - \frac{\partial g_{n+2}}{\partial t} - \frac{\partial g_n}{\partial t}. \tag{2.4}$$

Proof. We prove (2.4) by induction on n. Clearly, the result holds for $n = 1, 2$. Let $k \geq 2$. Assume that the result is true for $n \leq k$. We shall prove that it is true for $n = k+1$.

Taking $n = k+1$ in (1.6) and differentiating it with respect to x, we get

$$\frac{\partial g_{k+1}}{\partial x} = g_k + x\frac{\partial g_k}{\partial x} + \frac{\partial g_{k-1}}{\partial x}.$$

Therefore

$$\begin{aligned}(x^2+4)\frac{\partial g_{k+1}}{\partial x} &= (x^2+4)g_k + x(x^2+4)\frac{\partial g_k}{\partial x} + (x^2+4)\frac{\partial g_{k-1}}{\partial x}\\ &= (x^2+4)g_k + xk\big(g_{k+1}+g_{k-1}\big) - x^2 g_k - x\frac{\partial g_{k+2}}{\partial t} - x\frac{\partial g_k}{\partial t}\\ &+ (k-1)\big(g_k+g_{k-2}\big) - xg_{k-1} - \frac{\partial g_{k+1}}{\partial t} - \frac{\partial g_{k-1}}{\partial t}.\end{aligned}$$

Further simplification gives

$$\begin{aligned}(x^2+4)\frac{\partial g_{k+1}}{\partial x} &= (k+1)\Big(xg_{k+1}+g_k+At^k+xg_{k-1}+g_{k-2}+At^{k-2}\Big)\\ &- x(xg_k+g_{k-1}+At^{k-1}) - \frac{\partial g_{k+3}}{\partial t} - \frac{\partial g_{k+1}}{\partial t}.\end{aligned}$$

This imples

$$(x^2+4)\ \frac{\partial g_{k+1}}{\partial x} = (k+1)\Big(g_{k+2}+g_k\Big) - x(g_{k+1}) - \frac{\partial g_{k+3}}{\partial t} - \frac{\partial g_{k+1}}{\partial t}.$$

Therefore, by induction, the theorem is proved. □

Next, if we differentiate the equation (2.4) both sides $(r-1)$ times, with respect to x, and rearrange the terms then we obtain the following result.

Theorem 2.3. *For* $n, r \in \mathbb{Z}^+$,

$$\begin{aligned}(x^2+4)\frac{\partial^r g_n}{\partial x^r} &= n\Big(\frac{\partial^{r-1}g_{n+1}}{\partial x^{r-1}} + \frac{\partial^{r-1}g_{n-1}}{\partial x^{r-1}}\Big) - (2r-1)x\frac{\partial^{r-1}g_n}{\partial x^{r-1}}\\ &- (r-1)^2\ \frac{\partial^{r-2}g_n}{\partial x^{r-2}} - \frac{\partial^r g_{n+2}}{\partial t\partial x^{r-1}} - \frac{\partial^r g_n}{\partial t\partial x^{r-1}}.\end{aligned} \tag{2.5}$$

Theorem 2.4. *For* $n, r \in \mathbb{Z}^+$,

$$\frac{\partial^r g_{n+1}}{\partial x^r} = \begin{cases} 0, n < r;\\ r!, n = r;\\ \frac{1}{n-r}\Big[nx\frac{\partial^r g_n}{\partial x^r} + (n+r)\frac{\partial^r g_{n-1}}{\partial x^r} + r\frac{\partial^r g_{n+1}}{\partial t\,\partial x^{r-1}}\Big], n > r.\end{cases} \tag{2.6}$$

Proof. Note that g_{n+1} is the $(n+1)^{th}$ polynomial having n as the highest degree in x. Therefore, $\frac{\partial^r g_{n+1}}{\partial x^r} = 0$, for $r > n$ and $\frac{\partial^r g_{n+1}}{\partial x^r} = r!$, for $r = n$.
If $n > r$, then we prove the result by induction on r.
Differentiating equation (1.6) and then multiplying the resulting equation throughout by n,we get

$$n\frac{\partial g_{n+1}}{\partial x} = ng_n + nx\frac{\partial g_n}{\partial x} + n\frac{\partial g_{n-1}}{\partial x}.$$

Therefore, from equation (2.3), we get

$$(n-1)\frac{\partial g_{n+1}}{\partial x} = nx\frac{\partial g_n}{\partial x} + (n+1)\frac{\partial g_{n-1}}{\partial x} + \frac{\partial g_{n+1}}{\partial t}.$$

Thus, the result is true for $r = 1$.
Assume that it is true for $r = k$. Therefore,

$$\frac{\partial^k g_{n+1}}{\partial x^k} = \frac{1}{n-k}\Big[nx\frac{\partial^k g_n}{\partial x^k} + (n+k)\frac{\partial^k g_{n-1}}{\partial x^k} + k\frac{\partial^k g_{n+1}}{\partial t\,\partial x^{k-1}}\Big].$$

Differentiating with respect to x

$$(n-k)\ \frac{\partial^{k+1} g_{n+1}}{\partial x^{k+1}} = \Big[n\frac{\partial^k g_n}{\partial x^k} + nx\frac{\partial^{k+1} g_n}{\partial x^{k+1}} + (n+k)\frac{\partial^{k+1} g_{n-1}}{\partial x^{k+1}} + k\frac{\partial^{k+1} g_{n+1}}{\partial t\,\partial x^k}\Big]. \tag{2.7}$$

Differentiating (2.3) k times, with respect to x, we get

$$n\frac{\partial^k g_n}{\partial x^k} = \frac{\partial^{k+1} g_{n+1}}{\partial x^{k+1}} + \frac{\partial^{k+1} g_{n-1}}{\partial x^{k+1}} + \frac{\partial^{k+1} g_{n+1}}{\partial t \partial x^k}. \tag{2.8}$$

Substituting R.H.S. of (2.8) in (2.7) in place of $n\frac{\partial^k g_n}{\partial x^k}$, we get

$$(n-k-1)\ \frac{\partial^{k+1} g_{n+1}}{\partial x^{k+1}} = \Big[nx\frac{\partial^{k+1} g_n}{\partial x^{k+1}} + (n+k+1)\frac{\partial^{k+1} g_{n-1}}{\partial x^{k+1}} + (k+1)\frac{\partial^{k+1} g_{n+1}}{\partial t\,\partial x^k}\Big].$$

Thus, the result is true for $n = k+1$. Hence, by induction on r the result follows. □

Note that using (2.1) and (2.6), we obtain the following.

$$(n-r)\frac{\partial^{r-1} g_n}{\partial x^{r-1}} = x\frac{\partial^r g_n}{\partial x^r} + 2\frac{\partial^r g_{n-1}}{\partial x^r} + \frac{\partial^r g_{n+1}}{\partial t \partial x^r}, \text{ for all } n \geq r.$$

Theorem 2.5. *For all $n, r \in \mathbb{Z}^+$ and $n \geq r$,*

$$\frac{\partial^r g_{n+1}}{\partial x^r} = \sum_{j=0}^{[\frac{n-r}{2}]} (-1)^j \Big[(n-2j)\frac{\partial^{r-1} g_{n-2j}}{\partial x^{r-1}} - \frac{\partial^r g_{n+1-2j}}{\partial t \partial x^{r-1}}\Big]. \tag{2.9}$$

Proof. We first prove that

$$\frac{\partial g_{n+1}}{\partial x} = \sum_{j=0}^{[\frac{n-1}{2}]} (-1)^j \Big[(n-2j)g_{n-2j} - \frac{\partial g_{n+1-2j}}{\partial t}\Big], \text{for all } n \geq 1. \tag{2.10}$$

We prove (2.10) by induction on n. For $n = 1$, the result is true. Assume that (2.10) is true for $n \leq m$. Note that m may be an even or odd positive integer, accordingly, we have two cases, $m = 2k$ and $m = 2k+1$, where $k \in \mathbb{Z}^+$.

Let $m = 2k$. Then, we have

$$\frac{\partial g_{2k+1}}{\partial x} = \sum_{j=0}^{k-1} (-1)^j \Big[(2k-2j) g_{2k-2j} - \frac{\partial g_{2k+1-2j}}{\partial t}\Big]. \tag{2.11}$$

Also,

$$\frac{\partial g_{2k}}{\partial x} = \sum_{j=0}^{k-1} (-1)^j \Big[(2k-1-2j) g_{2k-1-2j} - \frac{\partial g_{2k-2j}}{\partial t}\Big]. \tag{2.12}$$

Equation (1.6) implies

$$\begin{aligned}
\frac{\partial g_{2k+2}}{\partial x} &= g_{2k+1} + x\frac{\partial g_{2k+1}}{\partial x} + \frac{\partial g_{2k}}{\partial x} \\
&= g_{2k+1} + x\sum_{j=0}^{k-1} (-1)^j \Big[(2k-2j) g_{2k-2j} - \frac{\partial g_{2k+1-2j}}{\partial t}\Big] \\
&+ \sum_{j=0}^{k-1} (-1)^j \Big[(2k-1-2j) g_{2k-1-2j} - \frac{\partial g_{2k-2j}}{\partial t}\Big] \\
&= g_{2k+1} + \sum_{j=0}^{k-1} (-1)^j \Big[(2k-2j)\Big(g_{2k+1-2j} - At^{2k-1-2j}\Big) \\
&\qquad - \frac{\partial}{\partial t}\Big(x g_{2k+1-2j} + g_{2k-2j} - At^{2k-2j}\Big)\Big] \\
&= g_{2k+1} + \sum_{j=0}^{k-1} (-1)^j \Big[(2k+1-2j)\Big(g_{2k-2j} - At^{2k-1-2j}\Big) \\
&\qquad - \frac{\partial g_{2k+2-2j}}{\partial t} + (2k-2j)At^{2k-1-2j}\Big].
\end{aligned}$$

Further simplification yields

$$\frac{\partial g_{2k+2}}{\partial x} = \sum_{j=0}^{k} (-1)^j \Big[(2k+1-2j) g_{2k+1-2j} - \frac{\partial g_{2k+2-2j}}{\partial t}\Big].$$

Similarly, the result can be proved for $m = 2k+1$. Thus, the result is valid for $n = m+1$. Hence, by induction on n, (2.10) is proved.

Now differentiate the equation (2.10), $(r-1)$ times, both sides with respect to x, to obtain

$$\frac{\partial^r g_{n+1}}{\partial x^r} = \sum_{j=0}^{[\frac{n-1}{2}]} (-1)^j \Big[(n-2j)\frac{\partial^{r-1} g_{n-2j}}{\partial x^{r-1}} - \frac{\partial^r g_{n+1-2j}}{\partial t \partial x^{r-1}}\Big]. \tag{2.13}$$

Equation (2.6) implies $\frac{\partial^r g_{n+1}}{\partial x^r} = 0$, if $n < r$. Therefore, we have

$$\frac{\partial^r g_{n+1}}{\partial x^r} = \sum_{j=0}^{[\frac{n-r}{2}]} (-1)^j \Big[(n-2j)\frac{\partial^{r-1} g_{n-2j}}{\partial x^{r-1}} - \frac{\partial^r g_{n+1-2j}}{\partial t \partial x^{r-1}}\Big], \text{ for all } n \geq r.$$

□

Using (2.1) and (2.9), we obtain the following.

$$\frac{\partial^r g_{n-1}}{\partial x^r} = \sum_{j=1}^{[\frac{n-r}{2}]} (-1)^{j+1} \Big[(n-2j)\frac{\partial^{r-1} g_{n-2j}}{\partial x^{r-1}} - \frac{\partial^r g_{n+1-2j}}{\partial t \partial x^{r-1}}\Big], \text{ for all } n \geq r+2.$$

Convolved Fibonacci numbers and polynomials have been considered in [12, 14] to derive various properties. We derive the convolution formula for derivatives of pseudo Fibonacci polynomials.

Theorem 2.6. ***Convolution property:***

(i) For all $n, r \in \mathbb{Z}^+$,

$$\frac{\partial^r g_n}{\partial x^r} = r \sum_{i=0}^{n} \Big(\frac{d^{r-1} F_i}{dx^{r-1}}\Big) g_{n-i}. \tag{2.14}$$

(ii) For all $n, r \in \mathbb{Z}^+$ and $n \geq r$,

$$\frac{\partial^r g_n}{\partial t^r} = r! \sum_{i=0}^{n-r} \Big(g_{n-r-i} - F_{n-r-i}\Big) \binom{i+r-1}{r-1} t^i. \tag{2.15}$$

Proof. (i) Let $n \geq 0$ and $r \in \mathbb{Z}^+$.

Differentiating (1.11) r times, with respect to x, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\partial^r g_n}{\partial x^r} u^n &= \frac{r!\, u^{r+1}(1+(A-t)u)}{(1-xu-u^2)^{r+1}(1-tu)} \\ &= \frac{r\,(r-1)!\, u^r}{(1-xu-u^2)^r} \frac{u(1+(A-t)u)}{(1-xu-u^2)(1-tu)} \\ &= r\Big(\sum_{n=0}^{\infty} \frac{d^{r-1}F_n}{dx^{r-1}} u^n\Big)\Big(\sum_{n=0}^{\infty} g_n u^n\Big).\end{aligned}$$

Equating the coefficient of u^n, we get

$$\frac{\partial^r g_n}{\partial x^r} = r \sum_{i=0}^{n} \Big(\frac{d^{r-1} F_i}{dx^{r-1}}\Big) g_{n-i}.$$

(ii) Differentiating (1.11) both sides r times, with respect to t, we get

$$\sum_{n=0}^{\infty} \frac{\partial^r g_n}{\partial t^r} u^n = \frac{u}{(1-xu-u^2)} \frac{\partial^r}{\partial t^r}\Big[\frac{\Big(1+(A-t)u\Big)}{(1-tu)}\Big]$$

$$= \frac{u}{(1-xu-u^2)} \Big\{\Big((1+(A-t)u\Big)\Big[\frac{r!u^r}{(1-tu)^{r+1}}\Big] - ru\Big[\frac{(r-1)!u^{r-1}}{(1-tu)^r}\Big]\Big\}$$

$$= \frac{r!u^{r+1}\Big((1+(A-t)u\Big)}{(1-xu-u^2)(1-tu)} \Big[\frac{1}{(1-tu)^r}\Big] - \frac{r!u^{r+1}}{(1-xu-u^2)}\Big[\frac{1}{(1-tu)^r}\Big]$$

$$= r!\Big[\Big(\sum_{n=0}^{\infty} g_n u^{n+r}\Big)\Big(\sum_{n=0}^{\infty}\binom{n+r}{r} t^n u^n\Big)\Big]$$

$$- r!\Big[\Big(\sum_{n=0}^{\infty} F_n u^{n+r}\Big)\Big(\sum_{n=0}^{\infty}\binom{n+r}{r} t^n u^n\Big)\Big].$$

Equating the coefficient of u^n, we get

$$\frac{\partial^r g_n}{\partial t^r} = r!\sum_{i=0}^{n-r}\binom{i+r-1}{r-1}\Big[g_{n-r-i} - F_{n-r-i}\Big]t^i.$$

□

3. Pseudo Fiboanacci polynomials and Differential equations

Hypergeometric functions play an important role in Mathematics and Physics. Many special functions can be deduced from it. Euler introduced this function as a power series defined by

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},$$

where a, b, c are rational parameters, $c \neq 0, -1, -2, \cdots$, $Re(c) > Re(b) > 0$ and $|z| < 1$. He also proved that this series satisfies hypergeometric equation, which is a second-order linear differential equation

$$z(1-z)\frac{d^2y}{dz^2} + [c-(a+b+1)z]\frac{dy}{dz} - aby = 0. \tag{3.1}$$

This differential equation occurs in many branches of mathematics, physics, and other sciences; see [13]. In [2], the author uses linear and quadratic transformations of hypergeometric functions to derive various representations of Fibonacci numbers in terms of hypergeometric functions. In [3], the authors prove that the n^{th} Fibonacci polynomial F_n, satisfy the hypergeometric equation

$$(x^2+4)\frac{d^2y}{dx^2}+3x\,\frac{dy}{dx}-(n^2-1)y=0$$

and further showed that $F_n(z)$ can be written as $F_n(z)= {}_2F_1\left(\frac{1-n}{2},\frac{1+n}{2};\frac{3}{2};z\right)$, where $z=1+\frac{x^2}{4}$. In [10], the general solution of the second-order non-homogeneous k-hypergeometric differential equation

$$kz(1-z)\frac{d^2y}{dz^2}+[c-(a+b+1)kz]\frac{dy}{dz}-aby=f(z) \tag{3.2}$$

is obtained, where $a,b,c\in\mathbb{R}, k\in\mathbb{R}^+$, $c\neq 0,-1,-2,\cdots$, $Re(c)>Re(b)>0$, $|z|<1$ and $f(z)=\sum_{i=1}^{m} d_i z^i$, where d_i, $i=0,1,2,\cdots,m$ are real or complex constants.

In this section, using the results of sections 1 and 2, we show that the n^{th} pseudo-Fibonacci polynomial is a solution of a non-homogeneous second-order linear hypergeometric differential equation.

Theorem 3.1. *The pseudo Fibonacci polynomials $g_n(x,t)$, satisfies the non-homogeneous differential equation*

$$(x^2+4)\frac{\partial^2y}{\partial x^2}+3x\,\frac{\partial y}{\partial x}-(n^2-1)y=-A\sum_{i=0}^{n-1}(i+1)(2n-1-i)F_{n-1-i}(x)\,t^i, \tag{3.3}$$

where $F_{n-1-i}(x)$ is $(n-1-i)^{th}$, Fibonacci polynomial, $i=0,1,\cdots,n-1$.

Proof. Differentiating equation (2.4) with respect to x on both sides, rearranging the terms and then using equation (2.3), we get

$$(x^2+4)\frac{\partial^2g_n}{\partial x^2}+3x\frac{\partial g_n}{\partial x}-(n^2-1)g_n=-(2n+1)\frac{\partial g_{n+1}}{\partial t}+\frac{\partial^2g_{n+2}}{\partial t^2}.$$

From equation (1.8), we have

$$(x^2+4)\frac{\partial^2g_n}{\partial x^2}+3x\frac{\partial g_n}{\partial x}-(n^2-1)g_n=-(2n+1)A\sum_{i=0}^{n-1}(i+1)F_{n-1-i}(x)t^i$$
$$+A\sum_{i=0}^{n-1}(i+2)(i+1)F_{n-1-i}(x)t^i.$$

Simplifying further, we get

$$(x^2+4)\frac{\partial^2g_n}{\partial x^2}+3x\frac{\partial g_n}{\partial x}-(n^2-1)g_n=-A\sum_{i=0}^{n-1}(1+i)(2n-1-i)F_{n-1-i}(x)t^i.$$

This completes the proof. □

Theorem 3.2. *For $n \geq 1$, the n^{th} pseudo-Fibonacci polynomial $g_n(x,t)$, satisfies the non-homogeneous second-order linear hypergeometric differential equation*

$$z(1-z)\frac{\partial^2 y}{\partial z^2}+\big(\frac{3}{2}-2z\big)\frac{\partial y}{\partial z}+\frac{n^2-1}{4}y=\frac{A}{4}\sum_{i=0}^{n-1}(i+1)(2n-1-i)F_{n-1-i}(z)\,t^i, \tag{3.4}$$

where $F_{n-1-i}(z) = {}_2F_1\big(1-\frac{n-i}{2}, \frac{n-i}{2}; \frac{3}{2}; z\big)$, $i = 0, 1, \cdots, n-1$.

Proof. Take $z = 1+\frac{x^2}{4}$. Therefore $\frac{\partial y}{\partial x} = \sqrt{z-1}\,\frac{\partial y}{\partial z}$, $\frac{\partial^2 y}{\partial x^2} = (z-1)\frac{\partial^2 y}{\partial z^2}+\frac{1}{2}\frac{\partial y}{\partial z}$. Using equation (3.3), we get

$$\begin{aligned}4z\Big((z-1)\frac{\partial^2 y}{\partial z^2}+\frac{1}{2}\frac{\partial y}{\partial z}\Big)+6(z-1)\frac{\partial y}{\partial z}-(n^2-1)y\\ =-A\sum_{i=0}^{n-1}(i+1)(2n-1-i)F_{n-1-i}(z)\,t^i.\end{aligned}$$

Dividing throughout by -4 and simplifying further we get the required result. □

Acknowledgement

The authors are grateful to the anonymous referee for his/her comments and useful suggestions to improve the quality of this paper.

References

[1] Arolkar, S., On the derivatives of B-Tribonacci Polynomials, *Notes Number Theory Discrete Math.* **28 (3)**(2022), 491–499.

[2] Dilcher, K. Hypergeometric functions and Fibonacci numbers *Fibonacci Quart.* **38 (4)** 2000, 342–363.

[3] Erkus-Duman, E. and Cifti, H., Fibonacci and Lucas Differential Equations, *Appl. Appl. Math.* **13(2)** (2018), 756–763.

[4] Falcon, S. and Plaza, A´., On k-Fibonacci sequences and polynomials and their derivatives, *Chaos Solit. Fractals* **39 (3)**(2009), 1005–1019.

[5] Filipponi, P. and Horadam, A., Derivative sequences of Fibonacci and Lucas polynomials, *In: Bergum, G. E., Philippou, A. N., Horadam, A. F. (eds.) Applications of Fibonacci numbers* Springer, Dordrecht **4**(1991), 99-108.

[6] Filipponi, P. and Horadam, A., Second derivative sequences of Fibonacci and Lucas polynomials, *Fibonacci Quart.* **31(3)**(1993), 194-204.

[7] Horadam, A. F., Jacobsthal Representation polynomials, *Fibonacci Quart.* **35(2)**(1997), 137-148.

[8] Horadam, A. F. and Filipponi, P., Morgan-Voyce polynomial derivative sequences, *Fibonacci Quart.* **39(2)**(2001), 116–122.

[9] Koshy T., *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, Inc., N. York, 2001.

[10] Liu, S. and Dong, Y., k-Hypergeometric Series Solutions to One Type of Non-Homogeneous k-Hypergeometric Equations, *Symmetry* **11** (2)(2019),262. https://doi.org/10.3390/sym11020262.

[11] Phadte, C. N. and Valaulikar Y. S., Pseudo Fibonacci Polynomials and Some Properties, *Bull. Marathwada Math. Soc.* **16(2)**(2015), 13-18.

[12] Ramírez, J. L., On convolved generalized Fibonacci and Lucas polynomials, *Appl. Math. Comput.* **229**(2014), 208–213.

[13] Rao, K. S. and Lakshminarayanan, V., Generalized Hypergeometric Functions: Transformations and Group Theoretical Aspects; *IOP Publishing*: Bristol, UK, 2018.

[14] Ricci, P. E., and Natalini, P., General Linear Recurrence Sequences and Their Convolution Formulas, *Axioms* **8(4)**(2019), 132; https://doi.org/10.3390/axioms8040132.

[15] Swamy, M. N. S., Generalized Fibonacci and Lucas polynomials, and their associated diagonal polynomials, *Fibonacci Quart.* **37(3)**(1999), 213–222.

[16] Wang, J., On the k^{th} Derivative sequences of Fibonacci and Lucas polynomials, *Fibonacci Quart.* **33(2)**(1995), 174–178.

Suchita Arolkar
Department of Mathematics and Statistics
Dnyanprassarak Mandal's College and Research Centre
Assagao-Goa, 403 507 India.
E-mail: suchita@dmscollege.com, suchitagolatkar2911@gmail.com

C. N. Phadte
Department of Mathematics
G. V. M.'s College
Ponda-Goa, 403 206 India.
E-mail: dbyte09@gmail.com

Y. S. Valaulikar
Ex-faculty, Department of Mathematics
Goa University
Taleigao Plateau Goa, 403 206 India.
E-mail: valaulikarys@gmail.com

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July-December (2024), 28–48

THE PHYSIOGNOMY OF THE ERDOS-SZEKERES CONJECTURE (HAPPY ENDING PROBLEM)

PRASAD BALAKRISHNAN WARRIER
(Received : 01 - 09 - 2023 ; Revised : 02 - 10 - 2024)

Abstract. The Erdos-Szekeres conjecture provides a functional relationship between the number of sides of a convex polygon and the minimum number of points in general position on a plane required to construct it. While the conjecture is true for small values of n, it remains unproven. This paper studies the physiognomy of the arrangement of points in relation to this conjecture and uses a shading technique to determine the number of points needed for hexagons and heptagons. The findings indicate that the conjecture may not be true and that the relation may be governed by another series.

1. Introduction

Mathematician Esther Klein observed that four out of any five points on a plane in general position are the vertices of a convex polygon. Following this, Klein brought up a more generic problem statement with her then fellow mathematicians, Paul Erdos and George Szekeres, who were part of the group she was working with at that time. Klein's problem stated: "What is the smallest number $P(n)$ such that any set of $P(n)$ points in the plane in general position has a subset of size n that are the vertices of a convex polygon?" Erdos and Szekeres worked on this problem extensively and coined the conjecture in 1937 based on observations made with three, four and five sided convex polygons.

2010 Mathematics Subject Classification: 52C10
Key words and phrases: Erdos-Szekeres conjecture, Happy ending problem, Convex polygons, Geometry, Proof

The conjecture stated that if the number of sides of a convex polygon is n, then the minimum number of points on a plane in general position (i.e., no 3 points are collinear) required to construct this polygon is given by $P(n) = 2^{n-2} + 1$ for all $n \geq 3$.

n	$P(n)$
3	3
4	5
5	9
6	17
7	33
8	65

TABLE 1. Values of n and $P(n)$ as per the conjecture

The conjecture is illustrated for $n = 4$ here below. The left side of the below figure shows an arrangement of 4 points on a plane from which a convex quadrilateral cannot be constructed. The right side of the figure shows how the addition of a 5th point to the same arrangement allows the construction of a convex quadrilateral.

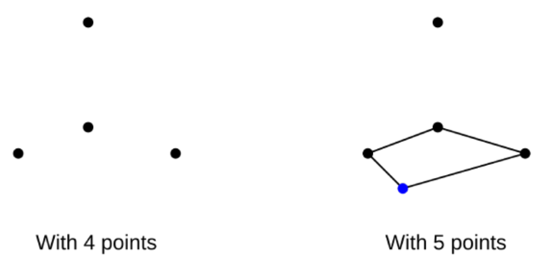


FIGURE 1. Example for P(4) = 5

By constructing explicit examples, Erdos and Szekeres later proved that:
$P(n) \geq 2^{n-2} + 1$

In 2016, Andrew Suk, proved that
$P(n) \leq 2^{n+o(n)}$, for $n \geq 7$

Suk also proved that for a sufficiently large n,
$P(n) \leq 2^{n+6n^{2/3}logn}$

Andreas F. Holmsen, Hossein Nassajian Mojarrad, János Pach and Gábor Tardos [?] claimed an improvement over Suk's proof in 2020:
$P(n) \leq 2^{n+O(\sqrt{nlogn})}$

Erdos named the problem the Happy Ending Problem since it led to the marriage of Klein and Szekeres in 1937.

2. Methods

2.1. **Convex Polygon.** A polygon in which all interior angles have a measure of less than 180° is a convex polygon. In such a polygon, all vertices point outwards, i.e., away from the center of the polygon. In other words, the edges of a convex polygon always turn in the same direction; clockwise or anti-clockwise. In contrast, a concave polygon has at least one internal angle that is greater than 180°. In the figure below, the interior angle at vertex F, ($\angle AFE$) is less than 180° in the convex polygon, and greater than 180° in the concave polygon.

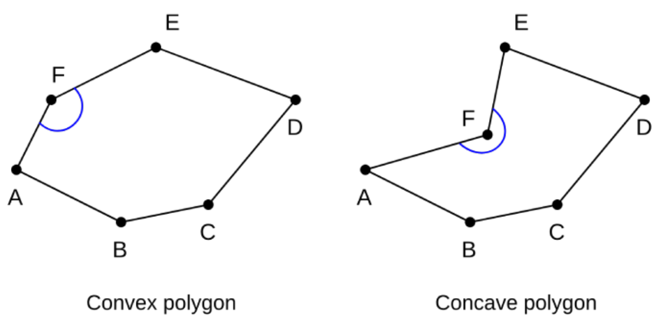


Figure 2. Difference between a Convex and a Concave polygon

2.2. **Regions around a convex polygon.** Any point on a plane can be added as a vertex to a convex n-gon. Depending on the region from which this new point is selected, the new $(n+1)$-gon may become convex or concave. In other words, the construction of a convex polygon splits the plane into two mutually exclusive regions based on the potential they have to permit the addition of a vertex to the polygon maintaining its convexity. This paper will examine these regions and will refer to these often. For convenience, I will refer to the regions that permit the addition of a vertex to the polygon (and retain convexity) as **Happy Regions**, named after this conjecture. Happy regions are obtained by extending alternate sides of a polygon. Extended alternate sides of a convex polygon may be parallel or intersecting lines. If the lines intersect on the side between the alternate sides, the region is finite and triangular in shape. If they do not intersect on the side or are parallel, they expand to ∞.

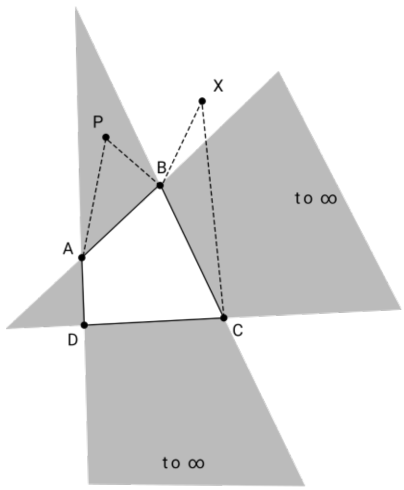


FIGURE 3. Happy regions around a quadrilateral

In the above illustration, the shaded areas around quadrilateral $ABCD$ are Happy regions. The point P in this region may be added so as to create the convex polygon $APBCD$. On the other hand, point X cannot be added since $ABXCD$ is a concave polygon. As mentioned above, Happy regions are either triangular in shape as seen on the sides of AD and AB or expand to ∞ as seen on the sides of BC and CD.

Lemma 2.1. *If a point on a plane lies either inside a convex polygon or in the region of the vertically opposite angle arising from the intersection of its* *any two extended edges, then the point cannot be added as a vertex to the convex polygon retaining its convexity.*

Important: Note that Happy regions must be re-evaluated after the addition of every new vertex.

2.3. **Plane of saturation.** The minimum number of points required to draw an n-gon is one more than the points required to saturate a plane with $(n-1)$-gons. For example, to draw a pentagon you must have exhausted all possible ways to draw a quadrilateral. In other words, the plane must be saturated with quadrilaterals first. Keeping this in mind, lets look at the classic example discussed earlier in Figure **??**. To draw a quadrilateral, we must first exhaust the plane with triangles. This can be explored by drawing Happy regions.

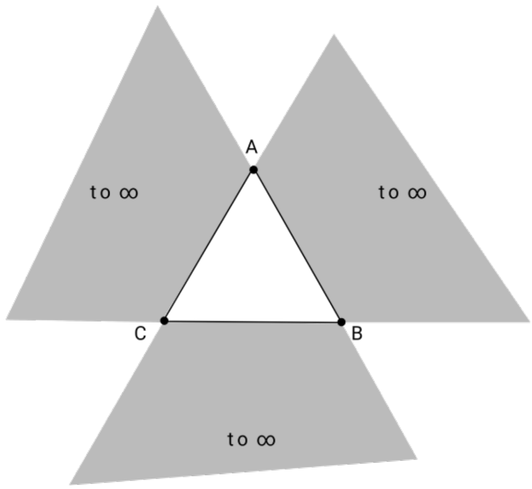


FIGURE 4. Happy regions around a triangle

Figure **??** shows a triangle with Happy regions around it. If a point is added to the unshaded region, it cannot change the triangle to a convex quadrilateral. This means that the plane is not saturated with triangles yet and there is opportunity to add an additional point. Let us add a point D inside the triangle.

Adding point D, (Refer: Figure **??**) we get 4 triangles, namely, $\triangle ABC$, $\triangle ABD$, $\triangle DBC$ and $\triangle DCA$. Figure **??** shows the Happy regions around the triangle $\triangle DCA$. If we also draw the Happy regions for triangles $\triangle ABD$ and $\triangle DBC$, they will cover the whole plane as shown in Figure **??**.

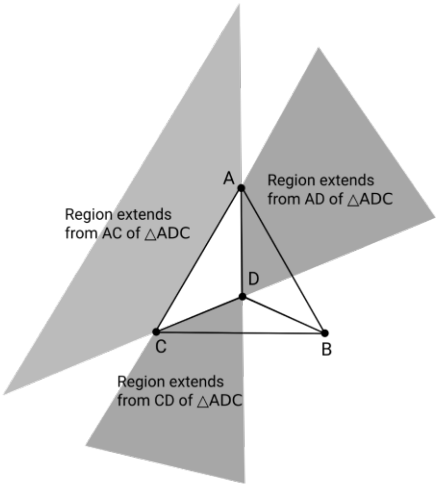


FIGURE 5. Happy regions around $\triangle ADC$

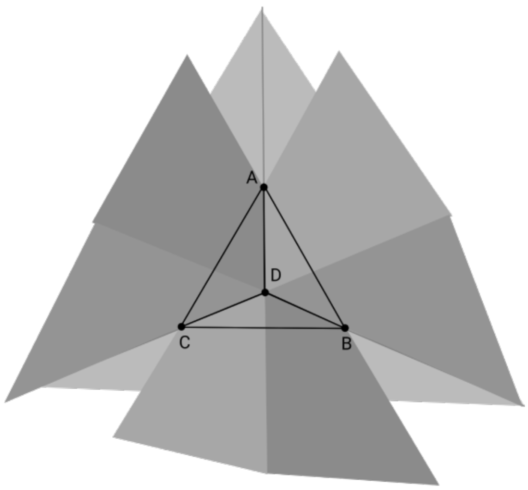


FIGURE 6. Happy Plane of Triangles

This means that adding a point anywhere on the plane now will create at least one quadrilateral. For the purpose of this paper, we will refer to such a plane as a **Happy Plane**. This is how Figure **??** came about. One way to determine the arrangment that saturates a plane with $(n-1)$-gon is to iteratively add points to non-Happy regions and re-draw Happy regions with each point. Although the method is largely inconvenient and does not serve as a proof, it still might help us find these arrangements.

Definitions:

Happy Region: Regions between any two extended alternate sides outside a convex polygon is referred to as a Happy Region. A point selected from this region can be added as a vertex to the convex n-gon to create a convex $(n+1)$-gon.

Happy Plane: If Happy Regions of convex n-gons cover the whole plane and no point lies in the happy regions of any of the convex n-gons, then the plane is said to be saturated with only n-gons. Such a plane is referred to a Happy Plane of n-gons. It must be noted that since no point lies on the Happy Regions, no subset of $(n+1)$-points is the vertex-set of a convex $(n+1)$-gon. Thus, happy planes of $(n-1)$-gons are maximal configurations that do not admit an n-gon.

2.4. **Happy plane of Quadrilaterals.** A convex pentagon can be constructed from 9 points (in general position) on a plane (Refer Table **??**). Therefore, there must be an arrangement of 8 points in which a convex pentagon cannot be constructed. In other words, there must be an arrangment of 8 points that saturates the construction of convex quadrilaterals. The arrangement of 8 points shown in Figure **??** does not allow the construction of a convex pentagon.

To understand why this arrangement is saturated with convex quadrilaterals, we will shade the Happy Regions until we get a Happy Plane or there are no more Happy Regions to shade. To do this, we will connect the points and begin by shading the quadrilaterals facing outwards as shown in Figure **??**. The below images show the shading of Happy regions of quadrilaterals starting with the outward facing quadrilaterals and then the inner regions after drawing the diagonal AC.

As it can be seen, this arrangment creates a Happy Plane of convex quadrilaterals. Adding a point anywhere on the plane will turn at least one of the quadrilaterals into a convex pentagon.

2.5. **Arrangement of points to saturate a plane with convex pentagons.** The conjecture states that a convex hexagon can be constructed with 17 points. In other words, there can be an arrangement of 16 points in which there are no convex hexagons. To check this, we will start

FIGURE 7. An arrangement of 8 points with no convex pentagons

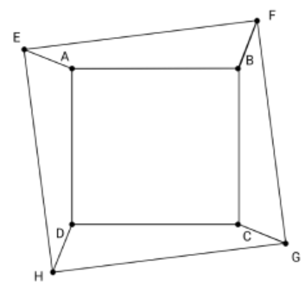
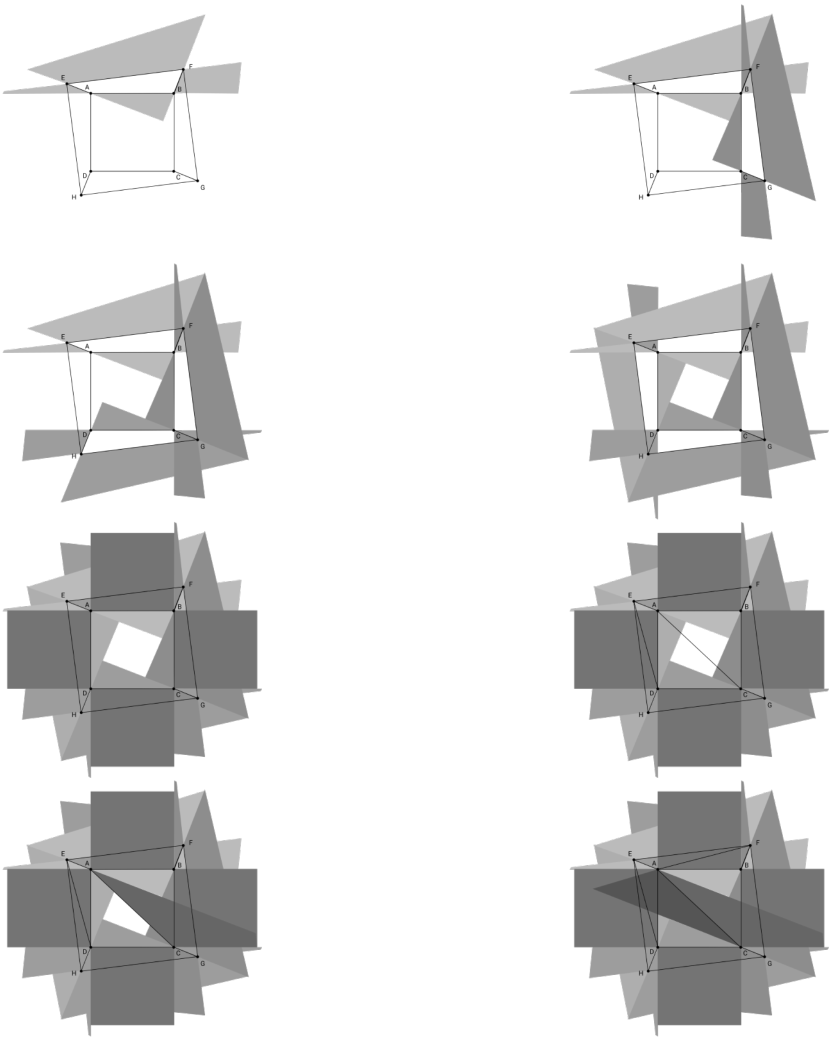


FIGURE 8. The 8 points creates 4 outward facing quadrilaterals when connected

the construction with 5 points at the vertices of a regular convex pentagon and then use trial and error to fix an additional 2 points along each vertex. Since the vertices of a regular pentagon are not in line with its center, we will be able to add one additional point at the center. We will then shade the Happy Regions and if there are non-Happy regions, we will add points in these gaps and iteratively shade the regions until we have a Happy Plane.


E
A
F
B
D
C
H
G

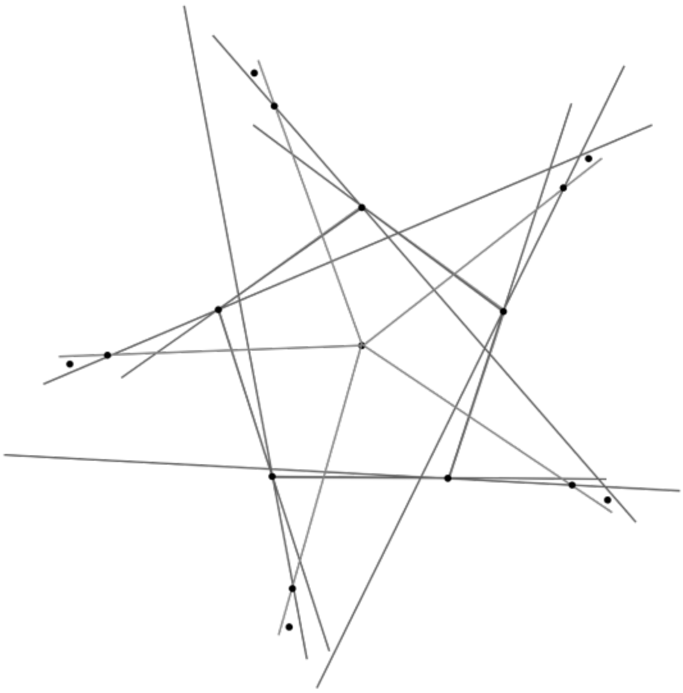


FIGURE 9. Construction of a pentagon with a point at the center and 2 additional points along each vertex

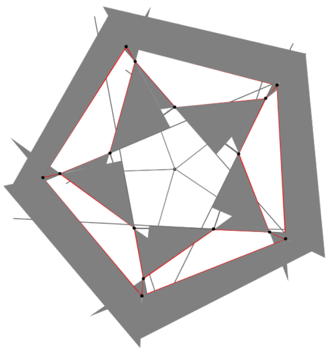


FIGURE 10. Shading Happy Regions of outward facing pentagons

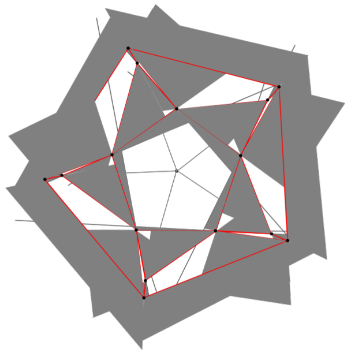


FIGURE 11. Shading Happy Regions of other pentagons (partly)

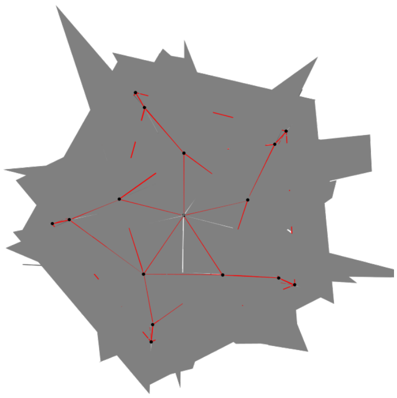


FIGURE 12. A Happy Plane of convex Pentagons is achieved at 16 points and no further shading is needed.

2.5.1. *Observations on Convex Pentagons.* In the above construction, we started off with 5 points and then with trial and error we were able to create an arrangment of 16 points that saturates the plane with convex Pentagons. A Happy Plane was achieved at 16 points and no further shading was required. Given below are the coordinates of the 16 points in the arrangment above for further research:

Vertices: {{x:240, y:186}, {x:319, y:244}, {x:288, y:337}, {x:190, y:336}, {x:160, y:243}}
Along Vertex 1:{{x:191, y:129},{x:180, y:111}}
Along Vertex 2:{{x:352, y:175},{x:367, y:159}}
Along Vertex 3:{{x:357, y:341},{x:377, y:349}}
Along Vertex 4:{{x:201, y:399},{x:199, y:420}}
Along Vertex 5:{{x:98, y:268},{x:77, y:273}}
Center:{{x:240, y:263}}

2.6. **Arrangement of points to saturate a plane with convex hexagons.** The conjecture states that a convex heptagon may be constructed with 33 points. In other words, there can be an arrangement with 32 points in which there are no convex heptagons. We will begin our construction with fewer points, i.e., we will use 6 points to form a regular hexagon.

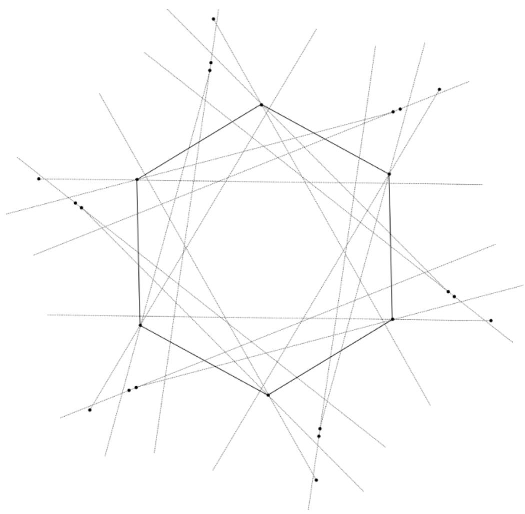


FIGURE 13. Arrangement of 24 points that contains only hexagons.

We will then add 3 additional points near each vertex in a manner that no convex heptagons are created. To accomplish this we will make positional adjustments using trial and error along two vertices and then copy the arrangment to the remaining vertices. This will give us a total of 24 points, a number that is lower than 32 as shown in Figure **??**.

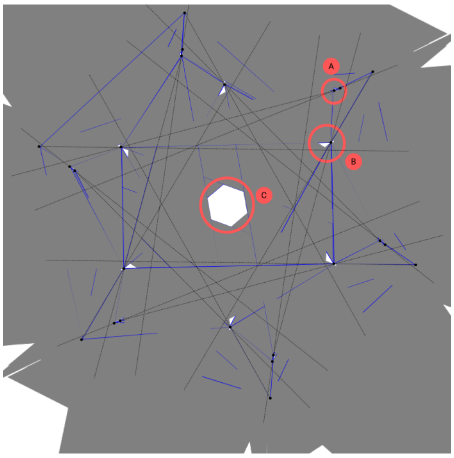


FIGURE 14. Non-Happy Regions: A and B along the vertices and C at the center.

After this, we will shade the Happy regions of hexagons and add new points in the gaps (non-Happy regions) and iteratively reshade them until we have a Happy Plane of convex hexagons. When we finish, we will count the total number of points in the arrangement and verify it with the number stated by the conjecture.

2.6.1. *Observations on Convex Hexagons.* In the above construction, we started off with 24 points and then were able to add only a maximum of 6 points. The Happy Plane was achieved with just 30 points (Refer Figure **??**). This means that this arrangement of 30 points saturates hexagons on a plane and the addition of one more point would result in a convex heptagon. Given below are the coordinates of the 30 points of the arrangment:

Vertices: {{x:396, y:143},{x:586, y:246},{x:591, y:462},{x:406, y:575}, {x:216, y:471},{x:211, y:254}}
Along Vertex 1: {{x:319, y:91},{x:320, y:80},{x:324, y:15}}
Along Vertex 2: {{x:591, y:153},{x:602, y:149},{x:660, y:120}}
Along Vertex 3: {{x:673, y:421},{x:682, y:428},{x:737, y:464}}
Along Vertex 4: {{x:483, y:625},{x:481, y:636},{x:477, y:701}}

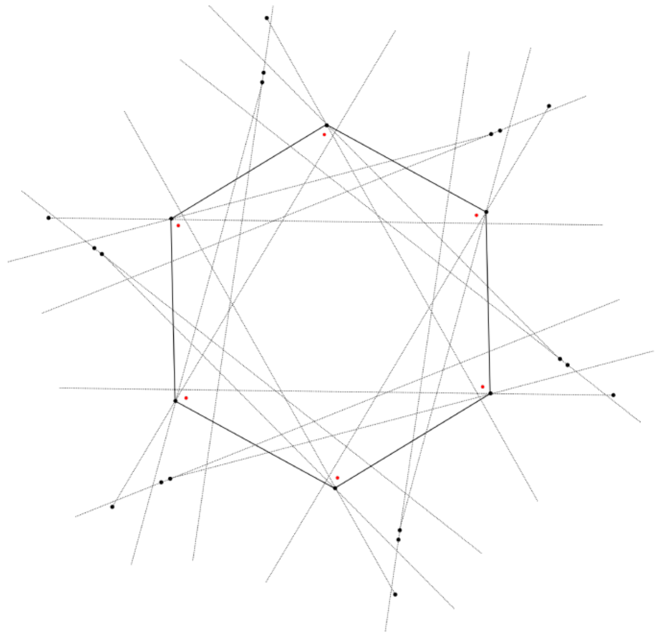


FIGURE 15. Adding 6 points inside B creates a Happy Plane i.e., the new Happy Regions covers A, B and C.

Along Vertex 5: {{x:209, y:563},{x:199, y:568},{x:140, y:597}}
Along Vertex 6: {{x:128, y:296},{x:119, y:289},{x:64, y:253}}
Inner points: {{x:393, y:154},{x:574, y:250},{x:581, y:454},{x:408, y:562}, {x:228, y:467},{x:219, y:262}}

2.7. **Arrangement of points to saturate a plane with convex heptagons.** The conjecture states that a convex octagon may be constructed with 65 points. In other words, there can be an arrangment of 64 points in which there are no convex octagons. To validate this, we will begin our construction in the same way as we did for convex hexagons. We will create an arrangment that makes up a regular heptagon and then add 3 points along each vertex using trial and error. Since we are using a regular convex heptagon, the vertices are not in line with the center of the heptagon. Taking advantage of this, we should be able to add one point at the center as well.

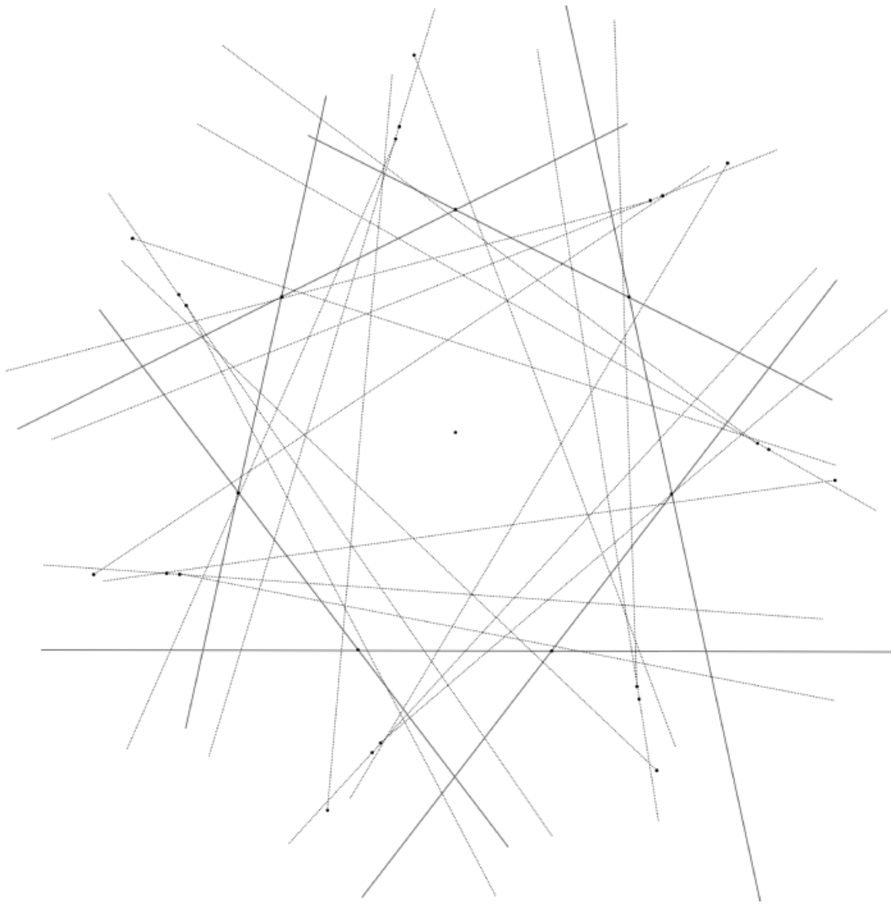


FIGURE 16. Arrangement of 29 points that contains only heptagons.

We will then shade the Happy regions of convex heptagons and add new points in the non-Happy regions and iteratively reshade them until we have a Happy Plane.

Note: In the images, the heptagon drawn was not perfectly regular. Due to this the non-Happy Regions are not perfectly symmetrical. Irregularities have been taken into account when shading the regions and shading is performed as per the structure of the heptagon drawn.

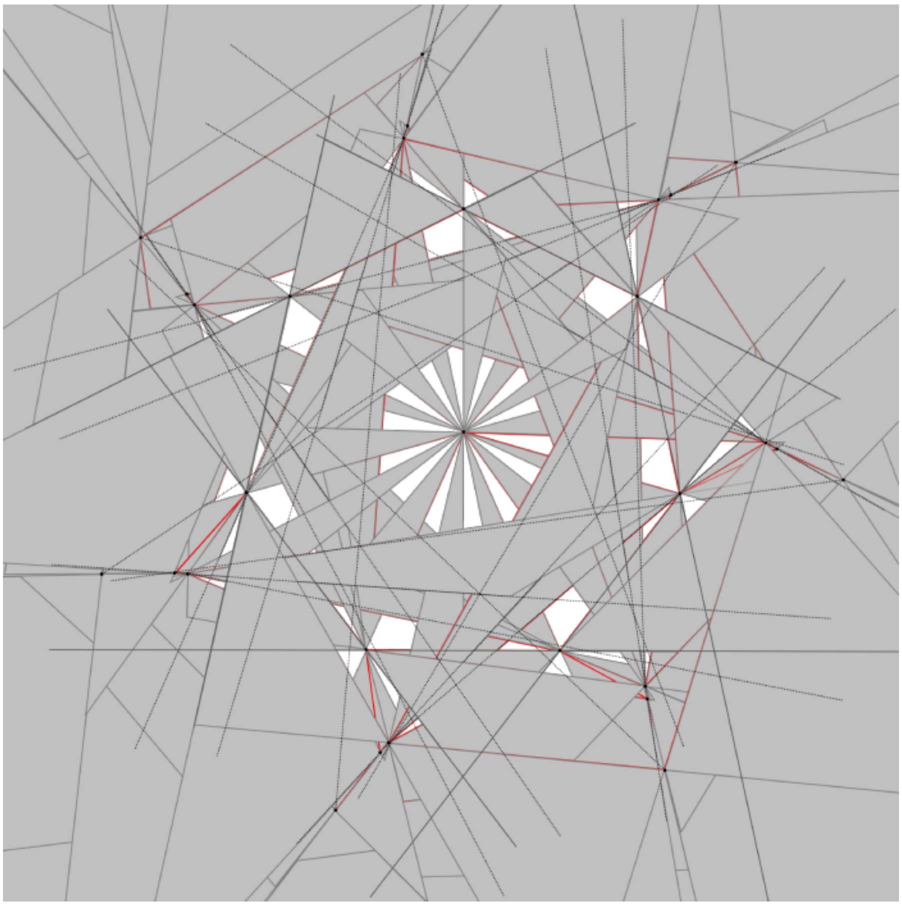


FIGURE 17. Non-Happy Regions are the white regions.

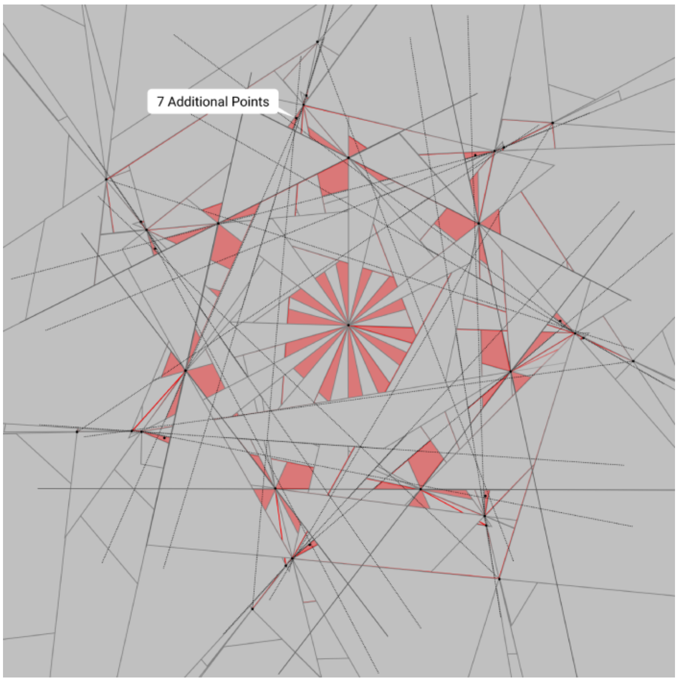


FIGURE 18. Adding the 7 points as shown created a Happy Plane.

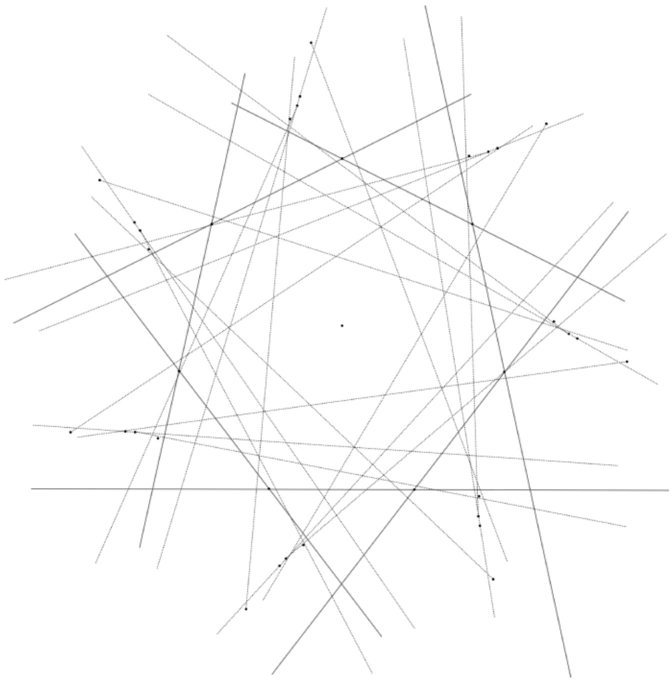


FIGURE 19. Final arrangement of points.

2.7.1. *Observations on Convex Heptagons.* In the above construction, we started off with 29 points and then we were able to add a maximum of 7 more points. The Happy Plane of convex heptagons was achieved with just 36 points (Refer: Figure **??**). This means that this arrangement of 36 points saturates convex heptagons on a plane and the addition of one more point would result in a convex octagon. Given below are the coordinates of the 36 points in the arrangment above for further research:

Vertices: {{x:411, y:182},{x:566, y:260},{x:604, y:436},{x:497, y:576}, {x:324, y:575},{x:217, y:435},{x:256, y:260}}

Along Vertex 1: {{x:358, y:119},{x:361, y:108},{x:374, y:44},{x:349, y:135}}

Along Vertex 2: {{x:585, y:174},{x:596, y:170},{x:654, y:141},{x:562, y:179}}

Along Vertex 3: {{x:681, y:391},{x:691, y:396},{x:750, y:424},{x:663, y:376}}

Along Vertex 4: {{x:573, y:608},{x:575, y:619},{x:591, y:683},{x:574, y:584}}

Along Vertex 5: {{x:344, y:658},{x:337, y:667},{x:297, y:718},{x:365, y:642}}

Along Vertex 6: {{x:165, y:508},{x:153, y:507},{x:88, y:508},{x:192, y:515}}

Along Vertex 7: {{x:171, y:268},{x:164, y:258},{x:123, y:208},{x:181, y:290}}

Central point: {{x:411, y:381}}

2.8. **Indications of a failing hypothesis.** I started off with the hypothesis that the conjecture is true. The points required to saturate a plane with convex pentagons (16) doubled from the number of points required to saturate a plane with convex quadrilaterals (8) and it appeared that the conjecture was true until this point. Continuing on the same hypothesis, I made several attempts to create an arrangment that saturates a plane with convex hexagons using 32 points. It was possible to contain the arrangement to convex hexagons until 24 points. However, 32 points always gave way to convex heptagons. During my study, some arrangements seemed to suggest that this number could only be a maximum of 30. A rigourous exercise of shading the Happy Regions for convex hexagons confirmed this number to be 30. Following this, I then proceeded to shade the Happy Regions for convex heptagons and the Happy Plane was achieved at 36 points.

The table below compares the number of points required to **saturate** a plane ($S(n)$) with convex $n - gons$ as per my constructions against the value proposed by the ES conjecture.

n	$S(n)$ by Construction	$S(n)$ by ES-Conjecture
3	4	4
4	8	8
5	16	16
6	30	32
7	36	64

TABLE 2. Values of n and $S(n)$ by Construction vs by Conjecture

As shown in the table above, the resulting series of saturation points from my constructions is 4,8,16,30,36... . The minimum number of points required to construct an $n - gon$ would therefore be one more than these, i.e., 5,9,17,31,37... . From these constructions and my observation, it appears that the number of points required to create a specific $n - gon$ is lesser than what is described by the conjecture. The next question that comes to mind is: How many points does the conjecture say is required to create a slightly larger polygon, say, a 15-gon (pentadecagon) or a 25-gon (pentacosagon)? If we used the formula in the ES conjecture, this would be 8,193 points for a 15-gon and 83,88,609 points for a 25-gon. A number close to 1 crore is arguably large for a rather small convex construction of a 25-gon, although that can never be a reason for why it should not be.

The series 4,8,16,30,36... was not easy to determine. However, the OEIS Foundation Inc. (2023) has a reference to this series and terms it as a bisection [**?**] of a series 1, 2, 4, 6, 8, 12, 16, 24, 30, 32, 36, 48, 60, 64, 72, 96, 120, 128, 144, 180, 192, 210, 216..., which is the least integer of each prime signature [**?**]. Our constructions confirm with reasonable accuracy that $P(4) = 5$, $P(5) = 9$, $P(6) = 17$, $P(7) = 31$ and $P(8) = 37$ (Saturations at 4,8,16,30,36). However, we cannot be sure if the series would diverge from $P(9)$. With some more research it should be possible to reason why this series presents in this manner and provide a generic proof of the same.

Nonetheless, if we extended our series to a 25-gon, our saturation points $S(n)$, would compare to those in the conjecture as shown in the table below.

n	$S(n)$ by Construction	$S(n)$ by ES-Conjecture
3	4	4
4	8	8
5	16	16
6	30	32
7	36	64
8	60	128
9	72	256
10	120	512
11	144	1024
12	192	2048
13	216	4096
14	256	8192
15	360	16384
16	420	32768
17	480	65536
18	576	131072
19	768	262144
20	864	524288
21	960	1048576
22	1080	2097152
23	1260	4194304
24	1440	8388608
25	1680	16777216

TABLE 3. Values of n and $S(n)$ by Construction vs by Conjecture

3. Concluding Comments

The relationship between the number of sides of a convex polygon and the minimum number of points in general position required to construct it as suggested by the Erdos–Szekeres conjecture (Happy ending problem) is incorrect. Although the ES-conjecture is valid for $n = 3$ (trivial), $n = 4$, $n = 5$ and $n = 6$, constructions indicate that the value of $P(n)$ is lower for $n = 7$ and $n = 8$ when compared to the conjecture. The series 5,9,17,31,37 that we finally have from our constructions is related to prime numbers.

The discussion regarding the significance of this series to this conjecture is beyond the scope of this paper and nothing further needs to be said.

Acknowledgement: We are grateful to the referee for the comments which improved the quality of the paper.

References

[1] P. Erdos and G. Szekeres, *A combinatorial problem in geometry*, Compos. Math. 2 (1935), 463–470.

[2] P. Erdos and G. Szekeres, *On some extremum problems in elementary geometry*, Ann. Univ. Sci. Budapest. E¨otv¨os Sect. Math. 3-4 (1960/1961), 53–62.

[3] https://www.quantamagazine.org/a-puzzle-of-clever-connections-nears-a-happy-end-20170530/

[4] OEIS Foundation Inc. (2023), Bisection of A025487, Entry A099992 in The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A099992

[5] OEIS Foundation Inc. (2023), Least integer of each prime signature A124832; also products of primorial numbers A002110., Entry A025487 in The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A025487

[6] OEIS Foundation Inc. (2023), Number of regions in regular n-gon with all diagonals drawn, Entry A007678 in [7] The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A007678

[7] OEIS Foundation Inc. (2023), Join n equal points around circle in all ways, count regions, Entry A006533 in The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A006533

[8] OEIS Foundation Inc. (2023), Maximal number of regions obtained by joining n points around a circle by straight lines. Also number of regions in 4-space formed by n-1 hyperplanes, Entry A000127 in The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A000127

[9] Wikipedia contributors. (2023, July 22). Dividing a circle into areas. In Wikipedia, The Free Encyclopedia. Retrieved 11:18, August 3, 2023, from `https://en.wikipedia.org/w/index.php?title=Dividing_a_circle_into_areas&oldid=1166504432`

[10] Wikipedia contributors. (2023, August 13). Happy ending problem. In Wikipedia, The Free Encyclopedia. Retrieved 11:32, March 29, 2024, from `https://en.wikipedia.org/w/index.php?title=Happy_ending_problem&oldid=1170093500`

Prasad Balakrishnan Warrier
Design Thinking and Product Management Consultant
Velachery, Chennai
Tamil Nadu - 600 042 INDIA.
E-mail:pbw_in@yahoo.com

The Mathematics Student ISSN: 0025-5742
Vol. 92, Nos. 3-4, July-December (2024), 49–60

INEQUALITIES CONCERNING RATIONAL FUNCTIONS WITH PRESCRIBED POLES

M. Y. MIR, W. M. SHAH
(Received : 14 - 09 - 2023 ; Revised : 18 - 08 - 2024)

Abstract. For a rational function $r \in \mathcal{R}_n$ Wali and Shah [The J. of Anal., **25**(1):(2017), 43-53] proved:

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| + \frac{|c_n| - |c_0|}{|c_n| + |c_0|}\right\}|r(z)|.$$

In this paper, we consider a more general class of rational functions $r(s(z))$ of degree mn, where $s(z)$ is a polynomial of degree m. We use simple techniques to strengthen generalizations of certain results, which extend some well known polynomial inequalities due to Turán and Erdös-Lax to a special class of rational functions $r(s(z))$. In particular we improve as well as generalize the results due to Qasim and Liman [Indian. J. Pure Appl. Math. **46**(3): (2015), 337-348].

1. Introduction

Let $\mathcal{P}_n$ denote the class of all complex polynomials $p(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n and p' be its derivative. Also let $\mathcal{R}_n = \mathcal{R}_n(\alpha_1, ..., \alpha_n) := \left\{\frac{p(z)}{w(z)} : p \in \mathcal{P}_n, w(z) = \prod_{j=1}^{n}(z - \alpha_j), |\alpha_j| > 1, 1 \leq j \leq n\right\}$ denote the class of rational functions with poles $\alpha_1, \alpha_2, ..., \alpha_n$ and with finite limit at infinity. Let D_k^- represent the set of all points which lie inside $T_k := \{z : |z| = k > 0\}$ and D_k^+ be the set of points which lie outside T_k. Also

$$\mathcal{B}(z) := \prod_{j=1}^{n}\left(\frac{1 - \overline{\alpha_j} z}{z - \alpha_j}\right) = \frac{w^*(z)}{w(z)},$$

is known as Blaschke product satisfying $|\mathcal{B}(z)| = 1$ for $z \in T_1$. We observe that $\mathcal{B} \in \mathcal{R}_n$. Concerning the estimate of $\max_{z\in T_1} |p'(z)|$ in terms of $\max_{z\in T_1} |p(z)|$,

2020 Mathematics Subject Classification: 30A10, 26D10, 30C10
Key words and phrases: Rational functions, Zeros, Poles, Inequalities.

Bernstein [2] proved the following:

If $p \in \mathcal{P}_n$, then for any $z \in \mathbb{C}$

$$\max_{z\in T_1}|p'(z)| \leq n \max_{z\in T_1}|p(z)|. \tag{1.1}$$

This inequality can be sharpened if there is a restriction on the zeros of $p(z)$. In fact, if $p(z) \neq 0$ in D_1^-, then

$$\max_{z\in T_1}|p'(z)| \leq \frac{n}{2} \max_{z\in T_1}|p(z)|, \tag{1.2}$$

whereas in reverse direction, if $p(z) \neq 0$ in D_1^+, then (1.2) can be replaced by

$$\max_{z\in T_1}|p'(z)| \geq \frac{n}{2} \max_{z\in T_1}|p(z)|. \tag{1.3}$$

Both the inequalities are sharp and equality in each holds for the polynomials of the form $p(z) = az^n + b$, where $|a| = |b|$. Inequality (1.2) was conjectured by Erdös and latter verified by Lax [4], whereas inequality (1.3) is due to Turán [10]. Concerning the estimate of $\min\limits_{z\in T_1}|p'(z)|$, Aziz and Dawood [1] proved:

If $p \in \mathcal{P}_n$ has all zeros in $T_1 \cup D_1^-$, then

$$\min_{z\in T_1}|p'(z)| \geq n \min_{z\in T_1}|p(z)|. \tag{1.4}$$

Li, Mohapatra and Rodriguez [3] gave a new perspective to Bernstein-type inequalities and extended them to rational functions $r \in \mathcal{R}_n$ with prescribed poles $\alpha_1, \alpha_2, ..., \alpha_n$ replacing z^n by $\mathcal{B}(z)$. Among other things they proved the following:

Theorem 1.1. *If all the zeros of $r \in \mathcal{R}_n$ lie in $T_1 \cup D_1^-$, then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2}|\mathcal{B}'(z)||r(z)|. \tag{1.5}$$

The result is sharp and equality holds for the rational function

$$r(z) = a\mathcal{B}(z) + b, |a| = |b| = 1.$$

Recently Wali and Shah [11] improved inequality (1.5) by taking into consideration the coefficients of a polynomial $p(z) := \sum\limits_{j=0}^{n} c_j z^j$ and proved the following:

Theorem 1.2. *If $r \in \mathcal{R}_n$ and all zeros of r lie in $T_1 \cup D_1^-$, then for $z \in T_1$*

$$|r'(z)| \geq \frac{1}{2}\left\{|\mathcal{B}'(z)| + \frac{|c_n| - |c_0|}{|c_n| + |c_0|}\right\}|r(z)|.$$

Qasim and Liman [9] considered a specialized class of rational functions $(r \circ s)z = r(s(z))$ defined by

$$(r \circ s)z = \frac{p(s(z))}{w(s(z))}\ ,$$

where $p(z) = \sum\limits_{j=0}^{n} a_j z^j$, $s(z) = \sum\limits_{j=0}^{m} b_j z^j$, and $p \circ s \in \mathcal{P}_{mn}$ is defined by:

$$\begin{aligned}(p \circ s)(z) &= p(s(z))\\ &= a_n\big(b_m z^m + b_{m-1}z^{m-1} + ... + b_0\big)^n +\\ &\qquad a_{n-1}\big(b_m z^m + b_{m-1}z^{m-1} + ... + b_0\big)^{n-1} + ... + a_0\\ &= a_n\left[\binom{n}{0}b_m^n z^{mn} + \left(\binom{n}{1}b_m^{n-1}b_{m-1}\right)z^{mn-1} + ... + b_0^n\right]\\ &\qquad + \cdots + a_1 b_0 + a_0\\ &= \sum_{j=0}^{mn} c_j z^j,\ \text{where } c_0 = \sum_{j=0}^{n} a_j b_0^j,\ ...,\ c_{mn} = a_n b_m^n.\end{aligned}$$

Also if $r \in \mathcal{R}_n$, then $r \circ s \in \mathcal{R}_{nm}$ and corresponding Blaschke product is given by

$$B(z) := \frac{w^*(s(z))}{w(s(z))} = z^{mn}\frac{\overline{w(s(\frac{1}{\bar{z}}))}}{w(s(z))} = \prod_{j=1}^{mn}\frac{(1 - \overline{\alpha_j}z)}{(z - \alpha_j)},$$

where

$$w(s(z)) = c_{mn}\prod_{j=1}^{mn}(z - \alpha_j).$$

Throughout this paper we shall assume that all poles $\alpha_1, \alpha_2, ..., \alpha_{mn}$ of $r(s(z))$ lie in D_1^+. For this class of rational functions Qasim and Liman [9] among other things proved the following:

Theorem 1.3. *Let $r \circ s \in \mathcal{R}_{nm}$ and $r(s(z)) \neq 0$ in D_1^+, then for $z \in T_1$,*

$$|r'(s(z))| \geq \frac{1}{2mM'}|B'(z)||r(s(z))|,$$

where $M' = \max\limits_{z \in T_1}|s(z)|$.

Theorem 1.4. *Let $r \circ s \in \mathcal{R}_{nm}$ be such that $r(s(z)) \neq 0$ in D_1^-, then for $z \in T_1$, $|r'(s(z))| \leq \dfrac{1}{2mm'}|B'(z)||r(s(z))|$, where $m' = \min\limits_{z \in T_1} |s(z)|$.*

Mir et al. [5] improved Theorem 1.3 by proving the following:

Theorem 1.5. *Let $r \circ s \in \mathcal{R}_{nm}$, and if all the zeros of $r(s(z))$ lie in $T_1 \cup D_1^-$, then for some $z \in T_1$*

$$|r'(s(z))| \geq \frac{1}{2mM'}\left\{\big|B'(z)\big| + \frac{|a_n b_m^n| - \big|\sum\limits_{j=0}^{n} a_j b_0^j\big|}{|a_n b_m^n| + \big|\sum\limits_{j=0}^{n} a_j b_0^j\big|}\right\}|r(s(z))|,$$

where $M' = \max\limits_{z \in T_1} |s(z)|$.

2. Basic Lemmas

Lemma 2.1. *If $(y_j)_{j=1}^{\infty}$ be a sequence of real numbers then for all $n \in \mathbb{N}$*

$$\sum_{j=1}^{n} \frac{1 - y_j}{1 + y_j} \geq \frac{1 - \prod\limits_{j=1}^{n} y_j}{1 + \prod\limits_{j=1}^{n} y_j}, 0 \leq y_j \leq 1. \tag{2.1}$$

$$\sum_{j=1}^{n} \frac{1 - y_j}{1 + y_j} \leq \frac{1 - \prod\limits_{j=1}^{n} y_j}{1 + \prod\limits_{j=1}^{n} y_j}, y_j \geq 1. \tag{2.2}$$

The proof of Lemma 2.1 is a simple consequence of the principle of mathematical induction.

Lemma 2.2. *Let $z \in T_1$, then*

$$Re\left(\frac{z\big(w(s(z))\big)'}{w(s(z))}\right) = \frac{nm - |B'(z)|}{2}. \tag{2.3}$$

The above Lemma is due to Mir [6].

Lemma 2.3. *Let $r \circ s \in \mathcal{R}_{nm}$ and all zeros of $s(z)$ lie in $T_1 \cup D_1^-$, then for $z \in T_1$*

$$|(r^*(s(z)))'| + |(r(s(z)))'| \leq |B'(z)| \sup_{z \in T_1} |r(s(z))|.$$

where $r^*(s(z)) = B(z)\overline{r(s(1/\overline{z}))}$.
The result is sharp and equality holds for $r(s(z)) = uB(z)$ *with* $u \in T_1$, *where* $s(z) = z^m$.

The above lemma is also due to Qasim and Liman [9].

3. Main Results

Theorem 3.1. *Suppose* $r \circ s \in \mathcal{R}_{nm}$ *be such that* $(r \circ s)z = \dfrac{p(s(z))}{w(s(z))}$, *where* $p \in \mathcal{P}_n$ *and* $s \in \mathcal{P}_m$. *If all the zeros of* $r(s(z))$ *lie in* $T_k \cup D_k^-, k \leq 1$, *then for* $z \in T_1$

$$|r'(s(z))| \geq \frac{1}{2mM'}\left\{|B'(z)| + \frac{mn(1-k)}{1+k} + 2\left(\sum_{j=1}^{mn}\frac{1}{1+|z_j|} - \frac{mn}{1+k}\right)\right\}|r(s(z))|, \tag{3.1}$$

where $M' = \max_{z \in T_1}|s(z)|$.

Since $|z_j| \leq k$, we have $\dfrac{|z_j|}{k} \leq 1$ and therefore from Theorem 3.1

$$\begin{aligned}
|r'(s(z))| &\geq \frac{1}{2mM'}\left\{|B'(z)| + \frac{mn(1-k)}{1+k} + 2\left(\sum_{j=1}^{mn}\frac{1}{1+|z_j|} - \frac{mn}{1+k}\right)\right\}|r(s(z))| \\
&= \frac{1}{2mM'}\left\{|B'(z)| + \frac{mn(1-k)}{1+k} + 2\sum_{j=1}^{mn}\left(\frac{1}{1+|z_j|} - \frac{1}{1+k}\right)\right\}|r(s(z))| \\
&= \frac{1}{2mM'}\left\{|B'(z)| + \frac{mn(1-k)}{1+k} + \frac{2k}{1+k}\sum_{j=1}^{mn}\frac{k-|z_j|}{k+k|z_j|}\right\}|r(s(z))| \\
&\geq \frac{1}{2mM'}\left\{|B'(z)| + \frac{mn(1-k)}{1+k} + \frac{2k}{1+k}\sum_{j=1}^{mn}\frac{k-|z_j|}{k+|z_j|}\right\}|r(s(z))| \\
&= \frac{1}{2mM'}\left\{|B'(z)| + \frac{mn(1-k)}{1+k} + \frac{2k}{1+k}\sum_{j=1}^{mn}\frac{1-\frac{|z_j|}{k}}{1+\frac{|z_j|}{k}}\right\}|r(s(z))|.
\end{aligned}$$

Using inequality (2.1), we get

$$|r'(s(z))| \geq \frac{1}{2mM'}\left\{|B'(z)| + \frac{mn(1-k)}{1+k} + \frac{2k}{1+k}\left(\frac{1-\prod_{j=1}^{mn}\frac{|z_j|}{k}}{1+\prod_{j=1}^{mn}\frac{|z_j|}{k}}\right)\right\}|r(s(z))|$$

$$= \frac{1}{2mM'}\left\{|B'(z)| + \frac{mn(1-k)}{1+k} + \frac{2k}{1+k}\left(\frac{1-\frac{1}{k^{mn}}\frac{|\sum_{j=1}^{n} a_j b_0^j|}{|a_n b_m^n|}}{1+\frac{1}{k^{mn}}\frac{|\sum_{j=1}^{n} a_j b_0^j|}{|a_n b_m^n|}}\right)\right\}|r(s(z))|.$$

Equivalently

$$|r'(s(z))| \geq \frac{1}{2mM'}\left\{|B'(z)| + \frac{mn(1-k)}{1+k} + \frac{2k}{1+k}\left(\frac{k^{mn}|a_n b_m^n| - |\sum_{j=1}^{n} a_j b_0^j|}{k^{mn}|a_n b_m^n| + |\sum_{j=1}^{n} a_j b_0^j|}\right)\right\}|r(s(z))|.$$

Hence, we have the following:

Corollary 3.2. *Suppose $r \circ s \in \mathcal{R}_{nm}$ has all zeros in $T_k \cup D_k^-, k \leq 1$, and $s(z)$ is a polynomial of degree m, then for $z \in T_1$*

$$|r'(s(z))| \geq \frac{1}{2mM'}\left\{|B'(z)| + \frac{mn(1-k)}{1+k} + \frac{2k}{1+k}\left(\frac{|a_n b_m^n|k^{mn} - |\sum_{j=0}^{n} a_j b_0^j|}{|a_n b_m^n|k^{mn} + |\sum_{j=0}^{n} a_j b_0^j|}\right)\right\}|r(s(z))|.$$

Remark 3.3. Since all the zeros of $r(s(z))$ lie in $T_k \cup D_k^-, k \leq 1$. If $z_1, z_2, ..., z_{mn}$ are zeros of $r(s(z))$, therefore $|z_j| \leq k$ for all $j = 1, 2, ..., mn$. Therefore it can be easily verified that

$$\sum_{j=1}^{mn} \frac{1}{1+|z_j|} - \frac{mn}{1+k} \geq 0.$$

This shows that inequality (3.1) improves a result due to Mir [6, Theorem 1].

Remark 3.4. Since $r(s(z))$ has all zeros in $T_k \cup D_k^-, k \leq 1$, therefore $|z_j| \leq 1$. Hence it can be easily verified that

$$|a_n b_m^n| - |\sum_{j=1}^{n} a_j b_0^j| \geq 0.$$

This shows that Corollary 3.2 improves a result due to Mir [6, Theorem 1].

A result of Mir et al.[5, Theorem 4.1] is a special case of Corollary 3.2 when $k = 1$.

Remark 3.5. Also for $s(z) = z$ and $k = 1$, Corollary 3.2 in particular reduces to a result due to Wali and Shah [11, Theorem 2].

Proof of Theorem 3.1. Let $r \circ s \in \mathcal{R}_{nm}$, so that $r(s(z)) = \dfrac{p(s(z))}{w(s(z))}$. If $z_1, z_2, ..., z_{mn}$, are zeros of $p(s(z))$ with $|z_j| \leq k \leq 1, j = 1, 2, ..., mn$, therefore, it can be easily verified that

$$Re\left(\frac{z\big(r(s(z))\big)'}{r(s(z))}\right) = Re\left(\frac{z\big(p(s(z))\big)'}{p(s(z))}\right) - Re\left(\frac{z\big(w(s(z))\big)'}{w(s(z))}\right).$$

By Lemma 2.2, we get for $z \in T_1$

$$\begin{aligned} Re\left(\frac{z\big(r(s(z))\big)'}{r(s(z))}\right) &= \sum_{j=1}^{mn} Re\left(\frac{z}{z - z_j}\right) - \frac{nm - |B'(z)|}{2} \\ &= \sum_{j=1}^{mn} Re\left(\frac{z}{z - z_j}\right) - \frac{nm}{2} + \frac{|B'(z)|}{2} \\ &\geq \sum_{j=1}^{mn} \frac{1}{1 + |z_j|} + \frac{2mn - nm(1 + k)}{2(1 + k)} + \frac{|B'(z)|}{2} - \frac{mn}{1 + k}. \end{aligned}$$

Since

$$\left|\frac{z\big(r(s(z))\big)'}{r(s(z))}\right| \geq Re\left(\frac{z\big(r(s(z))\big)'}{r(s(z))}\right)$$

Therefore we have for $z \in T_1$

$$\left|\frac{z\big(r(s(z))\big)'}{r(s(z))}\right| \geq \sum_{j=1}^{mn} \frac{1}{1 + |z_j|} + \frac{2mn - nm(1 + k)}{2(1 + k)} + \frac{|B'(z)|}{2} - \frac{mn}{1 + k}.$$

Equivalently for $z \in T_1$,

$$|r'(s(z))||s'(z)| \geq \frac{1}{2}\left\{|B'(z)| + \frac{mn(1 - k)}{1 + k} + 2\left(\sum_{j=1}^{mn} \frac{1}{1 + |z_j|} - \frac{mn}{1 + k}\right)\right\}|r(s(z))|.$$

Since by inequality (1.1) $|s'(z)| \leq mM'$, where $M' = \max_{z \in T_1} |s(z)|$, therefore we conclude

$$|r'(s(z))| \geq \frac{1}{2mM'}\left\{|B'(z)| + \frac{mn(1 - k)}{1 + k} + 2\left(\sum_{j=1}^{mn} \frac{1}{1 + |z_j|} - \frac{mn}{1 + k}\right)\right\}|r(s(z))|.$$

This completely proves Theorem 3.1.

As an improvement of Theorem 1.4, we next prove the following:

Theorem 3.6. *Suppose* $r\circ s\in\mathcal{R}_{mn}$ *be such that* $(r\circ s)z=\dfrac{p(s(z))}{w(s(z))}$, *where* $p\in\mathcal{P}_n$ *and* $s\in\mathcal{P}_m$. *If all the zeros of* $r(s(z))$ *lie in* $T_k\cup D_k^+, k\geq 1$ *and* $s(z)$ *has all zeros in* $T_1\cup D_1^-$. *Then for any* $z\in T_k\cup D_k^+$ *and for* $z\in T_1$

$$|r'(s(z))|\leq\frac{1}{2mm'}\Bigg\{|B'(z)|-\frac{mn(k-1)|r(s(z))|^2}{(1+k)\big(\sup_{z\in T_1}|r(s(z))|\big)^2} - 2\frac{|r(s(z))|^2}{\big(\sup_{z\in T_1}|r(s(z))|\big)^2}\left(\frac{mn}{1+k}-\sum_{j=1}^{mn}\frac{1}{1+|z_j|}\right)\Bigg\}\sup_{z\in T_1}|r(s(z))|, \tag{3.2}$$

where $m'=\min_{z\in T_1}|s(z)|$.

From inequality (3.2), we have

$$\begin{aligned}|r'(s(z))|&\leq\frac{1}{2mm'}\Bigg\{|B'(z)|-\frac{mn(k-1)|r(s(z))|^2}{(1+k)\big(\sup_{z\in T_1}|r(s(z))|\big)^2}\\&-2\frac{|r(s(z))|^2}{(1+k)\big(\sup_{z\in T_1}|r(s(z))|\big)^2}\left(\sum_{j=1}^{mn}\frac{|z_j|-k}{|z_j|+1}\right)\Bigg\}\sup_{z\in T_1}|r(s(z))|\\&\leq\frac{1}{2mm'}\Bigg\{|B'(z)|-\frac{mn(k-1)|r(s(z))|^2}{(1+k)\big(\sup_{z\in T_1}|r(s(z))|\big)^2}\\&-2\frac{|r(s(z))|^2}{(1+k)\big(\sup_{z\in T_1}|r(s(z))|\big)^2}\left(\sum_{j=1}^{mn}\frac{|z_j|-k}{|z_j|+k}\right)\Bigg\}\sup_{z\in T_1}|r(s(z))|\\&=\frac{1}{2mm'}\Bigg\{|B'(z)|-\frac{mn(k-1)|r(s(z))|^2}{(1+k)\big(\sup_{z\in T_1}|r(s(z))|\big)^2}\\&-2\frac{|r(s(z))|^2}{(1+k)\big(\sup_{z\in T_1}|r(s(z))|\big)^2}\left(\sum_{j=1}^{mn}\frac{\frac{|z_j|}{k}-1}{\frac{|z_j|}{k}+1}\right)\Bigg\}\sup_{z\in T_1}|r(s(z))|.\end{aligned}$$

Now using inequality (2.2), with $\dfrac{|z_j|}{k} \geq 1, j = 1, 2, ..., mn$, we get

$$
\begin{aligned}
|r'(s(z))| \leq \frac{1}{2mm'}\Bigg\{ & |B'(z)| - \frac{mn(k-1)|r(s(z))|^2}{(1+k)\big(\sup\limits_{z\in T_1}|r(s(z))|\big)^2} \\
& + 2\frac{|r(s(z))|^2}{(1+k)\big(\sup\limits_{z\in T_1}|r(s(z))|\big)^2}\left(\frac{1-\prod\limits_{j=1}^{mn}\frac{|z_j|}{k}}{1+\prod\limits_{j=1}^{mn}\frac{|z_j|}{k}}\right)\Bigg\}\sup_{z\in T_1}|r(s(z))| \\
= \frac{1}{2mm'}\Bigg\{ & |B'(z)| - \frac{mn(k-1)|r(s(z))|^2}{(1+k)\big(\sup\limits_{z\in T_1}|r(s(z))|\big)^2} \\
& + 2\frac{|r(s(z))|^2}{(1+k)\big(\sup\limits_{z\in T_1}|r(s(z))|\big)^2}\left(\frac{|a_n b_m^n|k^{mn} - \big|\sum\limits_{j=0}^{n} a_j b_0^j\big|}{|a_n b_m^n|k^{mn} + \big|\sum\limits_{j=0}^{n} a_j b_0^j\big|}\right)\Bigg\}\sup_{z\in T_1}|r(s(z))|.
\end{aligned}
$$

Hence, we have the following:

Corollary 3.7. Suppose $r \circ s \in \mathcal{R}_{nm}$ has all its zeros in $T_k \cup D_k^+, k \geq 1$, and $s(z)$ is a polynomial of degree m having all zeros in $T_1 \cup D_1^-$. Then for $z \in T_1$

$$
\begin{aligned}
|r'(s(z))| \leq \frac{1}{2mm'}\Bigg\{ & |B'(z)| - \frac{mn(k-1)|r(s(z))|^2}{(1+k)\big(\sup\limits_{z\in T_1}|r(s(z))|\big)^2} \\
& - 2\frac{|r(s(z))|^2}{\big(\sup\limits_{z\in T_1}|r(s(z))|\big)^2}\left(\frac{\big|\sum\limits_{j=0}^{n} a_j b_0^j\big| - |a_n b_m^n|k^{mn}}{\big|\sum\limits_{j=0}^{n} a_j b_0^j\big| + |a_n b_m^n|k^{mn}}\right)\Bigg\}\sup_{z\in T_1}|r(s(z))|.
\end{aligned}
$$

where $m' = \min\limits_{z\in T_1}|s(z)|$.

Remark 3.8. For $s(z) = z$ and $k = 1$, Corollary 3.7 reduces to a result due to Mir [7, Theorem 5].

Proof of Theorem 3.6. Since $p(s(z))$ is a polynomial of degree at-most mn. Let $z_1, z_2, ..., z_{mn}$, be the zeros of $p(s(z))$ with $|z_j| \geq k \geq 1, j =$

$1, 2, ..., mn$. Now we have

$$r(s(z)) = \frac{p(s(z))}{w(s(z))}\ ,$$

and it can be easily verified that

$$Re\left(\frac{z\big(r(s(z))\big)'}{r(s(z))}\right) = Re\left(\frac{z\big(p(s(z))\big)'}{p(s(z))}\right) - Re\left(\frac{z\big(w(s(z))\big)'}{w(s(z))}\right).$$

By Lemma 2.2, we get

$$\begin{aligned} Re\left(\frac{z\big(r(s(z))\big)'}{r(s(z))}\right) &= \sum_{j=1}^{mn} Re\left(\frac{z}{z-z_j}\right) - \frac{nm - |B'(z)|}{2} \\ &\leq \sum_{j=1}^{mn}\left(\frac{1}{1+|z_j|}\right) - \frac{nm}{2} + \frac{|B'(z)|}{2}. \end{aligned} \tag{3.3}$$

Also for $z \in T_1$, using the fact that $|B'(z)| = \dfrac{zB'(z)}{B(z)}$, and $|B(z)| = 1$, one can easily deduce that for $r^*(s(z)) = B(z)\overline{r(s(\frac{1}{\bar{z}}))}$,

$$\big|\big(r^*(s(z))\big)'\big| = \Big|\big|B'(z)\big|r(s(z)) - z\big(r(s(z))\big)'\Big|.$$

Hence for $z \in T_1$, with $r(s(z)) \neq 0$, we get by using inequality (3.4)

$$\begin{aligned} \left|\frac{z\big(r^*(s(z))\big)'}{r(s(z))}\right|^2 &= \left|\big|B'(z)\big| - \frac{z\big(r(s(z))\big)'}{r(s(z))}\right|^2 \\ &= \big|B'(z)\big|^2 + \left|\frac{z\big(r(s(z))\big)'}{r(s(z))}\right|^2 - 2\big|B'(z)\big| Re\left\{\frac{z\big(r(s(z))\big)'}{r(s(z))}\right\} \\ &\geq \big|B'(z)\big|^2 + \left|\frac{z\big(r(s(z))\big)'}{r(s(z))}\right|^2 \\ &\qquad - 2\big|B'(z)\big|\sum_{j=1}^{mn}\frac{1}{1+|z_j|} + \big|B'(z)\big|\big(nm - |B'(z)|\big) \\ &= \left|\frac{z\big(r(s(z))\big)'}{r(s(z))}\right|^2 + nm|B'(z)| - 2|B'(z)|\sum_{j=1}^{mn}\frac{1}{1+|z_j|}. \end{aligned}$$

This gives for $z \in T_1$

$$\left|\left(r^*(s(z))\right)^{'}\right|^2 \geq \left|\left(r(s(z))\right)^{'}\right|^2 + nm|B'(z)|\left|r(s(z))\right|^2 - 2|B'(z)|\left|r(s(z))\right|^2 \sum_{j=1}^{mn} \frac{1}{1+|z_j|}. \tag{3.4}$$

Now using Lemma 2.3, we get

$$\begin{aligned}
&\left|\left(r(s(z))\right)^{'}\right| + \left[\left|\left(r(s(z))\right)^{'}\right|^2 + nm|B'(z)|\left|r(s(z))\right|^2 - 2|B'(z)|\left|r(s(z))\right|^2 \sum_{j=1}^{mn} \frac{1}{1+|z_j|}\right]^{1/2} \\
&\leq \left|\left(r(s(z))\right)^{'}\right| + \left|\left(r^*(s(z))\right)^{'}\right| \leq |B'(z)| \sup_{z\in T_1} |r(s(z))|.
\end{aligned}$$

This gives on simplification, with $\tau = \sup_{z\in T_1} |r(s(z))|$,

$$\begin{aligned}
\left|\left(r(s(z))\right)^{'}\right| &\leq \frac{1}{2}\left\{|B'(z)| - \frac{mn|r(s(z))|^2}{\tau^2} + 2\frac{|r(s(z))|^2}{\tau^2}\sum_{j=1}^{mn}\frac{1}{1+|z_j|}\right\}\tau \\
&= \frac{1}{2}\left\{|B'(z)| - \frac{nm|r(s(z))|^2}{\tau^2}\left(1-\frac{2}{k+1}\right)\right. \\
&\quad \left. - \frac{mn|r(s(z))|^2}{\tau^2}\frac{2}{1+k} + 2\frac{|r(s(z))|^2}{\tau^2}\sum_{j=1}^{mn}\frac{1}{1+|z_j|}\right\}\tau \\
&= \frac{1}{2}\left\{|B'(z)| - \frac{nm|r(s(z))|^2}{\tau^2}\frac{k-1}{k+1}\right. \\
&\quad \left. - \frac{2|r(s(z))|^2}{\tau^2}\left(\frac{nm}{1+k} - \sum_{j=1}^{mn}\frac{1}{1+|z_j|}\right)\right\}\tau.
\end{aligned} \tag{3.5}$$

Hence required inequality is obtained by combining inequality (1.4) with inequality (3.5).

Acknowledgement: The authors are highly grateful to the referee for his/her useful suggestions.
Funding The first author is highly thankful to the funding agency DST-INSPIRE for their financial support.
Statement of Ethics. The authors have no conflicts of interest.

References

[1] A. Aziz and Q. M. Dawood, *Inequalities for polynomials and its derivative,* Indian. J.Approx. theory, **54**: (1988), 119-122.

[2] S. N. Bernstein, *Sur l' ordre de la meilleure approximation des fonctions contimues par des polynomes de degré donné* , Mem. Acad. R. Belg., **4** (1912), 1-103.

[3] X. Li, R. N. Mohapatra and R. S. Rodriguez, *Bernstein-type inequalities for rational functions with prescribed poles,* J. London Math. Soc., **51** (1995), 523-531.

[4] P. D. Lax, Proof of a conjecture of P. Erdös on the derivative of a polynomial, *Bull. Amer. Math. Soc.,* **50**(1944), 509-513.

[5] M. Y. Mir, S. L. Wali and W. M. Shah, *Inequalities for a Class of Rational Functions,* Int. J. of non linear Anal. and Appl.,**13**(2)(2022),609-617.

[6] A. Mir, *Inequalities for rational functions with prescribed poles,* Indian J. of pure and appl. Math., **50**(2): (2019), 315-331.

[7] A. Mir, *Some inequalities concerning rational functions with fixed poles,* J. of contemporary Math. Anal., **55**(2): (2020), 105-114.

[8] R. Osserman. *A sharp schwarz inequality on the boundary for functions regular in disk,* Proceedings of American Math. Soc. **12**, 3513-3517.

[9] I. Qasim and A. Liman, *Bernstein type inequalities for rational functions,* Indian. J. Pure Appl. Math. **46**(3): (2015), 337-348.

[10] P. Turán, *Über die ableitung von polynomen,* Compos. Math., **7** (1939), 89-95.

[11] S. L. Wali and W. M. Shah, *Applications of the schwarz lemma to inequalities for rational functions with prescribed poles,* The Journal of Anal., **25**(1):(2017), 43-53.

M. Y. Mir
Department of Mathematics
Central University of Kashmir,
191201, Ganderbal.
E-mail: myousf@cukashmir.ac.in

W. M. Shah
Department of Mathematics
Central University of Kashmir,
191201, Ganderbal.
E-mail: wali@cukashmir.ac.in

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July-December (2024), 61–73

PROPAGATION OF SH WAVES AT THE INTERFACE OF A PIEZO-ELECTRIC LAYER AND A PIEZO-MAGNETIC HALF SPACE

PAYEL MONDAL AND SARBANI CHAKRABORTY
(Received : 21 - 09 - 2023 ; Revised : 29 - 05 - 2024)

Abstract. The problem of propagation of SH waves in a piezo-electric layer overlying a piezo magnetic half space perfectly bonded with it has been investigated. The basic equations have been formulated under the assumption of continuity of displacement and other fields. Frequency equations are obtained for different cases of electric and magnetic boundary conditions and solved numerically to find possible existence.

1. Introduction

The mathematical theory of surface acoustic waves (SAW) in solids developed in the late nineteenth and early twentieth century, and it played a prominent role in explaining the nature of surface seismic waves. Such waves have been used to study the interior of the earth. An important field of application of these types of waves is heath monitoring of structures, and the other is non-destructive evaluation of materials.

In recent years, elastic wave propagation in piezoelectric or smart materials has attracted attention because of potential applications ([1]). It is seen that the electromechanical coupling can significantly alter the properties of elastic waves and new velocities of propagation can arise ([2], [3]).

Of particular interest is the propagation of electro-magneto-elastic waves in a composite structure consisting of two or more layers. Propagation characteristics of Love-type/ SH waves in piezoelectric/piezomagnetic plates or

2010 Mathematics Subject Classification: 11A41, 16N20
Key words and phrases: electro-magneto elasticity, piezo-electric materials, piezo-magnetic materials, SH waves

in layered composite structures have aroused considerable interest because of wide-spread applicability in semi conductor devices in recent years. In 2001, the existence of Bluestein-Gulyaev waves in a piezoelectric layered halfspace was investigated by Jin et al ([4]). Lee and Liu in 2004 studied plane waves in a infinite piezoelectric plate with dissipation ([5]). SH waves in a layered piezoelectric/piezomagnetic plate was discussed by Nie et al ([6]), and studies on Love waves in a piezoelectric layered structure with viscous dissipation have been carried-out by Du et al ([7]). The propagation of surface waves in a piezoelectric half space coated with a semi-conductor layer has been studied by Sharma et al ([8]). Love wave propagation in a functionally graded piezoelectric material layer was analysed by Du et al ([9]). Shear wave propagation in a composite layered structure consisting of two different piezoelectric materials has been investigated by Gaur and Rana ([10]). Recently, the effect of surface stress on waves in piezoelectric materials has been analysed by Zhang et al ([11]) and by Gour ([12]). Goyel et al ([13]) have discussed the effect of internal microstructure of a substrate on waves in a piezoelectric ceramic layer. Different aspects of waves propagating in piezoelectric/piezomagnetic layered structures have been discussed by Ezzin et al ([14]), Son and Kang ([15]), Soh and Lin ([16]), Melkumyan and Mai([17]).

In this present paper the problem of SH SAW propagation has been considered in Piezomagnetic half space with a piezoelectric layer perfectly bonded to it. The frequency equation has been formulated under the assumption of continuity at the interface and for different electrical and magnetic boundary conditions on the open surface. A numerical investigation has been done to determine the existence of real root.

2. Formulation and Basic Equations

The material under consideration is a piezo-electric layer (M_2) overlying a piezo-magnetic half space (M_1) perfectly bonded with it. Both the materials are with hexagonal symmetry (6mm) whose poling direction is along x_3 axis and $x_1 - x_2$ is the basal plane (Fig. 1). The thickness of the piezoelectric layer is d. Let x_1, x_2, x_3 denote the rectangular cartesian coordinates with x_3 oriented in the direction of the sixfold axis of a transversely isotropic material in class $6mm$. Let $\phi = \phi(x_1, x_2, t)$ be the electric

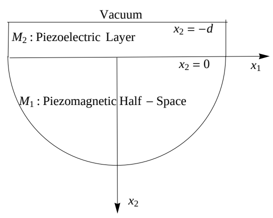


FIGURE 1. Geometry of the problem

potential and $\psi = \psi(x_1, x_2, t))$ the magnetic potential s.t.

$$E_i = -\phi_{,i} \text{ and } H_i = -\psi_{,i}$$

where E is the electric field vector and H is the magnetic field vector.

The mechanical displacement vectors are $(0, 0, u_3(x_1, x_2, t))$ in the x_1x_2 plane where $u_i(i = 1, 2, 3)$ is the mechanical displacement components. Let $D_i(i = 1, 2, 3)$ be the electric displacement, $B_i(i = 1, 2, 3)$ the magnetic induction, $\epsilon_{ij}(i, j = 1, 2, 3)$ be the mechanical strain, $\sigma_{ij}(i, j = 1, 2, 3)$ the mechanical stress. Then

$$u_1 = 0;\ u_2 = 0;\ u_3 = u_3(x_1, x_2, t)$$

$$E_1 = -\phi_{,1};\ E_2 = -\phi_{,2};\ E_3 = 0$$

$$H_1 = -\psi_{,1};\ H_2 = -\psi_{,2};\ H_3 = 0$$

$$\epsilon_{11} = 0,\ \epsilon_{22} = 0,\ \epsilon_{33} = 0,\ \epsilon_{31} = \frac{1}{2}u_{3,1},\ \epsilon_{23} = \frac{1}{2}u_{3,2},\ \epsilon_{12} = 0$$

Let the elastic stiffness coefficients be c_{ij}, piezoelectric coefficients be e_{ij} and the piezomagnetic coefficients be f_{ij}, $(i, j = 1, 2, 3)$. Also let χ_{ij} be the dielectric permittivity and μ_{ij} be the magnetic permeability $(i, j = 1, 2, 3)$.

For Piezomagnetic Half Space the constitutive equations are :

$$P^m = Q^m R^m$$

where $P^m = \begin{pmatrix} \sigma^m_{11}, \sigma^m_{22}, \sigma^m_{33}, \sigma^m_{23}, \sigma^m_{31}, \sigma^m_{12}, D^m_1, D^m_2, D^m_3, B^m_1, B^m_2, B^m_3 \end{pmatrix}^t$,
$R^m = \begin{pmatrix} 0,\ 0,\ 0,\ u^m_{3,2},\ u^m_{3,1},\ 0,\ \phi^m_{,1},\ \phi^m_{,2},\ 0,\ \psi^m_{,1},\ \psi^m_{,2},\ 0 \end{pmatrix}^t$,
$Q^m = \left(Q^m_{ij}\right)\ (i, j = 1, 2, \ldots 12)$
with $Q^m_{11} = Q^m_{22} = c^m_{11}$, $Q^m_{12} = Q^m_{21} = c^m_{12}$, $Q^m_{13} = Q^m_{31} = Q^m_{23} = Q^m_{32} = c^m_{13}$,
$Q^m_{33} = c^m_{33}$, $Q^m_{44} = Q^m_{55} = c^m_{44}$, $Q^m_{66} = c^m_{66} = \frac{1}{2}(c^m_{11} - c^m_{12})$,

$Q^m_{1\,12} = Q^m_{2\,12} = Q^m_{12\,1} = Q^m_{12\,2} = f^m_{31}$, $Q^m_{3\,12} = Q^m_{12\,3} = f^m_{33}$,
$Q^m_{4\,11} = Q^m_{5\,10} = Q^m_{10\,5} = Q^m_{11\,4} = f^m_{15}$,
$Q^m_{77} = Q^m_{88} = -\chi^m_{11}$, $Q^m_{99} = -\chi^m_{33}$, $Q^m_{10\,10} = Q^m_{11\,11} = -\mu^m_{11}$, $Q^m_{12\,12} = -\mu^m_{33}$
are the only non zero components of Q^m_{ij}.

Therefore,

$$\begin{aligned}
\sigma^m_{11} = 0,\ \sigma^m_{22} = 0,\ \sigma^m_{33} = 0,\ \sigma^m_{12} = 0;\\
\sigma^m_{23} = c^m_{44}u^m_{3,2} + f^m_{15}\psi^m_{,2};\\
\sigma^m_{13} = c^m_{44}u^m_{3,1} + f^m_{15}\psi^m_{,1};\\
D^m_1 = -\chi^m_{11}\phi^m_{,1};\\
D^m_2 = -\chi^m_{11}\phi^m_{,2};\\
D^m_3 = 0;\\
B^m_1 = f^m_{15}u^m_{3,1} - \mu^m_{11}\psi^m_{,1};\\
B^m_2 = f^m_{15}u^m_{3,2} - \mu^m_{11}\psi^m_{,2};\\
B^m_3 = 0
\end{aligned}$$

For Piezoelectric Layer the constitutive equations have the form :

$$P^e = Q^e R^e$$

where $P^e = \left(\ \sigma^e_{11},\, \sigma^e_{22},\, \sigma^e_{33},\, \sigma^e_{23},\, \sigma^e_{31},\, \sigma^e_{12},\, D^e_1,\, D^e_2,\, D^e_3,\, B^e_1,\, B^e_2,\, B^e_3\ \right)^t$,
$R^e = \left(\ 0,\, 0,\, 0,\, u^e_{3,2},\, u^e_{3,1},\, 0,\, \phi^e_{,1},\, \phi^e_{,2},\, 0,\, \psi^e_{,1},\, \psi^e_{,2},\, 0\ \right)^t$, $Q^e = \left(Q^e_{ij}\right)$ $(i, j = 1, 2, \ldots 12)$
with $Q^e_{11} = Q^e_{22} = c^e_{11}$, $Q^e_{12} = Q^e_{21} = c^e_{12}$, $Q^e_{13} = Q^e_{31} = Q^e_{23} = Q^e_{32} = c^e_{13}$,
$Q^e_{33} = c^e_{33}$, $Q^e_{44} = Q^e_{55} = c^e_{44}$, $Q^e_{66} = c^e_{66} = \frac{1}{2}(c^e_{11} - c^e_{12})$,
$Q^e_{19} = Q^e_{29} = Q^e_{91} = Q^e_{92} = e^e_{31}$, $Q^e_{39} = Q^e_{93} = e^e_{33}$,
$Q^e_{48} = Q^e_{57} = Q^e_{75} = Q^e_{84} = e^e_{15}$,
$Q^e_{77} = Q^e_{88} = -\chi^e_{11}$, $Q^e_{99} = -\chi^e_{33}$, $Q^e_{10\,10} = Q^e_{11\,11} = -\mu^e_{11}$, $Q^e_{12\,12} = -\mu^e_{33}$
are the only non zero components of Q^e_{ij}.

Therefore,

$$\begin{aligned}
\sigma^e_{11} = 0,\ \sigma^e_{22} = 0,\ \sigma^e_{33} = 0,\ \sigma^e_{12} = 0;\\
\sigma^e_{23} = c^e_{44}u^e_{3,2} + e^e_{15}\psi^e_{,2};\\
\sigma^e_{13} = c^e_{44}u^e_{3,1} + e^e_{15}\psi^e_{,1};
\end{aligned}$$

$$D_1^e = e_{15}^e u_{3,1}^e - \chi_{11}^e \phi_{,1}^e;$$
$$D_2^e = e_{15}^e u_{3,2}^e - \chi_{11}^e \phi_{,2}^e;$$
$$D_3^e = 0;$$
$$B_1^e = -\mu_{11}^e \psi_{,1}^e;$$
$$B_2^e = -\mu_{11}^e \psi_{,2}^e;$$
$$B_3^e = 0$$

The superscripts $'e'$ and $'m'$ refer to PE and PM materials respectively.

The equations of motion in the PE medium are :

$$\begin{aligned} c_{44}^e \nabla^2 u_3^e + e_{15}^e \nabla^2 \phi^e &= \rho^e \ddot{u}_3^e \\ e_{15}^e \nabla^2 u_3^e - \chi_{11}^e \nabla^2 \phi^e &= 0 \\ \nabla^2 \psi &= 0 \end{aligned} \tag{2.1}$$

And for PM medium are:

$$\begin{aligned} c_{44}^m \nabla^2 u_3^m + f_{15}^m \nabla^2 \psi^m &= \rho^m \ddot{u}_3^m \\ \nabla^2 \phi^m &= 0 \\ f_{15}^m \nabla^2 u_3^m - \mu_{11}^m \nabla^2 \psi^m &= 0 \end{aligned} \tag{2.2}$$

where ∇^2 is the Laplacian operator in two dimension, $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, ρ is the mass density.

The equations (2.1) and (2.2) are to be solved subject to the following conditions :

Conditions at Infinity:

For PM half space. As $x_2 \to \infty$, $u_3^m \to 0$, $\phi^m \to 0$, $\psi^m \to 0$

Boundary Conditions: On $(x_2 = -d)$, the following conditions are satisfied:

(1) Mechanical stress free condition: $\sigma_{23}^e = 0$

(2) The electric and magnetic fields satisfy any one of the following :

Case 1: Electrically closed Magnetically closed giving $\phi^e = 0$, $\psi^e = 0$

Case 2: Electrically open, Magnetically open giving $D_2^e = 0$, $B_2^e = 0$

Also we have, **Continuity Conditions at the interface:** At the interface ($x_2 = 0$) of the PE layer and the PM half-space, continuity of displacement and other fields gives,
$u_3^e = u_3^m,\ \phi^e = \phi^m,\ \psi^e = \psi^m,\ \sigma_{23}^e = \sigma_{23}^m,\ D_2^e = D_2^m,\ B_2^e = B_2^m$

3. Method of Solution:

Solution for SH waves propagating in the x_1 direction can be taken in the form:
In M_2

$$\begin{aligned} u_3^e(x_1, x_2, t) &= U_3^e(x_2)e^{ik(x_1-ct)} \\ \phi^e(x_1, x_2, t) &= \Phi^e(x_2)e^{ik(x_1-ct)} \\ \psi^e(x_1, x_2, t) &= \Psi^e(x_2)e^{ik(x_1-ct)} \end{aligned} \tag{3.1}$$

In M_1

$$\begin{aligned} u_3^m(x_1, x_2, t) &= U_3^m(x_2)e^{ik(x_1-ct)} \\ \phi^m(x_1, x_2, t) &= \Phi^m(x_2)e^{ik(x_1-ct)} \\ \psi^m(x_1, x_2, t) &= \Psi^m(x_2)e^{ik(x_1-ct)} \end{aligned} \tag{3.2}$$

where k is wave number, c is phase velocity.

Substituting (3.1) in equation (2.1) and (3.2) in equation (2.2),
for M_2

$$\begin{aligned} \frac{d^2U_3^e}{dx_2^2} - k^2\lambda^{e2}U_3^e &= 0 \\ \frac{d^2\Phi^e}{dx_2^2} - k^2\Phi^e &= \frac{e_{15}^e}{\chi_{11}^e}\left(\frac{d^2U_3^e}{dx_2^2} - k^2U_3^e\right) \\ \frac{d^2\Psi^e}{dx_2^2} - k^2\Psi^e &= 0 \end{aligned} \tag{3.3}$$

Where $\lambda^e = \sqrt{1 - \frac{c^2}{{c_{SH}^e}^2}},\ c_{SH}^e = \sqrt{\frac{c_{44}^e + \frac{{e_{15}^e}^2}{\chi_{11}^e}}{\rho^e}}$

for M_1

$$\begin{aligned}
\frac{d^2U_3^m}{dx_2^2} - k^2\lambda^{m2}U_3^m &= 0 \\
\frac{d^2\Phi^m}{dx_2^2} - k^2\Phi^m &= 0 \\
\frac{d^2\Psi^m}{dx_2^2} - k^2\Psi^m &= \frac{f_{15}^m}{\mu_{11}^m}\left(\frac{d^2U_3^m}{dx_2^2} - k^2U_3^m\right)
\end{aligned} \tag{3.4}$$

Where $\lambda^m = \sqrt{1 - \frac{c^2}{c_{SH}^{m\,2}}}$, $c_{SH}^m = \sqrt{\frac{c_{44}^m + \frac{f_{15}^{m\,2}}{\mu_{11}^m}}{\rho^m}}$

Here c_{SH}^e and c_{SH}^m are the bulk shear wave velocities of PE and PM materials respectively.

Therefore the solution (3.1) and (3.2) are written finally as:

For $c_{SH}^e < c < c_{SH}^m$

$$\begin{aligned}
u_3^e(x_1, x_2, t) &= (A_1^e \cos(-k\lambda^e x_2) + A_2^e \sin(k\lambda^e x_2))e^{ik(x_1-ct)} \\
\phi^e(x_1, x_2, t) &= (M_1^e e^{-kx_2} + M_2^e e^{kx_2})e^{ik(x_1-ct)} + \frac{e_{15}^e}{\chi_{11}^e}(A_1^e \cos(-k\lambda^e x_2) \\
&+ A_2^e \sin(k\lambda^e x_2))e^{ik(x_1-ct)} \\
\psi^e(x_1, x_2, t) &= (C_1^e e^{-kx_2} + C_2^e e^{kx_2})e^{ik(x_1-ct)}
\end{aligned} \tag{3.5}$$

And

$$\begin{aligned}
u_3^m(x_1, x_2, t) &= A^m e^{-k\lambda^m x_2} e^{ik(x_1-ct)} \\
\phi^m(x_1, x_2, t) &= M^m e^{-kx_2} e^{ik(x_1-ct)} \\
\psi^m(x_1, x_2, t) &= C^m e^{-kx_2} e^{ik(x_1-ct)} + \frac{f_{15}^m}{\mu_{11}^m} A^m e^{-k\lambda^m x_2} e^{ik(x_1-ct)}
\end{aligned} \tag{3.6}$$

For $c_{SH}^m < c < c_{SH}^e$

$$\begin{aligned}
u_3^e(x_1, x_2, t) &= (A_1^e e^{(-k\lambda^e x_2)} + A_2^e e^{(k\lambda^e x_2)})e^{ik(x_1-ct)} \\
\phi^e(x_1, x_2, t) &= (M_1^e e^{-kx_2} + M_2^e e^{kx_2})e^{ik(x_1-ct)} + \frac{e_{15}^e}{\chi_{11}^e}(A_1^e e^{(-k\lambda^e x_2)} \\
&+ A_2^e e^{(k\lambda^e x_2)})e^{ik(x_1-ct)} \\
\psi^e(x_1, x_2, t) &= (C_1^e e^{-kx_2} + C_2^e e^{kx_2})e^{ik(x_1-ct)}
\end{aligned} \tag{3.7}$$

And

$$\begin{aligned} u_3^m(x_1,x_2,t) &= A^m e^{-k\lambda^m x_2} e^{ik(x_1-ct)} \\ \phi^m(x_1,x_2,t) &= M^m e^{-kx_2} e^{ik(x_1-ct)} \\ \psi^m(x_1,x_2,t) &= C^m e^{-kx_2} e^{ik(x_1-ct)} + \frac{f_{15}^m}{\mu_{11}^m} A^m e^{-k\lambda^m x_2} e^{ik(x_1-ct)} \end{aligned} \quad (3.8)$$

4. Frequency Equations for Different Cases:

Substituting the solution (3.7) and (3.8) in the boundary conditions and continuity conditions at the interface we get,

The stress free boundary condition gives

$$A_1^e[(c_{44}^e + \frac{{e_{15}^e}^2}{\chi_{11}^e})\lambda^e e^{\lambda^e kd}] + A_2^e[-(c_{44}^e + \frac{{e_{15}^e}^2}{\chi_{11}^e})\lambda^e e^{\lambda^e kd}] + M_1^e[e_{15}^e e^{kd}] + M_2^e[e_{15}^e e^{-kd}] = 0 \quad (4.1)$$

For the electrically and magnetically closed boundary conditions:

$$A_1^e[\frac{e_{15}^e}{\chi_{11}^e} e^{\lambda^e kd}] + A_2^e[\frac{e_{15}^e}{\chi_{11}^e} e^{-\lambda^e kd}] + M_1^e[e^{kd}] + M_2^e[e^{-kd}] = 0 \quad (4.2)$$

$$C_1^e[e^{kd}] + C_2^e[e^{-kd}] = 0 \quad (4.3)$$

For the electrically and magnetically open boundary conditions:

$$M_1^e[e^{kd}] + M_2^e[-e^{-kd}] = 0 \quad (4.4)$$

$$C_1^e[e^{kd}] + C_2^e[-e^{-kd}] = 0 \quad (4.5)$$

The Continuity conditions lead to

$$A_1^e[1] + A_2^e[1] + A^m[-1] = 0 \quad (4.6)$$

$$A_1^e[\frac{e_{15}^e}{\chi_{11}^e}] + A_2^e[\frac{e_{15}^e}{\chi_{11}^e}] + M_1^e[1] + M_2^e[1] + M^m[-1] = 0 \quad (4.7)$$

$$C_1^e[1] + C_2^e[1] + A^m[-\frac{f_{15}^m}{\mu_{11}^m}] + C^m[-1] = 0 \quad (4.8)$$

$$A_1^e[(c_{44}^e + \frac{{e_{15}^e}^2}{\chi_{11}^e})\lambda^e] + A_2^e[-(c_{44}^e + \frac{{e_{15}^e}^2}{\chi_{11}^e})\lambda^e] \quad + \quad M_1^e[e_{15}^e] + M_2^e[-e_{15}^e]$$

$$+M^m[-1] + A^m[-(c_{44}^m + \frac{f_{15}^{m2}}{\mu_{11}^m})\lambda^m] + C^m[-f_{15}^m] \quad = \quad 0 \qquad (4.9)$$

$$M_1^e[-\chi_{11}^e] + M_2^e[\chi_{11}^e] + M^m[\chi_{11}^m] = 0 \qquad (4.10)$$

$$C_1^e[-\mu_{11}^e] + C_2^e[\mu_{11}^e] + C^m[\mu_{11}^m] = 0 \qquad (4.11)$$

Thus the frequency equation is obtained by eliminating the arbitrary constants from the magnetic and electric conditions given in the cases (1) and (2)

For

Case 1:

$$det(N_1) = 0$$

where N_1 is the coefficient matrix of equations (4.1), (4.2), (4.3), (4.6), (4.7), (4.8), (4.9), (4.10) and (4.11).

For

Case 2:

$$det(N_2) = 0$$

where N_2 is the coefficient matrix of equations (4.1), (4.4), (4.5), (4.6), (4.7), (4.8), (4.9), (4.10) and (4.11).

In all cases the waves are dispersive.

5. Numerical Results and Discussion:

Numerical results are obtained by taking the Piezoelectric material $PZT-4$ with Piezomagnetic material $CoFe_2O_4$ and Piezoelectric material $BaTiO_3$ with Piezomagnetic material $CoFe_2O_4$ with material properties given in (Table 1).

The frequency equation has been numerically solved for each of the stated electrical and magnetic boundary conditions and the wave velocities are plotted against the non-dimensional wave number $K = \frac{kd}{2\pi}$

Numerical investigations are carried out for two separate cases where the wave velocity lies between the two shear wave velocities of the two layers. For the $BaTiO_3/CoFe_2O_4$ composite $c^m_{SH} < c < c^e_{SH}$, and for the composite $PZT-4/CoFe_2O_4$, $c^e_{SH} < c < c^m_{SH}$.

Real wave velocity is found to exist for both situations. The wave velocity is plotted against the non-dimensional wave number for the cases as shown in (Fig.2), (Fig.3), (Fig.4).

6. Conclusions:

A problem on the propagation of guided SH waves in a piezoelectric-piezomagnetic composite layer is investigated in this paper. The frequency equation has been derived in closed form. The existence of real wave velocity has been established through numerical solution of the frequency equation.

The important results obtained could be incorporated as follows:

- The phase velocity of the waves lie in between the bulk shear wave velocities of the two materials.
- The velocity depends on the conditions on the boundary and real waves do not exist for every electric and magnetic boundary conditions.
- The nature of the dispersion curves are influenced by the properties of the piezoelectric layer. The velocities of the waves are higher for layers with larger shear wave velocities (This is in accordance with Nie et al [6]).
- The waves do not exist for low values of the non-dimensional wave number. This can also be interpreted as there does not exist any SH wave if a very thin piezoelectric layer is superposed on a piezomagnetic half space.
- The phase-velocities are influenced by the depth of the layer, which acts as a wave guide. The dispersion effect is more prominent for comparatively lower values of the non-dimensional wave number.

It is expected that these results will be of some use in construction of composite structures of piezoelectric-piezomagnetic materials.

The problem may be further extended to consider imperfect bondings at the interface and material inhomogeneity.

Acknowledgement: We are grateful to the referee for the comments which improved the quality of the paper.

$Material Properties$	$CoFe_2O_4$	$BaTiO_3$	$PZT-4$
$c_{44}(\times 10^9 N/m^2)$	45.3	44	25.6
$\rho(\times 10^3 kg/m^3)$	5.3	5.7	7.5
$\chi_{11}(\times 10^{-9} C^2/Nm^2)$	0.08	9.86	6.45
$\mu_{11}(\times 10^{-6} Ns^2/C^2)$	157	5	5
$e_{15}(C/m^2)$	-	11.4	12.7
$f_{15}(N/Am)$	550	-	-
$C_{SH}(m/s)$	2985.08	3167.28	2597.59

TABLE 1. Material Properties (SI units)

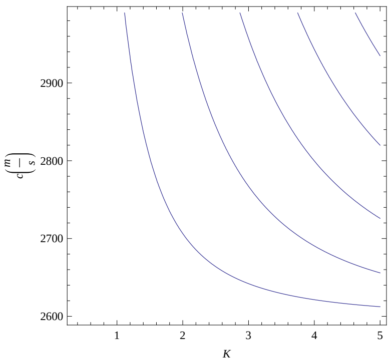


FIGURE 2. $PZT-4/CoFe_2O_4$: Case 1 (Electrically closed, Magnetically closed)

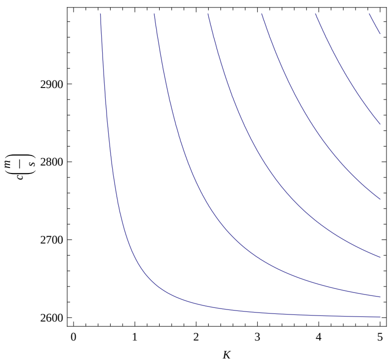


FIGURE 3. $PZT-4/CoFe_2O_4$: Case 2 (Electrically open, Magnetically open)

REFERENCES

[1] J. Holterman, P. Goren, Piezoelectric materials and applications, Applied-Piezo, The Netherlands, 2013.

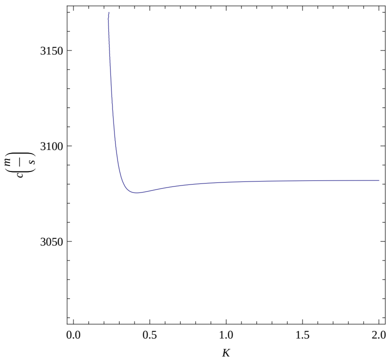


FIGURE 4. $BaTiO_3/CoFe_2O_4$: Case 1 (Electrically Closed, Magnetically Closed)

[2] Q. Wang, S. T. Quek, V. K. Varadan, Love waves in piezoelectric coupled solid media, *Smart materials and structures* **10** (2001) ,380-388.

[3] J. K. Du, X. Y. Jin, J. Wang, Love wave propagation in layered magneto electro elastic structures, *Science in China Series G: Physics, Mechanics and Astronomy* **Vol51, 6** (June 2008) ,617-631.

[4] F. Jin, Z. K. Wang, T. J. Wang, The Bluestein-Gulyaev wave in a piezoelectric layered halfspace, *Int, J, Engng. Sci* **39(11)** (2001) , 1271-1285.

[5] P. C. Y. Lee, N. H. Liu, Plane Harmonic Waves in an infinite piezoelectric plate with dissipation, *IEEE Transactions on Ultrasonics, Ferroelectrics and Frequency Control* **51(12)** (2004) ,1629-1638.

[6] Guoquan Nie, Zijun An, Jinxi Liu, SH guided waves in layered piezoelectric/piezomagnetic plates, *Progress in Natural Science* **19** (2009), 811-816.

[7] Jianke Du, Kai Xian, Ji Wang, Yook-Kong Yong, Love Wave propagation in Piezoelectric Layered Structure With Dissipation, *Ultrasonics* **49** (2009), 281-286.

[8] J. N. Sharma, K. K. Sharma, A. Kumar, Surface Waves in a piezoelectric-semiconductor composite structure, *Int. J. Solid Struc.* **47(6)** (2010), 816-826.

[9] J. Du, X. Jin, J. Wang, K. Xian, Love Wave propagation in a functionally graded Piezoelectric material, *Ultrasonics* **46:1** (2007), 13-22.

[10] A. M. Gaur, D. S. Rana, Shear Wave propagation in Piezoelectric- piezoelectric composite layered structure, *Latin Americal Journal of Solids and Structures* **11** (2014) , 2483-2496.

[11] L. L. Zhang, J. Zhao, X. L. Liu, J.X. Liu, Shear horizontal surface waves in piezoelectric materials with surface stress, *Philosophical magazine letters* **98:8** 2018.

[12] A. M. Gaur, Influence of stress on propagation of shear wave in piezoelectric-piezoelectric composite layered structure, *Latin Americal Journal of Solids and Structures* **15:1** 2018.

[13] R. Goyel, S. Kumar, V. Sharma, Microstructural considerations on SH-wave propagation in a piezoelectric layered structure, *J. of Theoretical and Applied Mechanics* **56(4)** (2018), 993-1004.

[14] Hamdi Ezzin, Morched Ben Amor, Mohamed Hedi Ben Ghozlen, Propagation behavior of SH waves in layered piezoelectric/piezomagnetic plates, *Acta Mechanica* November 2016, DOI 10.1007/s00707-016-1744-9 .

[15] M. J. Son, Y. J. Kang, Propagation behaviour of SH-waves in layered piezoelectric plates, *Journal of Mech. Sci. Tech* **25(3)** (2011) 613-619.

[16] A. K. Soh, J. X. Lin, Interfacial shear horizontal waves in a piezoelectric-piezomagnetic bi-material, *Philos. Mag. Lett* **86(1)** (2006), 31-35.

[17] A. Melkumyan, Y. W. Mai, Influence of imperfect bounding on interface waves guided by piezoelectric/piezomagnetic composites, *Philos Mag* **88(23)** (2008), 2965-2977.

Payel Mondal
Department of Mathematics
Vivekananda College
Madhyamgram
West Bengal, INDIA.
E-mail: mondalp1993@gmail.com

Sarbani Chakraborty
Department of Mathematics
West Bengal State University
Barasat
West Bengal, INDIA.
E-mail: b.sarbani@gmail.com

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July-December (2024), 74–90

APPROXIMATIONS, EXISTENCE AND UNIQUENESS OF THE INTEGRABLE LOCAL SOLUTION OF NONLINEAR VOLTERRA TYPE HYBRID INTEGRAL EQUATIONS

JANHAVI B. DHAGE AND BAPURAO C. DHAGE
(Received : 29 - 09 - 2023 ; Revised : 21 - 09 - 2024)

Abstract. In this paper, we prove a couple of approximation results for existence and uniqueness of the integrable local solutions of non-homogeneous nonlinear Volterra type hybrid integral equations under weaker partial compactness, partial Lipschitz and usual monotonicity type conditions. We employ the Dhage monotone iteration method based on the recent hybrid fixed point theorems of Dhage (2024) while establishing our main results. Our abstract result are also illustrated with a couple of numerical examples.

1. Introduction

Theoretical approximation results for existence and uniqueness of continuous and integrable local solutions for nonlinear differential and integral equations can be obtained under usual Lipschitz condition on the nonlinearity or monotonicity condition blending with the existence of upper and lower solutions of the related nonlinear problems. These results are achieved by the applications of Banach fixed point theorem or by monotone iteration method given in Ladde *et al.* [22] or generalized iteration method as depicted in Hekkilä and Lakshmikantham [20]. We observe that the hypotheses of continuity, boundedness and monotonicity of the nonlinearity are natural, but Lipschitzicity and existence of lower and upper solutions are stringent conditions which are rather difficult to hold for most of the nonlinear equations. Therefore, it is of interest to obtain such approximation results under weaker conditions or without the requirement of upper

2010 Mathematics Subject Classification: 45G10, 45I05, 47H10
Key words and phrases: Volterra integral equation; Hybrid fixed point principle; Dhage iteration method; Approximation result; Existence and uniqueness theorem.

and lower solutions which is the main motivation of the present paper. In the present study we obtain approximation results for existence and uniqueness of integrable solutions of a certain nonlinear hybrid Volterra integral equations.

Given a closed and bounded interval $J = [0, T]$ in $\mathbb{R}$, the set of real numbers, we consider a nonlinear hybrid Volterra integral equation (in short HVIE)

$$x(t) = q(t) + \lambda \int_0^t f(s, x(s))\, ds, \quad t \in J, \tag{1.1}$$

where $\lambda \in \mathbb{R}_+ = (0, \infty)$, and the functions $q : J :\to \mathbb{R}$, $f : J \times \mathbb{R} \to \mathbb{R}$ satisfy some hybrid conditions, that is, mixed conditions of "compactness, Lipschitz and monotonicity" to be specified later.

Definition 1.1. By an integrable solution of the nonlinear HVIE (1.1) we mean a function $x \in L^1(J, \mathbb{R})$ that satisfies the equation (1.1) defined on J, where $L^1(J, \mathbb{R})$ is the space of Lebesgue integrable functions on J. Furthermore, if a solution x of the HVIE (1.1) lies in the neighborhood of a point $x_0 \in L^1(J, \mathbb{R})$, we say it is a local or neighborhood solution of the HVIE (1.1) defined on J.

Remark 1.2. The concept of local or neighborhood solution of the HVIE (1.1) is different from that of usual notion of local solution as mentioned in Coddington [3]. In the terminology of Coddington [3], it is a nonlocal solution of the HVIE (1.1) defined on all of J.

The HVIE (1.1) is a nonlinear Volterra integral of second type which is very much common among the mathematicians working in the field of Volterra integral equations. It is needless to say the importance of the HVIE (1.1) and it appears in several biological and physical situations as mentioned in Banas [1], Mydlarczyk [23], Raffoul [24] and references therein. The HVIE (1.1) is studied very extensively in the literature for different aspects of the solution using different techniques from algebra, analysis and topology. Here, we discuss the HVIE (1.1) for approximation of the local solution via Dhage iteration method. In what follows, we discuss the existence, uniqueness and stability of integrable local solution by method of successive approximations using Dhage iteration method involving the recent hybrid fixed point theorems of Dhage [9, 10].

2. Preliminaries

We place the problem of HVIE (1.1) in the function space $L^1(J,\mathbb{R})$ of Lebesgue integrable real-valued functions defined on J. Now we introduce a norm $\|\cdot\|_{L^1}$ in $L^1(J,\mathbb{R})$ defined by

$$\|x\|_{L^1} = \int_0^T |x(t)|\,dt, \tag{2.1}$$

and an order relation $\preceq$ in $L^1(J,\mathbb{R})$ by the cone K given by

$$K = \{x \in L^1(J,\mathbb{R}) \mid x(t) \geq 0 \ \text{ a. e. } \ t \in J\}. \tag{2.2}$$

Thus,

$$x \preceq y \iff y - x \in K,$$

or equivalently,

$$x \preceq y \iff x(t) \leq y(t) \ \text{ a. e. } \ t \in J. \tag{2.3}$$

The details of order cones and related order relations may be found in Deimling [4], Guo and Lakshmikantham [19] and references therein. It is known that the Banach space $L^1(J,\mathbb{R})$ together with the order relations $\preceq$ becomes an ordered Banach space which we denote for convenience, by $\big(L^1(J,\mathbb{R}),K\big)$. We denote the open and closed spheres centered at $x_0 \in L^1(J,\mathbb{R})$ of radius r, by

$$B_r(x_0) = \{x \in L^1(J,\mathbb{R}) \mid \|x - x_0\|_{L^1} < r\} = B(x,r),$$

and

$$B_r[x_0] = \{x \in L^1(J,\mathbb{R}) \mid \|x - x_0\|_{L^1} \leq r\} = \overline{B(x,r)}, \tag{2.4}$$

respectively. It is clear that $B_r[x_0] = \overline{B_r(x_0)}$.

We need the following result concerning the compactness of a subset of $L^1(J,\mathbb{R})$ in what follows.

Lemma 2.1 (Kolmogorov compactness criterion [18])**.** *Let* $\Omega \subseteq L^p(J,\mathbb{R})$, $1 \leq p < \infty$. *If*

(i) Ω *is bounded, and*
(ii) $x_\eta \to x$ *as* $\eta \to 0$ *uniformly w.r.t.* $x \in \Omega$, *where* $x_\eta(t) = \dfrac{1}{\eta}\displaystyle\int_t^{t+\eta} x(s)\,ds.$

Then Ω *is a relatively compact subset of* $L^p(J,\mathbb{R})$.

It is well-known that the fixed point as well as hybrid fixed point theoretic techniques are very much useful in the subject of nonlinear analysis for dealing with the nonlinear equations qualitatively, see Granas and Dugundji [18], Raffoul [24] and the references therein. Here, we employ the Dhage monotone iteration method or simply *Dhage iteration method* based on the generalizations two hybrid fixed point theorems in the partially ordered abstract spaces. Generalizing the hybrid fixed point theorem of Dhage [10] and Dhage *et al.* [12], the present second author in [10] proved a Schauder type hybrid fixed point theorem in a partially ordered Banach space. Before stating this theorem, we give some preliminaries needed in the sequel.

Let $\big(E, d, \preceq\big)$ be a partially ordered metric space and let $S \subset E$. E is called **regular** if a monotone nondecreasing (resp. monotone nonincreasing) sequence $\{x_n\}$ in E converges to x_*, then $x_n \preceq x_*$ (resp. $x_* \preceq x_n$) for all $n \in \mathbb{N}$. The metric d and the order relation $\preceq$ are said to be **compatible** in S if a monotone sequence $\{x_n\}$ in S has a convergent subsequence, then the original sequence $\{x_n\}$ is convergent and converges to the same limit point. S is called a **Janhavi set** if d and $\preceq$ are compatible in it. S is called **partial bounded** (resp. partially closed, partially compact) if every chain C in S is bounded (resp. closed, compact).

A mapping $\mathcal{T} : S \to S$ is called **monotone nondecreasing** (resp. monotone nonincreasing) if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ (resp. $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$). $\mathcal{T}$ is **monotone** if it is either monotone nondereasing or monotone nonincreasing. $\mathcal{T}$ is called **partial bounded** (resp. partially totally bounded or partially precompact) if $\mathcal{T}(S)$ is partially bounded (resp. partially totally bounded or partially precompact for partially bounded S). T is **partially continuous** if $\{x_n\} \subset S$ converges to x_* with $x_n \preceq x_*$, then $\mathcal{T}x_n \to \mathcal{T}x$. T is called **partial completely continuous** if it is partially continuous and partially totally bounded.

Now we are equipped with all the necessary details to state our required hybrid fixed point theorems which are needed in what follows.

Theorem 2.2. *Let S be a non-empty, partial closed and partial bounded subset of a regular partially ordered Banach space $\big(E, \|\cdot\|, \preceq\big)$ and let every chain C in S be Janhavi set. Suppose that $\mathcal{T} : S \to S$ is a partial completely continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point ξ^* and* *the sequence* $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ *of successive iterations converges monotonically to* ξ^*.

Proof. The proof is similar to a hybrid fixed point theorem proved in Dhage [9] with obvious modifications, however the details appear in Dhage [10]. □

Theorem 2.3 (Dhage [9]). *Let* $B_r[x]$ *denote the partial closed ball centered at* x *of radius* r, *in a regular partially ordered Banach space* $\big(E, ||\cdot\|, \preceq,\big)$ *and let* $\mathcal{T} : E \to E$ *be a monotone nondecreasing and partial contraction operator with contraction constant* q. *If there exists an element* $x_0 \in X$ *such that* $x_0 \preceq \mathcal{T}x_0$ *or* $x_0 \succeq \mathcal{T}x_0$ *satisfying*

$$\|x_0 - \mathcal{T}x_0\| \leq (1-q)r, \tag{2.5}$$

for some real number $r > 0$, *then* $\mathcal{T}$ *has a unique comparable fixed point* x^* *in* $B_r[x_0]$ *and the sequence* $\{x_n\}_{n=0}^{\infty}$ *of successive iterations converges monotonically to* x^*. *Furthermore, if every pair of elements in* X *has a lower or upper bound, then* x^* *is unique.*

Remark 2.4. We note that every every pair of elements in a partially ordered set (in short poset) (**poset**) $(E, \preceq)$ has a lower or upper bound if $(E, \preceq)$ is a lattice, that is, $\preceq$ is a lattice order in E. In this case the poset $(E, \|\cdot\|, \preceq)$ is called a **partially lattice ordered Banach space**. There do exist several lattice partially ordered Banach spaces which are useful for applications in nonlinear analysis. For example, every Banach lattice is a partially lattice ordered Banach space. Notice that, $L^1(J, \mathbb{R})$ is a partially lattice ordered Banach space which is a complete lattice (see Dhage [5]). The details of the lattice structure of a Banach space appear in the monograph Birkhoff [2].

As a consequence of Remark 2.4, we obtain

Theorem 2.5. *Let* $B_r[x]$ *denote the partial closed ball centered at* x *of radius* r *for some real number* $r > 0$, *in a regular partially lattice ordered Banach space* $\big(E, ||\cdot\|, \preceq,\big)$ *and let* $\mathcal{T} : E \to E$ *be a monotone nondecreasing and partial contraction operator with contraction constant* q. *If there exists an element* $x_0 \in X$ *such that* $x_0 \preceq \mathcal{T}x_0$ *or* $x_0 \succeq \mathcal{T}x_0$ *satisfying (2.5), then* $\mathcal{T}$ *has a unique fixed point* ξ^* *in* $B_r[x_0]$ *and the sequence* $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ *of successive iterations converges monotonically to* ξ^*.

If a Banach space X is partially ordered by an order cone K in X, then in this case we simply say X is an **ordered Banach space** which we denote by (X,K). Similarly, if an ordered Banach space (X,K), where the partial order $\preceq$ defined by the con K is a lattice order, then (X,K) is called the **lattice ordered Banach space**. Clearly, an ordered Banach space $\big(L^1(J,\mathbb{R}),K\big)$ of Lebesgue integrable real-valued functions defined on the closed and bounded interval J is a lattice ordered Banach space, where the cone K is given by $K=\{x\in L^1(J,\mathbb{R})\mid x\succeq 0 \text{ a. e. on } J\}$. The details of the cones and their properties appear in Guo and Lakshmikantham [19]. Then, we have the following useful results concerning the ordered Banach spaces proved in Dhage [7, 8].

Lemma 2.6 (Dhage [7, 8])**.** *Every ordered Banach space (X,K) is regular.*

Lemma 2.7 (Dhage [7, 8])**.** *Every partially compact subset S of an ordered Banach space (X,K) is a Janhavi set in X.*

As a consequence of Lemmas 2.6 and 2.7, we obtain the following applicable hybrid fixed point theorems which we need in what follows.

Theorem 2.8. *Let S be a non-empty, partially closed and partially bounded subset of an ordered Banach space (X,K) and let $\mathcal{T}:S\to S$ be a partially completely continuous and monotone nondecreasing operator. If there exists an element $x_0\in S$ such that $x_0\preceq Tx_0$ or $x_0\succeq Tx_0$, then $\mathcal{T}$ has a fixed point $\xi^*\in S$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 2.9. *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r>0$, in a lattice ordered Banach space $\big(X,K\big)$ and let $\mathcal{T}:(X,K)\to(X,K)$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0\in X$ such that $x_0\preceq \mathcal{T}x_0$ or $x_0\succeq \mathcal{T}x_0$ satisfying (2.5), then $\mathcal{T}$ has a unique fixed point ξ^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

A few details of hybrid fixed point theorems and related applications appear in Deimling [4], Dhage [6, 7, 8], Dhage and Dhage [11], Dhage *et al.* [12, 14, 15], Dhage and Dhage [13] and references therein.

3. Local Approximation Results

We consider the following definition in the sequel.

Definition 3.1. A function $f : J \times \mathbb{R} \to \mathbb{R}$ is said to be $L^1_{\mathbb{R}}$-Carathéodory if

(i) the map $t \mapsto f(t,x)$ is measurable for each $x \in \mathbb{R}$,
(ii) the map $x \mapsto f(t,x)$ is continuous for almost everywhere $t \in J$, and
(iii) there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$|f(t,x)| \le h(t) \quad \text{a.e.} \quad t \in J, \text{ for all } x \in \mathbb{R}.$$

Lemma 3.2 (Granas and Dugundji [18]). *If $f(t,x)$ is $L^1_{\mathbb{R}}$-Carathéodory, then the function $t \mapsto f(t,x(t))$ is measurable and Lebesgue integrable for each $x \in L^1(J,,\mathbb{R})$.*

Lemma 3.3 (Krasnoselkii [21]). *If the function $f : J \times \mathbb{R} \to \mathbb{R}$ is $L^1_{\mathbb{R}}$-Carathéodory, then the superposition operator F defined by $(Fx)(t) = f(t,x(t))$ maps continuously the space $L^1(J,\mathbb{R})$ into itself.*

We need the following set of hypotheses in what follows.

(H_0) The function $q : J \to \mathbb{R}$ is Lebesgue integrable.
(H_1) There exists a constant $k > 0$ such that

$$0 \le f(t,x) - f(t,y) \le k(x-y) \quad \text{a. e.} \quad t \in J,$$

for all $x, y \in \mathbb{R}$ with $x \ge y$, where $\lambda\, k\, T < 1$.
(H_2) The function f is $L^1_{\mathbb{R}}$-Carathéodory.
(H_3) $f(t,x)$ is nondecreasing in x for almost everywhere on J.
(H_4) $f(t,q(t)) \ge 0$ a. e. $t \in J$, where q is given in hypothesis (H_0).

Theorem 3.4. *Suppose that the hypotheses (H_0), (H_2), (H_3) and (H_4) hold. If there exists a real number $r > 0$ such that $\lambda \|h\|_{L^1} T \le r$, then the HVIE (1.1) has an integrable solution x^* in $B_r[x_0]$, where, $x_0 \equiv q$, and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$\left.\begin{aligned} x_0(t) &= q(t), \quad t \in J, \\ x_{n+1}(t) &= q(t) + \lambda \int_0^t f(s, x_n(s))\, ds, \quad t \in J, \end{aligned}\right\} \tag{3.1}$$

where $n = 0, 1, \ldots$; is monotone nondecreasing and converges to x^*.

Proof. Set $X = L^1(J,\mathbb{R})$. Clearly, X is an ordered Banach space w.r.t. the norm $\|\cdot\|_{L^1}$ and the order relation $\preceq$ given by (2.1) and (2.3) respectively. Let x_0 be an initial function on J such that $x_0(t) = q(t)$ a. e. $t \in J$ and define a closed ball $B_r[x_0]$ in X defined by (2.4), where the number r satisfies the inequality $\lambda \|h\|_{L^1} T \leq r$. Now, define an operator $\mathcal{T}$ on $B_r[x_0]$ into X by

$$\mathcal{T}x(t) = q(t) + \lambda \int_0^t f(s, x(s))\, ds, \;\; t \in J.$$

Clearly, the integral and the consequently the operator $\mathcal{T}$ given in (3.2) is well defined in view of Lemma 3.2. Then the HFIE (1.1) is transformed into a hybrid operator equation (HOE),

$$\mathcal{T}x(t) = x(t), \quad \text{for all} \;\; t \in J. \tag{3.2}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.9 on $B_r[x_0]$ in the following series of steps.

Step I: *The operator $\mathcal{T}$ maps $B_r[x_0]$ into itself.*

Let $x \in B_r[x_0]$ be arbitrary. Then,

$$\begin{aligned}
|\mathcal{T}x(t) - x_0(t)| &= \lambda \left| \int_0^t f(s, x(s))\, ds \right| \\
&\leq \lambda \int_0^t \big| f(s, x(s)) \big|\, ds \\
&\leq \lambda \int_0^T h(s)\, ds \\
&= \lambda \|h\|_{L^1}.
\end{aligned}$$

Taking the integral on both sides from 0 to T w.r.t. t, we obtain

$$\begin{aligned}
\int_0^T \big|(\mathcal{T}x - x_0)(t)\big|\, dt &\leq \lambda \int_0^T \|h\|_{L^1}\, dt \\
&= \lambda \|h\|_{L^1} T.
\end{aligned}$$

Therefore,

$$\|\mathcal{T}x - q\|_{L^1} \leq \lambda \|h\|_{L^1} T \leq r.$$

This implies that $\mathcal{T}x \in B_r[x_0]$ for all $x \in B_r[x_0]$.

Step II: *$\mathcal{T}$ is a monotone nondecreasing operator on $B_r[x_0]$.*

Let $x, y \in B_r[x_0]$ be any two elements such that $x \succeq y$ almost everywhere on J. Then by (H_3) we obtain

$$\begin{aligned}
\mathcal{T}x(t) &= q(t) + \lambda \int_{t_0}^t f(s, x(s))\, ds \\
&\geq q(t) + \lambda \int_{t_0}^t f(s, y(s))\, ds
\end{aligned}$$

$$= \mathcal{T}y(t)$$

for almost every $t \in J$. So, $\mathcal{T}x \succeq \mathcal{T}y$ almost everywhere on J– that is, $\mathcal{T}$ is monotone nondecreasing on $B_r[x_0]$.

Step III: *$\mathcal{T}$ is a partially continuous operator on $B_r[x_0]$.*

Let C be a chain in $B_r[x_0]$ and let $\{x_n\}$ be a sequence in C converging almost everywhere to a point $x \in C$. Then by Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{T}x_n(t) &= \lim_{n\to\infty} \left[q(t) + \lambda \int_{t_0}^{t} f(s, x_n(s))\, ds \right] \\
&= q(t) + \lambda \lim_{n\to\infty} \int_{t_0}^{t} f(s, x_n(s))\, ds \\
&= q(t) + \lambda \int_{t_0}^{t} \Big[\lim_{n\to\infty} f(s, x_n(s)) \Big]\, ds \\
&= q(t) + \lambda \int_{t_0}^{t} f(s, x(s))\, ds = \mathcal{T}x(t)
\end{aligned}$$

for almost every $t \in J$. Therefore, $\mathcal{T}x_n \to \mathcal{T}x$ pointwise on J.

Next, we show that $\mathcal{T}x_n$ converges uniformly to $\mathcal{T}x$ in $L^!(J, \mathbb{R})$. Now, $\{\mathcal{T}x_n\}$ is a sequence of Lebesgue integrable functions, so it is also a sequence of measurable functions on J. Similarly, $\mathcal{T}x$ is also a measurable function on J. Moreover, we have

$$|\mathcal{T}x_n(t)| \le \|q\|_{L^1} + \lambda\, \|h\|_{L^1}$$

and

$$|\mathcal{T}x(t)| \le \|q\|_{L^1} + \lambda\, \|h\|_{L^1}.$$

Therefore, $\mathcal{T}x_n - \mathcal{T}x$ is measurable and

$$|\mathcal{T}x_n(t) - \mathcal{T}x(t)| \le |\mathcal{T}x_n(t)| + |\mathcal{T}x(t)| \le 2\big[\|q\|_{L^1} + \lambda\, \|h\|_{L^1}\big]$$

for almost every $t \in J$. Now, by definition of the norm $\|\cdot\|_{l^1}$, we obtain

$$\|\mathcal{T}x_n - \mathcal{T}x\|_{L^1} = \int_0^T |\mathcal{T}x_n(t) - \mathcal{T}x(t)|\, dt.$$

Again, applying the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|\mathcal{T}x_n-\mathcal{T}x\|_{L^1} &= \lim_{n\to\infty}\int_0^T |\mathcal{T}x_n(t)-\mathcal{T}x(t)|\,dt\\ &= \int_0^T \Big[\lim_{n\to\infty}|\mathcal{T}x_n(t)-\mathcal{T}x(t)|\Big]\,dt\\ &\to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

This shows that $\mathcal{T}x_n \to \mathcal{T}x$ uniformly. As a result $\mathcal{T}$ is a partially continuous operator on $B_r[x_0]$ into itself. We mention that the partial continuity of the operator $\mathcal{T}$ can also be obtained by giving different arguments and by using Lemma 3.3 as done in Banas [1], Emmanuel [17] and Krasnoselsdkii [21].

Step IV: *$\mathcal{T}$ is a partial compact operator on $B_r[x_0]$ into itself.*

To show $\mathcal{T}$ is a partial compact operator, it is enough to prove that $\mathcal{T}(B_r[x_0])$ is a partially compact subset of $B_r[x_0]$. Let C be a chain in $\mathcal{T}(B_r[x_0])$. We show that $\mathcal{T}(C)$ is a relatvely compact subset of $B_r[x_0]$. We apply the Kolomogorov theorem for compactness of a set in $L^1(J,\mathbb{R})$. Firstly, let $y\in\mathcal{T}(C)$ be any element. Then there is an element $x\in C$ such that $y=\mathcal{T}x$. Now, by hypothesis (H_2), we obtain

$$\begin{aligned}\|y\|_{L^1} &\le \|q\|_{L^1}+\lambda\int_0^T\left|\int_0^t |f(s,x(s))|\,ds\right|dt\\ &\le \|q\|_{L^1}+\lambda\int_0^T\int_0^t h(s)\,ds\,dt\\ &\le \|q\|_{L^1}+\lambda\|h\|_{L^1}T,\end{aligned}$$

for all $y\in\mathcal{T}(C)$. This shows that $\mathcal{T}(C)$ is uniformly bounded subset of $L^1(J,\mathbb{R})$. Next, we show that $(\mathcal{T}x)_\eta\to\mathcal{T}x$ as $\eta\to 0$ uniformly for every $x\in C$. Now,

$$\begin{aligned}\|(\mathcal{T}x)_\eta-\mathcal{T}x\|_{L^1} &= \int_0^T \big|(\mathcal{T}x)_\eta(t)-\mathcal{T}x(t)\big|\,dt\\ &= \int_0^T\left|\frac{1}{\eta}\int_t^{t+\eta}\mathcal{T}x(s)\,ds-\mathcal{T}x(t)\right|dt\\ &\le \int_0^T\frac{1}{\eta}\int_t^{t+h}\big|\mathcal{T}x(s)-\mathcal{T}x(t)\big|\,ds\,dt. \qquad (3.3)\end{aligned}$$

Since $\mathcal{T}x \in L^1(J,\mathbb{R})$, using the arguments that given in Swartz [25] (also see El-Sayed and Al-Issa [16]), we have

$$\frac{1}{\eta}\int_t^{t+\eta} \big|\mathcal{T}x(s)-\mathcal{T}x(t)\big|\,ds \to 0 \quad \text{as} \quad \eta\to 0,$$

uniformly for $x \in C$. Substituting the above estimate in (3.3), we obtain

$$\|\mathcal{T}x)_\eta - \mathcal{T}x\|_{L^1} \to 0 \quad \text{as} \quad \eta \to 0,$$

uniformly for $x \in C$. Therefore, $(\mathcal{T}x)_\eta \to \mathcal{T}x$ uniformly as $\eta \to 0$ for all $x \in C$. Now by an application of Kolomogorov theorem, we infer that $\mathcal{T}(C)$ is relatively compact subset of $B_r[x_0]$. Consequently, $\mathcal{T}$ is a partially compact operator on $B_r[x_0]$ into itself.

Step V: *The element $x_0 \equiv q \in B_r[x_0]$ satisfies the order relation $x_0 \preceq \mathcal{T}x_0$ almost everywhere on J.*

Since (H_4) holds, one has

$$\begin{aligned} x_0(t) &= q(t) + \lambda\int_{t_0}^t f(s,x_0(s))\,ds \\ &\leq x_0(t) + \lambda\int_{t_0}^t f(s,q(t))\,ds \\ &= q(t) + \lambda\int_{t_0}^t f(s,x_0(s))\,ds = \mathcal{T}x_0(t) \end{aligned}$$

for almost every $t \in J$. As a result, we have $x_0 \preceq \mathcal{T}x_0$ almost everywhere on J. This shows that the initial function x_0 in $B_r[x_0]$ serves to satisfy the operator inequality $x_0 \preceq \mathcal{T}x_0$.

Thus, the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.8, and so $\mathcal{T}$ has a fixed point x^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^\infty$ of successive iterations converges monotone nondecreasingly to x^* almost everywhere on J. This further implies that the HIE (1.1) and consequently the HVIE (1.1) has a integrable local solution x^* and the sequence $\{x_n\}_{n=0}^\infty$ of successive approximations defined by (3.1) converges monotone nondecreasingly to x^*. This completes the proof. □

Next, we prove an approximation result for existence and uniqueness of the solution simultaneously under weaker form of one sided or partial Lipschitz condition.

Theorem 3.5. *Suppose that the hypotheses (H_1), (H_2) and (H_4) hold. Furthermore, if*

$$\lambda \, \|h\|_{L^1} T \leq (1 - \lambda \, kT) r, \quad \lambda \, kT < 1, \tag{3.4}$$

for some real number $r > 0$, then the HVIE (1.1) has a unique integrable local solution x^* in $B_r[x_0]$ defined on J, where $x_0 \equiv q$ almost everywhere on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.1) converges monotone nondecreasingly to x^*.

Proof. Set $(X, K) = \big(L^1(J, \mathbb{R}), \preceq \big)$, which is a lattice w. r. t. the lattice operations *meet* ($\wedge$) and *join* ($\vee$) defined by $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$ respectively, and so every pair of elements of X has a lower and an upper bound. Let x_0 be an initial function on J such that $x_0(t) = q(t)$ for almost everywhere $t \in J$ and consider the closed ball $B_r[x_0]$ centered at $x_0 \in L^1(J, \mathbb{R})$ of radius r, in the lattice ordered Banach space (X, K).

Define an operator $\mathcal{T}$ on X into X by (3.2). Clearly, $\mathcal{T}$ is monotone nondecreasing on X. To see this, let $x, y \in X$ be two elements such that $x \succeq y$ almost everywhere on J. Then, by hypothesis (H_2), we have

$$\mathcal{T}x(t) - \mathcal{T}y(t) = \lambda \int_{t_0}^{t} \big[f(s, x(s)) - f(s, y(s))\big] \, ds \geq 0,$$

for almost every $t \in J$. Therefore, $\mathcal{T}x \succeq \mathcal{T}y$, and consequently $\mathcal{T}$ is monotone nondecresing on X.

Next, we show that $\mathcal{T}$ is a partial contraction on X. Let $x, y \in X$ be such that $x \succeq y$. Then, by hypothesis (H_2), we obtain

$$\begin{aligned} |\mathcal{T}x(t) - \mathcal{T}y(t)| &= \left| \int_{t_0}^{t} \lambda \big[f(s, x(s)) - f(s, y(s))\big] \, ds \right| \\ &\leq \lambda \left| \int_{t_0}^{t} k\big(x(s) - y(s)\big) \, ds \right| \\ &\leq \lambda \int_{t_0}^{T} k |x(s) - y(s)| \, ds \\ &= \lambda \, k \, \|x - y\|_{L^1} \end{aligned}$$

for almost every $t \in J$. Taking the integral from 0 to T on both sides of the above inequality yields

$$\|\mathcal{T}x - \mathcal{T}y\|_{L^1} \leq \lambda \, k \, T \, \|x - y\|_{L^1}, \quad \lambda \, k \, T < 1,$$

for all comparable elements $x, y \in X$. This shows that $\mathcal{T}$ is a partial contraction on X with contraction constant $\lambda \, k \, T$. Furthermore, it can be shown, as in the proof of Theorem 3.4, that the element $x_0 \in B_r[x_0]$ satisfies the relation $x_0 \preceq \mathcal{T} x_0$ in view of hypothesis (H_4). Finally, by hypothesis (H_1), one has

$$\begin{aligned} |x_0(t) - \mathcal{T} x_0(t)| = |q(t) - \mathcal{T} q(t)| &= \left| \lambda \int_0^t f(s, q(s))\, ds \right| \\ &\leq \lambda \int_0^T |f(s, q(s))|\, ds \\ &\leq \lambda \int_0^T h(s)\, ds = \lambda \, \|h\|_{L^1} \end{aligned}$$

for almost every $t \in J$. Now, from condition (3.4), we get

$$\begin{aligned} \|x_0 - \mathcal{T} x_0\|_{L^1} &= \int_0^T |x_0(t) - \mathcal{T} x_0(t)|\, dt \\ &\leq \int_0^T \lambda \, \|h\|_{L^1}\, dt \\ &= \lambda \, \|h\|_{L^1} T \leq (1 - \lambda \, k \, T) r, \end{aligned}$$

which shows that the condition (2.5) of Theorem 2.9 is satisfied. Hence, $\mathcal{T}$ has a unique fixed point x^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to x^*. This further implies that the HIE (3.2) and consequently the HVIE (1.1) has a unique integrable local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.1) converges monotone nondecreasingly to x^*. This completes the proof. □

Remark 3.6. The conclusion of Theorems 3.4 and 3.5 also remains true if we replace the hypothesis (H_4) with the following one.

(H_4') The function f satisfies inequality $f(t, q(t)) \leq 0 \quad$ a. e. $\; t \in J$, where the function q given in hypothesis (H_1).

In this case, the HVIE (1.1) has a integrable local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.1) is monotone nonincreasing and converges to x^*.

Remark 3.7. If the initial condition in the equation (1.1) is such that $q(t) > 0$ a. e. $t \in J$, then under the conditions of Theorem 3.4, the HVIE (1.1) has a integrable local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.1) converges monotone nondecreasingly to the positive solution x^*. Similarly, under the conditions of Theorem 3.5, the HVIE (1.1) has a unique integrable local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.1) converges monotone nondecreasingly to x^*.

Example 3.8. Let $J = [0,1] \subset \mathbb{R}$ and consider the HVIE

$$x(t) = t^2 + \int_0^t \tanh x(s)\, ds, \quad t \in [0,1]. \tag{3.5}$$

Here, the functions $q(t) = t^2 = x_0(t)$ and $f(t,x) = \tanh x$ satisfy all the hypotheses of Theorem 3.4 with $r = 1$. Hence, the HVIE (3.5) has a integrable nonnegative local solution x^* in $B_1[x_0]$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$\begin{aligned} x_0(t) &= t^2, \quad t \in [0,1], \\ x_{n+1}(t) &= t^2 + \int_0^t \tanh x_n(s)\, ds, \quad t \in [0,1], \end{aligned}$$

for $n = 0, 1, 2, \ldots$, is monotone nondeceasing and converges to x^*.

Example 3.9. Given $J = [0,1] \subset \mathbb{R}$, consider the HVIE

$$x(t) = \frac{t+1}{2} + \frac{1}{2}\int_0^t \tan^{-1} x(s)\, ds, \quad t \in [0,1]. \tag{3.6}$$

Here, the functions $q(t) = \dfrac{t+1}{2} = x_0(t)$ and $f(t,x) = \tan^{-1} x$ satisfy all the hypotheses of Theorem 3.5 with $r = 4$. Hence, the HVIE (3.5) has a unique integrable positive local solution x^* in $B_4[x_0]$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$\begin{aligned} x_0(t) &= \frac{t+1}{2}, \quad t \in [0,1], \\ x_{n+1}(t) &= \frac{t+1}{2} + \frac{1}{2}\int_0^t \tan^{-1} x_n(s)\, ds, \quad t \in [0,1], \end{aligned}$$

for $n = 0, 1, 2, \ldots$, is monotone nondeceasing and converges to x^*.

Remark 3.10. We observe that the existence and uniqueness results, Theroems 3.4 and 3.5 of this paper may be extended to the nonlinear Volterra type hybrid integral equation,

$$x(t) = q(t) + \lambda \int_0^t k(t,s) f(s, x(s))\, ds, \quad t \in J, \tag{3.7}$$

with appropriate modifications. In this case the desired approximation results for existence and uniqueness theorems are obtained under additional assumption that the kernel function $k : J \times J \to \mathbb{R}$ is $L^1_{\mathbb{R}}$-Carathéodory and nonnegative. The details of such criteria appears in Banas [1], Emmanuel [17] and references therein.

4. The Comparison

We observe that the existence of solutions of the HVIE (1.1) can also be obtained by an application of topological Schauder fixed point principle under the hypothesis (H_0) and (H_2), but in that case we do not get any sequence of successive approximations that converges to the solution. Again, we can not apply analytical or geometric Banach contraction mapping principle to the problem (3.1) under the considered hypotheses (H_0), (H_1) and (H_3) in order to get the desired conclusion, because here the nonlinear function f does not satisfy the usual Lipschitz condition on the domain $J \times \mathbb{R}$. Similarly, since $L^1(J, \mathbb{R})$ is a complete lattice w.r.t. the partial $\preceq$, we can apply algebraic Tarski fixed point theorem [26] or its extension obtained in Dhage [5] to HVIE (1.1) under the hypotheses (H_0), (H_1) and (H_3) for proving the existence of solution, but in that case also we do not get any sequence of successive approximations that converges to the solution. Therefore, all these arguments show that our hybrid fixed point principles, Theorems 2.8 and 2.9, have more advantages than other classical fixed point principles to get more information about the solution of nonlinear equations in the subject of nonlinear analysis. Finally, while concluding this paper, we mention that the integral equations (1.1) considered in this paper is very simple, however the method can be applied to other more complex nonlinear Volterra or Fredholm type integral equations involving integer or Riemann-Louville type fractional order of integration. The research in this direction forms the further scope for the work and some of the results along this line will be reported elsewhere.

References

[1] J. Banas, Integrable solutions of Hammerstein and Urysohn integral equations, *J. Austral. Math. Soc. (series A)*, **46** (1969), 61-68.

[2] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Coll. Publ. New York 1967.

[3] E. A. Coddington, *An Introduction to Ordinary Differential Equations*, Dover Publications Inc. New York, 1989.

[4] K. Deimling, *Nonlinear Functional Analysis*, Springer Verlag 1985.

[5] B. C. Dhage, An extension of lattice fixed point theorem and its applications, *Pure Appl. Math. Sci*, **25** (1987), 37-42.

[6] B. C. Dhage, Partially condensing mappings in partially ordered normed linear spaces and applications to functional integral equations, *Tamkang J. Math.*, **45** (4) (2014), 397-427.

[7] B. C. Dhage, A coupled hybrid fixed point theorem for sum of two mixed monotone coupled operators in a partially ordered Banach space with applications, *Tamkang J. Math.*, **50** (1) (2019), 1-36.

[8] B. C. Dhage, Coupled and mixed coupled hybrid fixed point principles in a partially ordered Banach algebra and PBVPs of nonlinear coupled quadratic differential equations, *Differ. Equ. Appl.*, **11** (1) (2019), 1-85.

[9] B. C. Dhage, A Schauder type hybrid fixed point theorem in a partially ordered metric space with applications to nonlinear functional integral equations, *Jñānābha*, **52** (2) (2022), 168-181.

[10] B. C. Dhage, Nonlinear partial completely continuous operators in a partially ordered Banach space and nonlinear hyperbolic partial differential equations, *Malaya J. Mat.*, **12** (2024) to appear.

[11] B. C. Dhage, S. B. Dhage, Approximating solutions of nonlinear first order ordinary differential equations, *GJMS Special issue for Recent Advances in Mathematical Sciences and Applications-13, GJMS*, **2** (2) (2013), 25-35.

[12] B. C. Dhage, J. B. Dhage, S.B. Dhage, Approximating existence and uniqueness of solution to a nonlinear IVP of first order ordinary iterative differential equations, *Nonlinear Studies*, **29** (1) (2022), 303-314.

[13] J. B. Dhage, B. C. Dhage, Approximating local solution of an IVP of nonlinear first order ordinary hybrid differential equations, *Nonlinear Studies* **30** (3) (2023), 721-732.

[14] J. B. Dhage, S. B. Dhage, B. C. Dhage, Approximation results for PBVPs of nonlinear first order ordinary functional differential equations in a closed subset of the Banach space, *Malaya Journal of Matematik*, vol. **11**, no. S, Oct. 2023, pp. 197-207.

[15] J. B. Dhage, S. B. Dhage, B. C. Dhage, An algorithmic approach to local solution of the nonlinear higher order ordinary hybrid differential equations, *Jñānābha* **54** (1) (2024), 29-40.

[16] A. M. A. El-Sayed, Sh. M. Al-Issa, Existence of integrable solutions for integrodifferential inclusions of fractional order: coupled system approach, *J. Nonlinear Sciences Appl.*, **13** (2020), 180-186.

[17] G. Emmanuel, Integrable solutions of a functional-integral equations, *J. Integral Equ. Appl.*, **4** (1) (1992), 89-94.

[18] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer 2003.

[19] D. Gua, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, London 1988.

[20] S. Heikkilä, V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker inc., New York, 1994.

[21] M. K. Krasnoselskii, On continuity of superposition operator $Fx(t) = f(t, x(t))$, *DOkl Akad Nauk SSSR* 77 (1951), 185-188 (In Russian).

[22] G. S. Ladde, V. Lakshmikantham, V, A. S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, Boston, 1985.

[23] W. Mydlarczyk, A system of Volterra integral equations with blowing up solutions, *Colloq. Math.*, **146** (2017), 99–110.

[24] Y. Raffoul, Classification of positive solutions of nonlinear systems of Volterra integral equations, *Ann. Funct. Anal.*, **2** (2011), 34-41.

[25] C. Swartz, *Measure, Integration and Function Spaces*, World Scientific Publishing Co. River Edge 1999.

[26] A. Tarski, A lattice theoretical fixed point theorem and its applications, *Pacific J. Math.*, **5** (1965), 285-309.

Jahavi B. Dhage
Kasubai, Gurukul Colony, Thodga Road
Ahmedpur - 413515, Dist. Latur
Maharashtra, India
E-mail: jbdhage@gmail.com

Bapurao C. Dhage
Kasubai, Gurukul Colony, Thodga Road
Ahmedpur - 413515, Dist. Latur
Maharashtra, India
E-mail: bcdhage@gmail.com

The Mathematics Student ISSN: 0025-5742
Vol. 92, Nos. 3-4, July-December (2024), 91–105

, , , , .

SPECTRAL DECOMPOSITION OF SOME TRIDIAGONAL MATRICES BY SECOND KIND OF CHEBYSHEV POLYNOMIALS

FAUZIA SHAHEEN AND NAJMUDDIN AHMAD

(Received : 30 - 10 - 2023 ; Revised : 14 - 05 - 2024)

Abstract. In this research, a particular kind of large and sparse Toeplitz matrix based on the arrangement of the entries of a tridiagonal matrix has been taken into consideration. Since we can directly relate the determinant of the corresponding tridiagonal matrix (with the same diagonal, subdiagonal, and superdiagonal entries) to Chebyshev polynomials. Utilizing this strategy, we have generalized the problem discussed by J.Rimes [16, 15] and Jesús Gutiérrez-Gutiérrez [10, 11]. The characteristic equation of a tridiagonal matrix, all eigenvalues, and associated eigenvectors along with the basic properties such as determinants, and the trace have been discussed in terms of the second kind of Chebyshev polynomials. At last, all the results have been applied successfully to the justifying examples.

1. Introduction

Tridiagonal matrices are commonly encountered in a wide range of Mathematical and Engineering applications such as solving ordinary and partial differential equations [8, 21], time series analysis [13], discrete ill-posed problems [12] and some other problems [9, 20, 3, 14, 1, 2, 18, 19, 4, 6] and [5].
Some authors have already used various Chebyshev polynomials on toeplitz matrices, based on the arrangement of sub-diagonal, diagonal, and super-diagonal elements. [22, 17, 7].

2010 Mathematics Subject Classification: 65F5, 65F15, 65F40, 65F50.
Key words and phrases: Spectral decomposition, Chebyshev polynomial, Eigen values, Eigen vectors, Toeplitz matrix

J. Rimes [16, 15] has applied in their research article the second kind of Chebyshev polynomials to tridiagonal matrices with elements -1; 0; 0; . . . , 0, 1 in principal and 1, 1, 1, . . . , 1 in neighbouring diagonals. Jesus Gutierrez-Gutierrez [10, 11] further derived a general expression for the entries of the q^{th} power of the $n \times n$ complex tridiagonal matrix, tridiagonal$_n(a_1;\ a_0;\ a_1)$ for all $n \in N$, in terms of the second kind Chebyshev polynomials.

In this research article, we have discussed the most general case of $n \times n$ matrices with elements $\gamma+\beta,\ \gamma+\beta,\ \beta,\ ...,\ \beta,\ \gamma+\beta$ in principal and $p,\ p,\ p,\ ...,\ p$ in sub diagonal and $q,\ q,\ q,\ ...,\ q$ in super diagonal ($p,\ q,\ \beta$ and γ are all real) for finding expressions for eigenvalues and associated eigenvectors.

The study is organized as follows: In the first section, we introduced the Toeplitz matrix that we are going to discuss in this research article. Preliminary definitions of Chebyshev polynomials are provided in the second part, along with some common results and lemmas. In the third section, we arrived at several results regarding eigenvalues, associated eigenvectors, and some fundamental properties of the given matrix. We have also discussed how to apply the results to various problems.

In this study, we take into account the n^{th} order near-Toeplitz tridiagonal matrices with the same precise perturbations in the first, second, and last main diagonal entries as follows:

$$P = \begin{pmatrix} \gamma+\beta & q & & & & \\ p & \gamma+\beta & q & & & \\ & p & \beta & q & & \\ & \ddots & \ddots & \ddots & & \\ & & p & \beta & q & \\ & & & p & \gamma+\beta \end{pmatrix}. \tag{1.1}$$

2. Notation and Preliminaries

2.1. **Chebyshev Polynomials.** The Chebyshev polynomials $T_n(a),\ U_n(a),\ V_n(a)$ and $W_n(a)$ of the first, second, third and fourth kinds are polynomials in x

of degree n defined respectively by

$$\begin{aligned} T_n(a) &= \cos n\phi, \\ U_n(a) &= \frac{\sin(n+1)\phi}{\sin\phi}, \\ V_n(a) &= \frac{\cos\left(n+\frac{1}{2}\right)\phi}{\cos\frac{\phi}{2}}, \\ W_n(a) &= \frac{\sin\left(n+\frac{1}{2}\right)\phi}{\sin\frac{\phi}{2}}, \end{aligned}$$

when $a = \cos\phi$, $-1 \le a \le 1$.

Lemma 2.1. *The four kinds of Chebyshev polynomials satisfy the same recurrence relation:*

$$X_n(a) = 2aX_{n-1}(a) - X_{n-2}(a),$$

with $X_0(a) = 1$ *in each case and* $X_1(a) = a, 2a, 2a-1, 2a+1$ *respectively. Furthermore, three relationships can be derived from the above*

$$\begin{aligned} 2T_n(a) &= U_n(a) - U_{n-2}(a), \\ V_n(a) &= U_n(a) - U_{n-1}(a), \\ W_n(a) &= U_n(a) + U_{n-1}(a). \end{aligned}$$

$U_n(a)$ *can be expressed by the determinant, namely*

$$U_0(a) = 1, \quad U_1(a) = 2a, \quad \textit{and} \quad U_2(a) = \begin{vmatrix} 2a & c \\ b & 2a \end{vmatrix} = 2aU_1(a) - U_0(a),$$

$$U_n(a) = \begin{vmatrix} 2a & c & & & \\ b & 2a & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2a & c \\ & & & b & 2a \end{vmatrix} = 2aU_{n-1}(a) - U_{n-2}(a),$$

where $bc = 1$.

3. Spectral Decomposition

Lemma 3.1. *If* P *is a tridiagonal matrix of the form (1.1), then the trace of* P *is*

$$\operatorname{tr} P = n\beta + 3\gamma,$$

the determinant of P is

$$\det P = \gamma^n U_{n-1}\left(\frac{\beta}{2\gamma}\right) + (\gamma+\beta)^2(\gamma)^{n-2}U_{n-2}\left(\frac{\beta}{2\gamma}\right), \tag{3.1}$$

and characteristic equation of P is

$$\begin{aligned}\det(P-\lambda I) = ((\gamma+\beta)-\lambda)\Bigg[&((\gamma+\beta)-\lambda)^2\gamma^{n-3}U_{n-3}\left(\frac{\beta-\lambda}{2\gamma}\right)\\ &- 2((\gamma+\beta)-\lambda)\gamma^{n-2}U_{n-4}\left(\frac{\beta-\lambda}{2\gamma}\right) + \gamma^{n-1}U_{n-5}\left(\frac{\beta-\lambda}{2\gamma}\right)\Bigg]\\ &- ((\gamma+\beta)-\lambda)\gamma^{n-1}U_{n-3}\left(\frac{\beta-\lambda}{2\gamma}\right) + \gamma^n U_{n-4}\left(\frac{\beta-\lambda}{2\gamma}\right). \end{aligned}\tag{3.2}$$

Proof. The trace of P is equal to the sum of all the diagonal entries, so obviously we have $trP = n\beta + 3\gamma$ from the form of P. Let's consider a tridiagonal matrix with constant diagonal entries as

$$A_n = \begin{pmatrix} \beta & q & & & & \\ p & \beta & q & & & \\ & p & \beta & q & & \\ & \ddots & \ddots & \ddots & & \\ & & p & \beta & q \\ & & & p & \beta \end{pmatrix}.$$

This determinant will be equivalent to the determinant obtained by expanding the Chebyshev polynomial when the condition that the product of subdiagonal and superdiagonal will be equal to one. So on modification of the terms of A_n, we will get

$$A_n = (\sqrt{pq})^n \begin{pmatrix} \frac{\beta}{\sqrt{pq}} & \sqrt{\frac{q}{p}} & & & & \\ \sqrt{\frac{p}{q}} & \frac{\beta}{\sqrt{pq}} & \sqrt{\frac{q}{p}} & & & \\ & \sqrt{\frac{p}{q}} & \frac{\beta}{\sqrt{pq}} & \sqrt{\frac{q}{p}} & & \\ & \ddots & \ddots & \ddots & & \\ & & \sqrt{\frac{p}{q}} & \frac{\beta}{\sqrt{pq}} & \sqrt{\frac{q}{p}} \\ & & & \sqrt{\frac{p}{q}} & \frac{\beta}{\sqrt{pq}} \end{pmatrix},$$

$$\det A_n = (\sqrt{pq})^n U_n\left(\frac{\beta}{2\sqrt{pq}}\right).$$

Now solving the determinant of P we have

$$\det P = (\gamma+\beta)\Big[(\gamma+\beta)^2 \det A_{n-3} - 2(\gamma+\beta)pq \det A_{n-4} + p^2q^2 \det A_{n-5}\Big] - pq(\gamma+\beta)\det A_{n-3} + p^2q^2 \det A_{n-4},$$

$$= (\gamma+\beta)\Big[(\gamma+\beta)^2(\sqrt{pq})^{n-3}U_{n-3}\left(\frac{\beta}{2\sqrt{pq}}\right) - 2(\gamma+\beta)pq(\sqrt{pq})^{n-4}U_{n-4}\left(\frac{\beta}{2\sqrt{pq}}\right) + p^2q^2(\sqrt{pq})^{n-5}U_{n-5}\left(\frac{\beta}{2\sqrt{pq}}\right)\Big] - pq(\gamma+\beta)(\sqrt{pq})^{n-3}U_{n-3}\left(\frac{\beta}{2\sqrt{pq}}\right) + p^2q^2(\sqrt{pq})^{n-4}U_{n-4}\left(\frac{\beta}{2\sqrt{pq}}\right),$$

$$= (\gamma+\beta)\Big[(\gamma+\beta)^2(\gamma)^{n-3}U_{n-3}\left(\frac{\beta}{2\gamma}\right) - 2(\gamma+\beta)(\gamma)^{n-2}U_{n-4}\left(\frac{\beta}{2\gamma}\right) + (\gamma)^{n-1}U_{n-5}\left(\frac{\beta}{2\gamma}\right)\Big] - (\gamma+\beta)(\gamma)^{n-1}U_{n-3}\left(\frac{\beta}{2\gamma}\right) + (\gamma)^{n}U_{n-4}\left(\frac{\beta}{2\gamma}\right),$$

$$\det P = \gamma^n U_{n-1}\left(\frac{\beta}{2\gamma}\right) + (\gamma+\beta)^2\gamma^{n-2}U_{n-2}\left(\frac{\beta}{2\gamma}\right).$$

Similar to the determinant of P the characteristic polynomial will be

$$\det(P-\lambda I) = ((\gamma+\beta)-\lambda)\Big[((\gamma+\beta)-\lambda)^2 \det A_{n-3} - 2pq((\gamma+\beta)-\lambda)\det A_{n-4} + p^2q^2 \det A_{n-5}\Big] - pq((\gamma+\beta)-\lambda)\det A_{n-3} + p^2q^2 \det A_{n-4},$$

$$= ((\gamma+\beta)-\lambda)\Big[((\gamma+\beta)-\lambda)^2\gamma^{n-3}U_{n-3}\left(\frac{\beta-\lambda}{2\gamma}\right)$$
$$-2((\gamma+\beta)-\lambda)\gamma^{n-2}U_{n-4}\left(\frac{\beta-\lambda}{2\gamma}\right)+\gamma^{n-1}U_{n-5}\left(\frac{\beta-\lambda}{2\gamma}\right)\Big]$$
$$-((\gamma+\beta)-\lambda)\gamma^{n-1}U_{n-3}\left(\frac{\beta-\lambda}{2\gamma}\right)+\gamma^{n}U_{n-4}\left(\frac{\beta-\lambda}{2\gamma}\right).$$

□

It is generally known that the corresponding eigenvectors of P can be obtained by solving the equation

$$(P-\lambda I)x = 0,\ x \neq 0 \tag{3.3}$$

In which the coefficient matrix $(P-\lambda I)$ is non symmetric. It is more convenient to solve the equation system if we change the coefficient matrix into a symmetric matrix. Let $D = \text{diag}(d_0,\ d_1\ , ...,\ d_{n-1})$ and $d_k = \left(\frac{p}{q}\right)^{k/2}$. Suppose y solves equation

$$(P-\lambda I)Dy = 0, \tag{3.4}$$

which can be deduced from the linear system of equations with the symmetric tridiagonal matrix, then $x = Dy$ is an eigenvector of P.

Let $a_i = \cos\frac{i\pi}{n},\ i = 1, 2, ..., n-1$.
When $\gamma = -\sqrt{pq}$, the equation (3.4) can be written as

$$\left(\frac{\beta-\lambda}{\gamma}-1\right)y_1 + y_2 = 0,$$
$$y_1 + \left(\frac{\beta-\lambda}{\gamma}-1\right)y_2 + y_3 = 0,$$
$$y_2 + \left(\frac{\beta-\lambda}{\gamma}\right)y_3 + y_4 = 0,$$
$$y_{n-2} + \left(\frac{\beta-\lambda}{\gamma}\right)y_{n-1} + y_n = 0,$$
$$y_{n-1} + \left(\frac{\beta-\lambda}{\gamma}-1\right)y_n = 0.$$

Solving the above equations, we have some solutions

$$y^{(i)} = [y_1,\ y_2,\ y_3,\ ...,\ y_n].$$

Hence, solutions of equation (3.3), the eigenvectors of P with $\gamma = -\sqrt{pq}$, are

$$x^{(i)} = [d_0 y_1,\ d_1 y_2,\ d_2 y_3,\ ...,\ d_{n-1} y_n]^T,$$

When $\gamma = \sqrt{pq}$, the equation (3.4) can be written as

$$\begin{aligned}
\left(\frac{\beta-\lambda}{\gamma}+1\right) y_1 + y_2 &= 0,\\
y_1 + \left(\frac{\beta-\lambda}{\gamma}+1\right) y_2 + y_3 &= 0,\\
y_2 + \left(\frac{\beta-\lambda}{\gamma}\right) y_3 + y_4 &= 0,\\
y_{n-2} + \left(\frac{\beta-\lambda}{\gamma}\right) y_{n-1} + y_n &= 0,\\
y_{n-1} + \left(\frac{\beta-\lambda}{\gamma}+1\right) y_n &= 0.
\end{aligned}$$

The system has solutions

$$y^{(i)} = [y_1,\ y_2,\ y_3,\ ...,\ y_n].$$

Therefore, the solutions of equation (3.3) are

$$x^{(i)} = [d_0 y_1,\ d_1 y_2,\ d_2 y_3,\ ..., d_{n-1} y_n]^T,$$

which are the eigenvectors of P with $\gamma = \sqrt{pq}$.

Lemma 3.2. *Using the above results of the eigenvalues and the corresponding eigenvectors of P, we give the spectral decomposition of P.*
Note that $E = diag(\lambda_1,\ \lambda_2,\ ...,\ \lambda_n)$ and $\lambda_1,\ ...,\ \lambda_n$ are eigenvalues of P.

If P has n linearly independent eigenvectors $x^{(1)},\ x^{(2)},\ ...,\ x^{(n)}$, form a non singular matrix X with them as columns, then $P = XEX^{-1}$, where

$$E = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

4. Examples

Example 4.1. Let's consider a tridiagonal square matrix of order 5 of the form (1.1) and applied the results for eigenvalues and eigenvectors which we have obtained as above.

$$P=\begin{pmatrix} 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & 1 & 1 \end{pmatrix}.$$

The Trace of the matrix P is

$$\texttt{tr}(P)=n\beta+3\gamma=3.$$

The determinant of P is

$$\det P=U_4\left(\frac{0}{2}\right)+U_3\left(\frac{0}{2}\right),$$

$$\det P=\prod_{i=1}^{4}\left(2\cos\frac{i\pi}{5}\right)+\prod_{i=1}^{3}\left(2\cos\frac{i\pi}{4}\right),$$

$$\det P=1.$$

Eigenvalues are given by the following relation

$$\begin{aligned}\det(P-\lambda I)=(1-\lambda)\Big[&(1-\lambda)^2\gamma^{n-3}U_{n-3}\left(\frac{-\lambda}{2}\right)\\ &-2(1-\lambda)\gamma^{n-2}U_{n-4}\left(\frac{-\lambda}{2}\right)+\gamma^{n-1}U_{n-5}\left(\frac{-\lambda}{2}\right)\Big]\\ &-(1-\lambda)\gamma^{n-1}U_{n-3}\left(\frac{-\lambda}{2}\right)+\gamma^{n}U_{n-4}\left(\frac{-\lambda}{2}\right)=0,\end{aligned}$$

$$\left(1-\lambda+2\lambda^2-\lambda^3\right)U_{n-1}\left(\frac{-\lambda}{2}\right)-\left(1-2\lambda+\lambda^2\right)U_{n}\left(\frac{-\lambda}{2}\right)=0,$$

$$\begin{aligned}\lambda\Big[2U_{n-2}\left(\frac{-\lambda}{2}\right)+U_{n-3}\left(\frac{-\lambda}{2}\right)+U_{n-1}\left(\frac{-\lambda}{2}\right)\Big]&\\ -\Big[U_{n-2}\left(\frac{-\lambda}{2}\right)+U_{n-1}\left(\frac{-\lambda}{2}\right)\Big]&=0,\end{aligned}$$

$$\lambda=\frac{U_{n-2}\left(\frac{-\lambda}{2}\right)+U_{n-1}\left(\frac{-\lambda}{2}\right)}{U_{n-3}\left(\frac{-\lambda}{2}\right)+2U_{n-2}\left(\frac{-\lambda}{2}\right)+U_{n-1}\left(\frac{-\lambda}{2}\right)}.$$

And corresponding eigenvectors are

$$x^{(i)} = \begin{pmatrix} 1 \\ (\lambda - 1) \\ (\lambda^2 - 2\lambda) \\ (\lambda^3 - 2\lambda^2 - \lambda + 1) \\ (\lambda^4 - 2\lambda^3 - 2\lambda^2 + 3\lambda) \end{pmatrix},$$

where

$$\lambda = \frac{U_3\left(\frac{-\lambda}{2}\right) + U_4\left(\frac{-\lambda}{2}\right)}{U_2\left(\frac{-\lambda}{2}\right) + 2U_3\left(\frac{-\lambda}{2}\right) + U_4\left(\frac{-\lambda}{2}\right)}.$$

And

$$U_2\left(\frac{-\lambda}{2}\right) = \prod_{i=1}^{2}\left(-\lambda + 2\cos\frac{i\pi}{3}\right),\ i = 1,2,3,4,5$$

$$U_3\left(\frac{-\lambda}{2}\right) = \prod_{i=1}^{3}\left(-\lambda + 2\cos\frac{i\pi}{4}\right),\ i = 1,2,3,4,5$$

$$U_4\left(\frac{-\lambda}{2}\right) = \prod_{i=1}^{4}\left(-\lambda + 2\cos\frac{i\pi}{3}\right),\ i = 1,2,3,4,5.$$

Example 4.2. Another tridiagonal square matrix has been subjected to the same applications, and the findings are as follows.

$$P = \begin{pmatrix} 5 & 8 & & & \\ 2 & 5 & 8 & & \\ & 2 & 1 & 8 & \\ & & 2 & 1 & 8 \\ & & & 2 & 5 \end{pmatrix}.$$

The Trace of the matrix P is

$$\mathtt{tr}P = n\beta + 3\gamma = 17.$$

The determinant of P is

$$\det P = (4)^5 U_4\left(\frac{1}{8}\right) + 25(4)^3 U_3\left(\frac{1}{8}\right),$$

$$\det P = 4\prod_{i=1}^{4}\left(1 + 8\cos\frac{i\pi}{5}\right) + 25\prod_{i=1}^{3}\left(1 + 8\cos\frac{i\pi}{5}\right),\ i = 1,2,3,4,5$$

$$\det P = 61.$$

Eigenvalues are given by the following characteristic equation

$$\det(P-\lambda I) = (5-\lambda)\Bigg[(5-\lambda)^2\gamma^{n-3}U_{n-3}\left(\frac{1-\lambda}{8}\right) - 2(5-\lambda)\gamma^{n-2}U_{n-4}\left(\frac{1-\lambda}{8}\right) + \gamma^{n-1}U_{n-5}\left(\frac{1-\lambda}{8}\right)\Bigg] - (5-\lambda)\gamma^{n-1}U_{n-3}\left(\frac{1-\lambda}{8}\right) + \gamma^{n}U_{n-4}\left(\frac{1-\lambda}{8}\right) = 0,$$

$$(5-\lambda)\Bigg[(5-\lambda)^2 4^2 U_{n-3}\left(\frac{1-\lambda}{8}\right) - 2(5-\lambda)4^3U_{n-4}\left(\frac{1-\lambda}{8}\right) + 4^4U_{n-5}\left(\frac{1-\lambda}{8}\right)\Bigg] - (5-\lambda)4^4U_{n-3}\left(\frac{1-\lambda}{8}\right) + 4^5U_{n-4}\left(\frac{1-\lambda}{8}\right) = 0,$$

$$1424U_4\left(\frac{1-\lambda}{8}\right) - 1600U_n\left(\frac{1-\lambda}{8}\right) - 560\lambda U_{n-1}\left(\frac{1-\lambda}{8}\right) + 640\lambda U_n\left(\frac{1-\lambda}{8}\right) + 176\lambda^2U_{n-1}\left(\frac{1-\lambda}{8}\right) - 16\lambda^3U_{n-1}\left(\frac{1-\lambda}{8}\right) - 64\lambda^2U_n\left(\frac{1-\lambda}{8}\right) = 0,$$

$$\lambda\Bigg[6U_{n-1}\left(\frac{1-\lambda}{8}\right) + 16U_{n-3}\left(\frac{1-\lambda}{8}\right) + 36U_{n-2}\left(\frac{1-\lambda}{8}\right)\Bigg] - \Bigg[54U_{n-1}\left(\frac{1-\lambda}{8}\right) + 140U_{n-2}\left(\frac{1-\lambda}{8}\right) + 40U_n\left(\frac{1-\lambda}{8}\right)\Bigg] = 0,$$

$$\lambda = \frac{27U_{n-1}\left(\frac{1-\lambda}{8}\right) + 70U_{n-2}\left(\frac{1-\lambda}{8}\right) + 20U_n\left(\frac{1-\lambda}{8}\right)}{3U_{n-1}\left(\frac{1-\lambda}{8}\right) + 8U_{n-3}\left(\frac{1-\lambda}{8}\right) + 18U_{n-2}\left(\frac{1-\lambda}{8}\right)}.$$

And corresponding eigen vectors are

$$x^{(i)} = \begin{pmatrix} (\frac{1}{4})^0 \\ (\frac{1}{4})^{\frac{1}{2}}\left(\frac{\lambda-5}{4}\right) \\ (\frac{1}{4})^1\left(\frac{\lambda^2-10\lambda+9}{16}\right) \\ (\frac{1}{4})^{\frac{3}{2}}\left(\frac{\lambda^3-11\lambda^2+3\lambda+71}{64}\right) \\ (\frac{1}{4})^2\left(\frac{\lambda^4-12\lambda^3-2\lambda^2+228\lambda-215}{256}\right) \end{pmatrix},$$

where

$$\lambda = \frac{27U_4\left(\frac{1-\lambda}{8}\right) + 70U_3\left(\frac{1-\lambda}{8}\right) + 20U_5\left(\frac{1-\lambda}{8}\right)}{3U_4\left(\frac{1-\lambda}{8}\right) + 8U_2\left(\frac{1-\lambda}{8}\right) + 18U_3\left(\frac{1-\lambda}{8}\right)}.$$

And

$$U_2\left(\frac{1-\lambda}{8}\right) = \frac{1}{4^2}\prod_{i=1}^{2}\left(1-\lambda+8\cos\frac{i\pi}{3}\right),\ i=1,2,3,4,5$$

$$U_3\left(\frac{1-\lambda}{8}\right) = \frac{1}{4^3}\prod_{i=1}^{3}\left(1-\lambda+8\cos\frac{i\pi}{4}\right),\ i=1,2,3,4,5$$

$$U_4\left(\frac{1-\lambda}{8}\right) = \frac{1}{4^4}\prod_{i=1}^{4}\left(1-\lambda+8\cos\frac{i\pi}{5}\right),\ i=1,2,3,4,5$$

$$U_5\left(\frac{1-\lambda}{8}\right) = \frac{1}{4^5}\prod_{i=1}^{5}\left(1-\lambda+8\cos\frac{i\pi}{6}\right),\ i=1,2,3,4,5.$$

Example 4.3. Let's consider the third example as follows:

$$P = \begin{pmatrix} -1 & 1 & & & \\ 1 & -1 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & 1 & -1 \end{pmatrix}.$$

The Trace of the matrix P is

$$\mathtt{tr}P = n\beta + 3\gamma = -3.$$

The determinant of P is

$$\det P = (-1)^5 U_4\left(\frac{0}{-2}\right) + (-1)^3 U_3\left(\frac{0}{-2}\right),$$

$$\det P = -1\prod_{i=1}^{4}\left(-2\cos\frac{i\pi}{5}\right) + \prod_{i=1}^{3}\left(-2\cos\frac{i\pi}{4}\right) = -1.$$

Eigenvalues are given by the following characteristic equation

$$(-1-\lambda)\Big[(-1-\lambda)^2(-1)^{n-3}U_{n-3}\left(\frac{\lambda}{2}\right) - 2(-1-\lambda)(-1)^{n-2}U_{n-4}\left(\frac{\lambda}{2}\right) + (-1)^{n-1}U_{n-5}\left(\frac{\lambda}{2}\right)\Big] - (-1-\lambda)(-1)^{n-1}U_{n-3}\left(\frac{\lambda}{2}\right) + (-1)^{n}U_{n-4}\left(\frac{\lambda}{2}\right) = 0,$$

$$U_n\left(\frac{\lambda}{2}\right) - U_{n-1}\left(\frac{\lambda}{2}\right) - \lambda U_{n-1}\left(\frac{\lambda}{2}\right) + 2\lambda U_n\left(\frac{\lambda}{2}\right) - 2\lambda^2 U_{n-1}\left(\frac{\lambda}{2}\right) - \lambda^3 U_{n-1}\left(\frac{\lambda}{2}\right) + \lambda^2 U_n\left(\frac{\lambda}{2}\right) = 0,$$

$$\left(3U_{n-1}\left(\frac{\lambda}{2}\right) + U_{n-2}\left(\frac{\lambda}{2}\right) + 2U_{n-3}\left(\frac{\lambda}{2}\right)\right) + \lambda\left(U_{n-1}\left(\frac{\lambda}{2}\right) + U_{n-3}\left(\frac{\lambda}{2}\right)\right) = 0,$$

$$\lambda = -\frac{3U_{n-1}\left(\frac{\lambda}{2}\right) + U_{n-2}\left(\frac{\lambda}{2}\right) + 2U_{n-3}\left(\frac{\lambda}{2}\right)}{U_{n-3}\left(\frac{\lambda}{2}\right) + U_{n-1}\left(\frac{\lambda}{2}\right)}.$$

And corresponding eigenvectors are

$$x^{(i)} = \begin{pmatrix} 1 \\ (1+\lambda) \\ (\lambda^2+2\lambda) \\ (\lambda^3+2\lambda^2-\lambda-1) \\ (\lambda^4+2\lambda^3-2\lambda^2-3\lambda) \end{pmatrix},$$

where

$$\lambda = \frac{-3U_4\left(\frac{\lambda}{2}\right) + U_3\left(\frac{\lambda}{2}\right) + 2U_2\left(\frac{\lambda}{2}\right)}{U_2\left(\frac{\lambda}{2}\right) + U_4\left(\frac{\lambda}{2}\right)}.$$

And

$$U_2\left(\frac{\lambda}{2}\right) = \frac{1}{(-1)^2}\prod_{i=1}^{2}\left(\lambda + 2\cos\frac{i\pi}{3}\right),\ i = 1,2,3,4,5$$

$$U_3\left(\frac{\lambda}{2}\right) = \frac{1}{(-1)^3}\prod_{i=1}^{3}\left(\lambda + 2\cos\frac{i\pi}{4}\right),\ i = 1,2,3,4,5$$

$$U_4\left(\frac{\lambda}{2}\right) = \frac{1}{(-1)^4}\prod_{i=1}^{4}\left(\lambda + 2\cos\frac{i\pi}{5}\right),\ i = 1,2,3,4,5.$$

5. Conclusion

In this research, a formulation for all the eigenvalues and corresponding eigenvectors of the $n \times n$ tridiagonal matrix P in terms of the second kind of Chebyshev polynomials has been presented. This technique is very simple and quick to apply because this study has explained all the eigenvalues and associated eigenvectors at once for very large and sparse matrices. Successful implementation of results to three arbitrary matrices has been given. Further, the study can also show how the m^{th} power of a triangular matrix can be used for calculating the inverse and solution of the tridiagonal matrices.

Acknowledgement (MCN)

IU/R & D/2022-MCN0001608 office of research and development Integral University, Lucknow.

References

[1] Ahmad, N. and Shaheen, F., Study of Numerical Solution of Linear System of Equations by Using SOR Algorithm with $0 < \omega < 1$, *Communications in Mathematics and Applications* **12** (2021), 853-865.

[2] Ahmad, N. and Shaheen, F., New Modified SSOR Method for Numerical Solution of Linear Augmented System, *Journal of Science and Arts* **21** (2021), 365-384.

[3] Anderson, C. R., A Rayleigh–Chebyshev procedure for finding the smallest eigenvalues and associated eigenvectors of large sparse Hermitian matrices, *Journal of Computational Physics* **229** (2010), 7477-7487.

[4] Bourne, D., Hydrodynamic stability, the Chebyshev tau method and spurious eigenvalues, *Continuum Mechanics and Thermodynamics* **15** (2003), 571-579.

[5] da, F., Carlos, M. and Ylmaz, F., Some comments on k-tridiagonal matrices: determinant, spectra, and inversion, *Applied Mathematics and Computation* **270** (2015), 644-647.

[6] da F., CM, The characteristic polynomial of some perturbed tridiagonal k-Toeplitz matrices, *Appl. Math. Sci* **1** (2007), 59-67.

[7] Da F., CM, On the eigenvalues of some tridiagonal matrices, *Journal of Computational and Applied Mathematics* **200** (2007), 283-286.

[8] Diele, F., Lopez, L., The use of the factorization of five-diagonal matrices by tridiagonal Toeplitz matrices,*Applied mathematics letters* **11** (1998), 61-69.

[9] Driessel, K. R., A relation between some special centro-skew, near-Toeplitz, tridiagonal matrices and circulant matrices, *arXiv preprint arXiv:1102.1953* **124** (2011).

[10] Gutirrez G., J., Powers of tridiagonal matrices with constant diagonals, *Applied Mathematics and Computation* **206** (2008), 885-891.

[11] Granville, A., Powers of real persymmetric anti-tridiagonal matrices with constant anti-diagonals, *Applied mathematics and computation* **206** (2008), 919-924.

[12] Hansen, P.,C., Rank-deficient and discrete ill-posed problems: numerical aspects of linear inversion, SIAM, 1998.

[13] Luati, A. and Proietti, T., On the spectral properties of matrices associated with trend filters, *Econometric Theory* **26** (2010), 1247-1261.

[14] Noschese, S., Pasquini, L., and Reichel, L., Tridiagonal Toeplitz matrices: properties and novel applications, *Numerical Linear Algebra with Applications* **20** (2013), 302-326.

[15] Rimas, J., On computing of arbitrary positive integer powers for tridiagonal matrices with elements-1, 0, 0,..., 0, 1 in principal and 1, 1, 1,..., 1 in neighbouring diagonals-II, *Applied Mathematics and Computation* **188** (2007), 2020-2024.

[16] Rimas, J., On computing of arbitrary positive integer powers for tridiagonal matrices with elements -1,0,0,....,0,1 in principal and 1,1,1,...,1 in neighbouring diagonals - I,*Applied Mathematics and Computation* **188** (2007), 634-637.

[17] Saad, Y., Chebyshev acceleration techniques for solving nonsymmetric eigenvalue problems, *Mathematics of Computation* **42** (1984), 567-588.

[18] Shaheen, F., Ahmad, N., Novel Approach for Eigenvalue Problems Using the Monte Carlo Method, *Journal of Science and Arts* **23** (2023), 953-964.

[19] Shaheen, F., Ahmad, N., Fermat Collocation Method For Solution Of A System Of Linear Ordinary Differential Equations, *Journal of the Calcutta Mathematical Society,* **20** (2024), 69-80.

[20] Shen, N., Jiang, Z. and Li, J., The Spectral Decomposition of near-Toeplitz tridiagonal matrices, *International Journal of Applied Mathematics and Informatics* **124** (2013), 115-122.

[21] Smith, G. D., Numerical solution of partial differential equations: finite difference methods, Oxford university press,1985.

[22] Wrigley, HE., Accelerating the Jacobi method for solving simultaneous equations by Chebyshev extrapolation when the eigenvalues of the iteration matrix are complex, *The Computer Journal* **6** (1963), 169-176.

Fauzia Shaheen
Department of Mathematics & Statistics
Faculty of Science
Integral University
Kursi road, Lucknow, UP, India.
E-mail: shaheenfauzia97@gmail.com

Najmuddin Ahmad
Department of Mathematics & Statistics
Faculty of Science

Integral University
Kursi road, Lucknow, UP, India.
E-mail: najmuddinahmad33@gmail.com

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July-December (2024), 106–114

A SOLUTION TO ANOTHER INTERESTING SUM INVOLVING CLASSICAL HARMONIC NUMBER AND CENTRAL BINOMIAL COEFFICIENT

NANDAN SAI DASIREDDY

(Received : 05 - 11 - 2023 ; Revised : 14 - 07 - 2024)

Abstract. This paper is a continuation of a previous one (Math. Student 93(1-2):132-137, 2024) in which an interesting sum considered by Nimbran has been solved. In the present paper we solve another interesting sum considered by Nimbran, providing explicit closed-form expression for the following series.

$$\sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{(2n+1)^2\, 2^{2n}}.$$

The key ingredients for obtaining our infinite series result (1.1) are some of the difficult definite integral formulas due to K. S. Kölbig and an obscure infinite series due to Khristo N. Boyadzhiev.

1. Introduction

Continuing the previous paper [3], which was devoted to solving an interesting sum posed by Nimbran in [4, p. 134], we discuss here another interesting sum considered by Nimbran in [4, p. 134].

Amrik Singh Nimbran left the following problem of evaluating a series involving classical harmonic number and central binomial coefficient as an interesting sum in [4, p. 134], which we will provide a solution to in this paper:

$$\sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{(2n+1)^2\, 2^{2n}} \tag{1.1}$$

2020 Mathematics Subject Classification: Primary 40A25, 65B10, 05A10; Secondary 11M06
Key words and phrases: Classical harmonic numbers, Riemann zeta function, central binomial coefficients, Catalan's constant, infinite summation formulas

We shall recall notation from [2], writing

$$\mathcal{G} := \Im\left(\mathrm{Li}_3\left(\frac{i+1}{2}\right)\right)$$

to denote the Catalan-like constant explored in [2], [5], letting
$G=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}$ denote the "original" Catalan's constant.

Throughout, we use the same notations and definitions as [3].

To evaluate (1.1), we shall establish some lemmas.

Lemma 1.1. *Let $n>-\frac{1}{2}$. The following equality holds:*

$$\int_0^1 x^{2n}\ln(x)\,dx=-\frac{1}{(2n+1)^2}.$$

Proof. We have, using integration by parts, that

$$\int_0^1 x^{2n}\ln(x)\,dx=\left[\frac{x^{2n+1}\ln(x)}{2n+1}\right]_0^1-\int_0^1\frac{x^{2n}}{2n+1}dx=-\frac{1}{(2n+1)^2}.$$

□

Lemma 1.2. *The following identity holds:*

$$\sum_{n=1}^{\infty}H_n\binom{2n}{n}\left(\frac{x^2}{4}\right)^n=\frac{2}{\sqrt{1-x^2}}\ln\left(\frac{1+\sqrt{1-x^2}}{2\sqrt{1-x^2}}\right).$$

Proof. In a wonderful paper [1, Thm. 1, p. 2], Khristo N. Boyadzhiev had evaluated the following generating function involving classical harmonic number and central binomial coefficient:

$$\sum_{n=1}^{\infty}H_n\binom{2n}{n}x^n=\frac{2}{\sqrt{1-4x}}\ln\left(\frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}\right),\quad x\in\left[-\frac{1}{4},\frac{1}{4}\right).\tag{1.2}$$

Replacing x with $\frac{x^2}{4}$ on both sides of (1.2), gives us the desired result.

□

Lemma 1.3. *The following equality holds:*

$$\int_0^1\frac{\ln(t)}{1+t^2}dt=-G.$$

Proof.
$$\int_0^1 \frac{\ln(t)}{1+t^2}dt = \int_0^1 \ln(t)\left(\sum_{i=0}^{\infty}(-1)^i t^{2i}\right)dt$$
$$= \sum_{i=0}^{\infty}(-1)^i\left(\int_0^1 t^{2i}\ln(t)dt\right)$$
$$= -\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)^2} = -G.$$

□

Lemma 1.4. *The following equality holds:*
$$\int_0^1 \frac{\ln\left(\frac{1+t^2}{t}\right)}{1+t^2}dt = \frac{\pi}{2}\ln 2.$$

Proof. We make the substitution: $t = \tan y$ so that $dt = \left(1+t^2\right)\, dy$

$$\text{Then, } \int_0^1 \frac{\ln\left(\frac{1+t^2}{t}\right)}{1+t^2}dt = \int_0^{\frac{\pi}{4}}\left(\ln 2 - \ln(\sin 2y)\right)dy$$
$$= \frac{\pi}{4}\ln 2 - \int_0^{\frac{\pi}{4}}\ln(\sin 2y)dy$$

Now, in the last definite integral on the right side, we make the substitution: $2y = a$ so that $da = 2dy$

$$\text{Then, } \int_0^1 \frac{\ln\left(\frac{1+t^2}{t}\right)}{1+t^2}dt = \frac{\pi}{4}\ln 2 - \frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\sin a)da.$$

In this equation, due to symmetry, recognizing that $\int_0^{\frac{\pi}{2}}\ln\left(\sin a\right)da = \int_0^{\frac{\pi}{2}}\ln\left(\cos a\right)da$, we conclude:
$$\int_0^1 \frac{\ln\left(\frac{1+t^2}{t}\right)}{1+t^2}dt = \frac{\pi}{4}\ln 2 - \frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\cos a)da.$$

Using the derivative of the Euler's beta function and the Leibniz formula for the differentiation of products, K.S.Kölbig [6, p. 25], had evaluated the definite integral

$$\int_0^{\frac{\pi}{2}} \ln(\cos a)\, da = -\frac{\pi}{2}\ln 2.$$

giving us that the following equality holds:

$$\int_0^1 \frac{\ln\left(\frac{1+t^2}{t}\right)}{1+t^2}dt = \frac{\pi}{4}\ln 2 - \frac{1}{2}\left(-\frac{\pi}{2}\ln 2\right) = \frac{\pi}{2}\ln 2.$$

□

Lemma 1.5. *The following equality holds:*

$$\int_0^1 \frac{\ln(1+t^2)}{1+t^2}dt = \frac{\pi}{2}\ln 2 - G.$$

Proof. $\displaystyle\int_0^1 \frac{\ln(1+t^2)}{1+t^2}dt = \int_0^1 \frac{\ln(t)}{1+t^2}dt + \int_0^1 \frac{\ln\left(\frac{1+t^2}{t}\right)}{1+t^2}dt = -G + \frac{\pi}{2}\ln 2.$

□

Lemma 1.6. *The following equality holds:*

$$\int_0^{\frac{\pi}{4}} \ln^2(\sin z)\, dz = \frac{9\pi}{32}\ln^2 2 + \frac{G\ln 2}{2} + \frac{23\pi}{64}\zeta(2) - \mathcal{G}.$$

Proof. The proof is detailed in [2, Proof of Lemma 1.1].

□

Lemma 1.7. *The following equality holds:*

$$\int_0^1 \frac{\ln^2(1+t^2)}{1+t^2}dt = 4\mathcal{G} - 2G\ln 2 - \frac{7\pi}{16}\zeta(2) + \frac{7\pi}{8}\ln^2 2.$$

Proof. In the recent article [5, Equation (46)], Campbell, Levrie and Nimbran have evaluated the following definite integral formula:

$$\int_0^1 \frac{\ln^2(1+t^2)}{1+t^2}dt = \pi\zeta(2) + 2\pi\ln^2(2) - 4\int_0^{\frac{\pi}{4}} \ln^2(\sin z)\, dz \tag{1.3}$$

Employing Lemma 1.6 on right side of (1.3), gives us the desired result.

□

Lemma 1.8. *The following equality holds:*

$$\int_0^1 \frac{\ln x}{\sqrt{1-x^2}} \ln\left(1+\sqrt{1-x^2}\right) dx = 4\mathcal{G} - \frac{11\pi}{16}\zeta(2) + \frac{3\pi}{8}\ln^2 2.$$

Proof. We make the substitution: $x = \cos v$ so that $dx = -\sin v \; dv$

Then, $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{1-x^2}} \ln\left(1+\sqrt{1-x^2}\right) dx = \int_0^{\frac{\pi}{2}} \ln\cos v \ln(1+\sin v)\, dv$

In this equation, due to symmetry, recognizing that,

$\displaystyle\int_0^{\frac{\pi}{2}} \ln\cos v \ln(1+\sin v)\, dv = \int_0^{\frac{\pi}{2}} \ln\sin v \ln(1+\cos v)\, dv$, we conclude:

$$\int_0^1 \frac{\ln x}{\sqrt{1-x^2}} \ln\left(1+\sqrt{1-x^2}\right) dx = \int_0^{\frac{\pi}{2}} \ln\sin v \ln(1+\cos v)\, dv$$

Now, in the definite integral on the right side, we make the Weierstrass substitution: $\tan\frac{v}{2} = t$, so that $\frac{dt}{dv} = \frac{1+t^2}{2}$. It follows that,

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \ln\sin v \ln(1+\cos v)\, dv &= 2\ln^2(2)\int_0^1 \frac{1}{1+t^2}dt \\
&+2\ln(2)\int_0^1 \frac{\ln t}{1+t^2}dt \\
&-2\int_0^1 \frac{\ln(t)\ln\left(1+t^2\right)}{1+t^2}dt \\
&-4\ln(2)\int_0^1 \frac{\ln\left(1+t^2\right)}{1+t^2}dt \\
&+2\int_0^1 \frac{\ln^2\left(1+t^2\right)}{1+t^2}dt \\
&= \frac{\pi}{4}\ln^2 2 - 2G\ln 2 - \frac{7\pi}{8}\zeta(2) + 8\mathcal{G} \\
&-2\int_0^1 \frac{\ln(t)\ln\left(1+t^2\right)}{1+t^2}dt
\end{aligned}$$

In the recent article [7, p. 69], Sofo had evaluated the following definite integral formula:

$$\int_0^1 \frac{\ln(t)\ln\left(1+t^2\right)}{1+t^2}dt = 2\mathcal{G} - G\ln 2 - \frac{3\pi}{32}\zeta(2) - \frac{\pi}{16}\ln^2 2.$$

giving us that the following equality holds:

$$\int_0^1 \frac{\ln x}{\sqrt{1-x^2}}\ln\left(1+\sqrt{1-x^2}\right)dx = \frac{\pi}{4}\ln^2 2 - 2G\ln 2 - \frac{7\pi}{8}\zeta(2) + 8\mathcal{G} - 4\mathcal{G}$$
$$-2\left(-G\ln 2 - \frac{3\pi}{32}\zeta(2) - \frac{\pi}{16}\ln^2 2\right)$$
$$= 4\mathcal{G} - \frac{11\pi}{16}\zeta(2) + \frac{3\pi}{8}\ln^2 2.$$

□

Lemma 1.9. *The following equality holds:*

$$\int_0^1 \frac{\ln x}{\sqrt{1-x^2}}dx = -\frac{\pi}{2}\ln 2.$$

Proof. We make the substitution: $x = \sin q$ so that $dx = \cos q\ dq$

Then, $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{1-x^2}}dx = \int_0^{\frac{\pi}{2}} \ln(\sin q)\, dq.$

In this expression, due to symmetry, we have,

$$\int_0^{\frac{\pi}{2}} \ln(\sin q)\, dq = \int_0^{\frac{\pi}{2}} \ln(\cos q)\, dq.$$

Using the derivative of the Euler's beta function and the Leibniz formula for the differentiation of products, K.S.Kölbig [6, p. 25], had evaluated the definite integral

$$\int_0^{\frac{\pi}{2}} \ln(\cos q)\, dq = -\frac{\pi}{2}\ln 2.$$

giving us that the following equality holds:

$$\int_0^1 \frac{\ln x}{\sqrt{1-x^2}} dx = -\frac{\pi}{2} \ln 2.$$

□

Lemma 1.10. *The following equality holds:*

$$\int_0^1 \frac{\ln x}{\sqrt{1-x^2}} \ln\left(\sqrt{1-x^2}\right) dx = \frac{\pi}{8}\left(-\zeta(2) + 4\ln^2 2\right).$$

Proof. We make the substitution: $x = \sin u'$ so that $dx = \cos u' \, du'$

Then, $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{1-x^2}} \ln\left(\sqrt{1-x^2}\right) dx = \int_0^{\frac{\pi}{2}} \ln\left(\sin u'\right) \ln\left(\cos u'\right) du'.$

Using the derivative of the Euler's beta function and the Leibniz formula for the differentiation of products, K.S.Kölbig [6, p. 25], had evaluated the definite integral

$$\int_0^{\frac{\pi}{2}} \ln\left(\sin u'\right) \ln\left(\cos u'\right) du' = \frac{\pi}{8}\left(-\zeta(2) + 4\ln^2 2\right).$$

giving us that the following equality holds:

$$\int_0^1 \frac{\ln x}{\sqrt{1-x^2}} \ln\left(\sqrt{1-x^2}\right) dx = \frac{\pi}{8}\left(-\zeta(2) + 4\ln^2 2\right).$$

□

Theorem 1.11. *The following identity holds:*

$$\sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{(2n+1)^2 \, 2^{2n}} = -8\mathcal{G} - \frac{3\pi}{4}\ln^2 2 + \frac{9\pi}{8}\zeta(2).$$

Proof.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{(2n+1)^2 \, 2^{2n}} &= \sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{2^{2n}} \left(\frac{1}{(2n+1)^2}\right) \\
&= \sum_{n=1}^{\infty} \frac{H_n \binom{2n}{n}}{2^{2n}} \left(-\int_0^1 x^{2n} \ln(x) \, dx\right) \\
&= -\int_0^1 \ln(x) \left(\sum_{n=1}^{\infty} H_n \binom{2n}{n} \left(\frac{x^2}{4}\right)^n\right) dx \\
&= -2\int_0^1 \left(\frac{\ln(x)}{\sqrt{1-x^2}} \ln\left(\frac{1+\sqrt{1-x^2}}{2\sqrt{1-x^2}}\right)\right) dx
\end{aligned}$$

$$= -2\int_0^1 \frac{\ln x}{\sqrt{1-x^2}} \ln\left(1+\sqrt{1-x^2}\right) dx$$
$$+2\ln 2 \int_0^1 \frac{\ln x}{\sqrt{1-x^2}} dx$$
$$+2\int_0^1 \frac{\ln x}{\sqrt{1-x^2}} \ln\left(\sqrt{1-x^2}\right) dx$$
$$= -2\left(4\mathcal{G} - \frac{11\pi}{16}\zeta(2) + \frac{3\pi}{8}\ln^2 2 + \frac{\pi}{8}\zeta(2)\right)$$
$$= -8\mathcal{G} - \frac{3\pi}{4}\ln^2 2 + \frac{9\pi}{8}\zeta(2),$$

and the theorem is proved.

Here in the proofs of Lemma 1.3 and Theorem 1.11, Bernstein's theorem [8, Thm. 9.30, p. 243] justifies interchanging the order of integration and summation because of the positivity of the coefficients.

□

Acknowledgement: I am indebted to the anonymous referee for carefully reading the manuscript and for a thorough, precise and helpful report which helped me to correct a mistake in the statement of Lemma 1.1.

References

[1] Boyadzhiev, K. N., Series with Central Binomial Coefficients, Catalan Numbers, and Harmonic Numbers, *J. Integer Seq.* **15**(1) (2012), Article 12.1.7, 11pp. MR: 2872464. Zbl: 1291.11046.

[2] Dasireddy, N. S., Solution to a problem involving central binomial coefficients, *Integral Transforms Spec. Funct.*, to appear (2024). doi: 10.1080/10652469.2024.2339854.

[3] Dasireddy, N. S., A Solution To An Interesting Sum Involving Classical Harmonic Number and Central Binomial Coefficient, *Math. Student* **93**(1-2) (2024), 132-137. https://indianmathsoc.org/pdf/Mathstudent-V93-part1-2-2024.pdf.

[4] Nimbran, A. S., Sums of series involving central binomial coefficients and harmonic numbers, *Math. Student* **88**(1-2) (2019), 125-135. https://indianmathsoc.org/ms/mathstudent-part-1-2019.pdf. Zbl: 07844181.

[5] Campbell, J. M., Levrie, P., and Nimbran, A. S., A natural companion to Catalan's constant, *J. Class. Anal.* **18**(2) (2021), 117-135. doi: 10.7153/jca-2021-18-09. MR: 4364726. Zbl: 1499.33016.

[6] Kölbig, K. S., On the value of a logarithmic-trigonometric integral, *BIT* **11** (1971), 21-28. doi: 10.1007/BF01935325. Zbl: 0216.48504.

[7] Sofo, A., A family of definite integrals, *Sci. Ser. A Math. Sci. (N.S.)* **31** (2021), 61-74. https://scientia.mat.utfsm.cl/archivos/vol31/5.pdf. Zbl: 1497.11209.

[8] Apostol, T. M., Mathematical Analysis, 2nd edn., Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1974. MR: 0344384. Zbl: 0309.26002.

Nandan Sai Dasireddy
House No. 3-11-363, Pavani Plaza, Flat No:401,
Road No. 2, Shivaganga Colony, Siris Road,
L.B. Nagar, Hyderabad, Telangana, India.
E-mail: dasireddy.1818@gmail.com

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July-December (2024), 115–133

DYNAMICS OF TWO BY TWO SYMMETRIC MATRICES OF TRACE ZERO

ARIJIT MUKHERJEE

(Received : 10 - 12 - 2023 ; Revised : 11 - 03 - 2024)

Abstract. In this paper, we describe the entire structure of the vector space Sym_2^0 of all symmetric matrices of size 2 having trace zero. This is motivated by the geometrical interpretation of any arbitrary element of Sym_2^0. We further study the orbits and stable sets of these elements. As an application of the obtained structure of Sym_2^0, we obtain the symmetric matrices of size 2, trace of whose product with any trace zero symmetric matrix is zero. Finally some well known trigonometric formulas are interpreted geometrically incorporating the anatomy of Sym_2^0.

1. Introduction

The study of symmetric matrices and trace zero matrices attracted considerable attention. Many mathematicians have researched on symmetric matrices to study SNIEP, symmetric non-negative inverse eigenvalue problem (cf. [2],[5],[6] & [7]). Also people have independently worked on trace zero matrices and found necessary and sufficient conditions for a matrix to have zero trace (cf. [1] & [8]). A study of SNIEP for trace zero symmetric matrices can be found in [9].

This paper is devoted to the study of trace zero symmetric matrices of size 2 and its applications. But we do it in a different context and therefore follow a different approach altogether.

We begin by fixing some notations which we are going to use repeatedly. Let $\mathbb{R}$ be the field of real numbers. In this paper, by a vector space we always mean a vector space over $\mathbb{R}$ and by a matrix we mean a matrix with real entries. Let Sym_n & $\mathcal{O}_n$ be the set of all symmetric matrices and

2010 Mathematics Subject Classification: 15A63, 15B10, 97G60, 97H60
Key words and phrases: Orthogonal matrix, symmetric matrix, trace, inner product, orbit, trigonometry

orthogonal matrices of size n respectively and Sym_n^0 be the subset of Sym_n consisting of all matrices having trace zero. By PSym_n^0 we denote the set of all $n \times n$ symmetric matrices, trace of whose product with any element of Sym_n^0 is zero. We reserve the notations $\mathbb{I}_n$ and $\mathbb{O}_n$ for identity matrix and zero matrix of size n respectively. We denote the trace of a given matrix A by $\mathrm{Tr}(A)$ and determinant of A by $\det(A)$. Given any two matrices A and B of size n, we denote the matrix multiplication $A \cdot B$ simply by their juxtaposition AB.

In this paper, our main aim is to describe the precise structure of Sym_2^0. In Theorem 2.4, we show that the elements of Sym_2^0 are precisely of the form $\left(\begin{smallmatrix} \lambda\cos(\theta) & \lambda\sin(\theta) \\ \lambda\sin(\theta) & -\lambda\cos(\theta) \end{smallmatrix}\right)$ for some $\lambda \in \mathbb{R}, \theta \in [0, 2\pi)$. Using the structure of Sym_2^0, we further show that the set of all trace zero symmetric matrices of size 2 having eigenvalues 1 and -1 is same as the set of all size 2 orthogonal matrices having determinant -1, (cf. Corollary 2.5).

We come up with the geometrical interpretation of the elements of Sym_2^0 as well. In fact, this is the motivation behind finding the anatomy of Sym_2^0. Moreover, we extensively discuss about the orbits, raise the questions about the finiteness of those orbits and answer those using the obtained structure of Sym_2^0. We obtain a necessary and sufficient condition for finiteness of the orbit $\mathrm{Or}_{(a,b)}(T_\theta^\lambda)$ of an arbitrary element T_θ^λ of Sym_2^0 starting at a point (a, b) (cf. Theorem 2.7).

We further study the dynamics of T_θ^λ and show how the stable set $\mathrm{Stab}_{T_\theta^\lambda}((a, b))$ of any point (a, b) with respect to T_θ^λ varies as λ varies. In the process, we obtain that the stable set $\mathrm{Stab}_{T_\theta^\lambda}((a, b))$ either contains only (a, b) or is the whole of $\mathbb{R}^2$ depending on whether λ lies in the open interval $(1, 1)$ or not, (cf. Theorem 2.10).

In section 3, we look upon a couple of applications of the structure of Sym_2^0. Firstly, we derive the structure of PSym_2^0 in subsection 3.1. To be precise, we show that PSym_2^0 consists of scalar matrices and scalar matrices only, (cf. Theorem 3.1). We then prove that the obtained anatomy of PSym_2^0 can be generalised for any $n \geq 2$ using Frobenius inner product on Sym_n (cf. Theorem 3.2).

As another application, we talk about two rigid motions, namely rotation and reflection, of any point of the Euclidean plane and show that rotating the point of reflection of a given point with respect to a line is same as reflecting it with respect to some other line, (cf. Theorem 3.3).

Though this can be proved using simple techniques of Euclidean geometry, but we do it incorporating the structure of Sym_2^0 and as a result the elucidation seems to be an elegant one. The given proof can also be thought of as a geometric interpretation of couple of well known and frequently used trigonometric formulas (cf. Remark 3.4).

In Section 4, we conclude by indicating that the structure of some subsets of Sym_n^0 can be obtained for $n > 2$ as well by adapting the method used in Theorem 2.4, provided some conditions being suitably put on the set of eigenvalues of its elements.

2. On the structure of Sym_2^0 and orbits of its elements

In this section, we provide the structure of Sym_2^0, analyse its elements from a geometric viewpoint and then discuss upon the orbits and the stable sets of those elements.

For any given real number θ, denote the line in $\mathbb{R}^2$ passing through origin and making an angle θ with the positive direction of x-axis in the anticlockwise direction by L_θ. Given any $\lambda \in \mathbb{R}$, define a map (geometrically), denoted by T_θ^λ, as follows : Given any point (x_1, x_2), the map T_θ^λ first reflects the point (x_1, x_2) with respect to the line L_θ and scales that by λ followed by that. Denoting the reflection of (x_1, x_2) with respect to the line L_θ by $R_\theta((x_1, x_2))$, the map T_θ^λ can be given as follows:

$$\begin{aligned} T_\theta^\lambda &: \mathbb{R}^2 \to \mathbb{R}^2 \\ (x_1, x_2) &\mapsto \lambda R_\theta((x_1, x_2)). \end{aligned} \tag{2.1}$$

Define the orbit $\mathrm{Or}_{(a,b)}(T_\theta^\lambda)$ of the map T_θ^λ starting at a point (a, b) as follows:

$$\mathrm{Or}_{(a,b)}(T_\theta^\lambda) := \{(T_\theta^\lambda)^n((a, b)) \mid n \in \mathbb{N}\}.$$

We now ask the following questions:

Question 2.1. (1) Given any $(a, b) \in \mathbb{R}^2$, for what values of θ and λ, the orbit $\mathrm{Or}_{(a,b)}(T_\theta^\lambda)$ is finite?

(2) For what values of θ and λ, the orbit $\mathrm{Or}_{(a,b)}(T_\theta^\lambda)$ is a singleton set, where $(a, b) \in \mathbb{R}^2$?

(3) Given any $(a, b) \in \mathbb{R}^2$, for what values of θ and λ, the orbit $\mathrm{Or}_{(a,b)}(T_\theta^\lambda)$ is a set having two points?

(4) For what values of θ and λ, $(T_\theta^\lambda)^n = \mathbb{I}_2$ for some positive integer n?

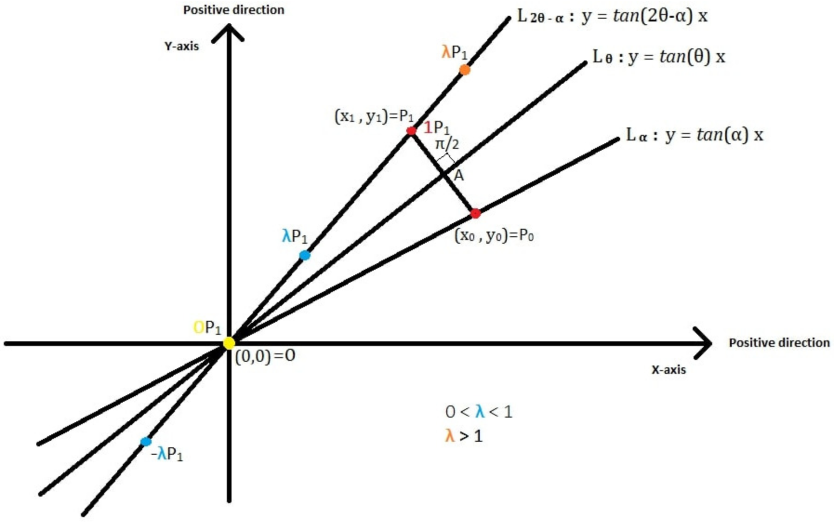


FIGURE 1. Image of a point P_0 under the map λR_θ for some values of λ

(5) Given any $(a, b) \in \mathbb{R}^2$, the sequence $\{(T_\theta^\lambda)^n((a, b))\}_{n\geq 0}$ is convergent in usual topology and in discrete topology for what values of θ and λ?

To answer these questions, we first calculate $T_\theta^\lambda(P_0)$ for any given point $P_0 \in \mathbb{R}^2$. We do the calculation assuming that both the coordinates of the point $P_0 = (x_0, y_0)$ are positive and the line joining P_0 and origin makes an angle α with the positive direction of x-axis. That is to say, P_0 lies in the line L_α. Moreover, we assume that $0 < \alpha \leq \theta \leq 2\theta - \alpha < \frac{\pi}{2}$, that is to say the lines L_α, L_θ and $L_{2\theta-\alpha}$ are having slopes in non-decreasing order, lies in first and third quadrant and none of those are x-axis or y-axis. The calculations are very much similar in the remaining cases as well.

As we mentioned, we start with a point $P_0 = (x_0, y_0)$ lying in the line L_α. We first want to determine the reflection of P_0 with respect to the line L_θ. For that we draw a perpendicular from the point P_0 to the line L_θ and denote the point of intersection of this with L_θ by A. Then we extend the line segment $\overline{P_0A}$ and suppose that intersects the line $L_{2\theta-\alpha}$ at the point $P_1 = (x_1, y_1)$. Then clearly the point P_1 is the reflection of P_0. Now we determine x_1 and y_1 in terms of the known quantities x_0, y_0, θ and α using the properties of reflection.

The equations of the line L_θ, L_α and $L_{2\theta-\alpha}$ are given as follows:

$$L_\theta : y = \tan(\theta)\ x,\ L_\alpha : y = \tan(\alpha)\ x,\ L_{2\theta-\alpha} : y = \tan(2\theta-\alpha)\ x.$$

By properties of reflection, we get that the triangle AP_0O and AP_1O are congruent to each other. That is to say, $\triangle AP_0O \cong \triangle AP_1O$. Therefore, the length $\mathcal{L}(\overline{OP_0})$ of the line segment $\overline{OP_0}$ is equal to that of $\overline{OP_1}$, that is to say, $\mathcal{L}(\overline{OP_0}) = \mathcal{L}(\overline{OP_1})$. We then have the following implications.

$$\begin{aligned}\mathcal{L}(\overline{OP_0}) = \mathcal{L}(\overline{OP_1}) &\Rightarrow \sqrt{x_0^2 + y_0^2} = \sqrt{x_1^2 + y_1^2}\\ &\Rightarrow \sqrt{x_0^2 + x_0^2\tan^2(\alpha)} = \sqrt{x_1^2 + x_1^2\tan^2(2\theta-\alpha)}\\ &\Rightarrow x_0\sec(\alpha) = x_1\sec(2\theta-\alpha) \Rightarrow x_1 = x_0\sec(\alpha)\cos(2\theta-\alpha).\end{aligned}$$

So the abscissa x_1 of the point of reflection P_1 of P_0 is given as follows:

$$x_1 = x_0\sec(\alpha)\cos(2\theta-\alpha). \tag{2.2}$$

As (x_1, y_1) lies in the line $L_{2\theta-\alpha}$, we have $y_1 = \tan(2\theta-\alpha)\ x_1$. We now have the following implications:

$$\begin{aligned}y_1 = \tan(2\theta-\alpha)\ x_1 &\Rightarrow y_1 = x_0\sec(\alpha)\cos(2\theta-\alpha)\tan(2\theta-\alpha)\ \text{(by (2.2))}\\ &\Rightarrow y_1 = x_0\sec(\alpha)\sin(2\theta-\alpha)\\ &\Rightarrow y_1 = y_0\,\mathrm{cosec}(\alpha)\sin(2\theta-\alpha)\ \text{(as } (x_0,y_0) \text{ lies in } L_\alpha).\end{aligned}$$

So the ordinate y_1 of the point of reflection P_1 of P_0 is given as follows:

$$y_1 = y_0\,\mathrm{cosec}(\alpha)\sin(2\theta-\alpha). \tag{2.3}$$

Remark 2.2. Taking $\theta = \alpha$ in equations (2.2) and (2.3), we have $P_0 = (x_0, y_0) = (x_1, y_1) = P_1$. On the other hand, if $\theta \neq \alpha$ and $P_0 \neq (0,0)$, then $P_0 \neq P_1$ as $P_0 \in L_\alpha$ and $P_1 \in L_{2\theta-\alpha}$. This can be justified also using (2.2) and (2.3). This depicts the fact that reflection of any point located on the axis of the reflection is the point itself. Moreover, this is true exclusively for points located on the line of reflection.

So we have obtained the coordinates of the point of reflection of (x_0, y_0) about the line L_θ. For the moment, we take a break from the discussion about the map T_θ^λ. We continue with the same after a while and answer the questions raised.

We now provide the description of Sym_2^0. Towards that, we have the following proposition which says about the structure of the set $\mathcal{O}_2$ of orthogonal matrices of size 2. Though this is a standard result (for example, see [3, p. 348] for determinant 1 orthogonal matrices), we include this over here for the sake of continuity.

Proposition 2.3. *The collection $\mathcal{O}_2$ of all orthogonal matrices of size 2 is given as follows:*

$$\mathcal{O}_2 = \{ \left(\begin{smallmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{smallmatrix}\right) \mid \alpha \in [0, 2\pi) \} \cup \{ \left(\begin{smallmatrix} \cos(\beta) & \sin(\beta) \\ \sin(\beta) & -\cos(\beta) \end{smallmatrix}\right) \mid \beta \in [0, 2\pi) \}.$$

Proof. Given $x = (x_1, x_2)$ and $y = (y_1, y_2)$, by $\langle x, y \rangle$ we denote the standard inner product of x and y in $\mathbb{R}^2$. That is to say, $\langle x, y \rangle = x_1 y_1 + x_2 y_2$. Let $M = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ be a given orthogonal matrix. As the columns of M are of unit norm with respect to the standard inner product of $\mathbb{R}^2$, we have

$$p^2 + r^2 = 1, q^2 + s^2 = 1. \tag{2.4}$$

Then (2.4) in turn imply that

$$\mid p \mid \leq 1, \ \mid r \mid \leq 1, \ \mid q \mid \leq 1, \ \mid s \mid \leq 1. \tag{2.5}$$

As the columns of M are orthogonal with respect to the standard inner product of $\mathbb{R}^2$, we have

$$pq + rs = 0. \tag{2.6}$$

Case - 1:
If $\det(A) = 1$, then we moreover have

$$ps - rq = 1. \tag{2.7}$$

So, conditions (2.5), (2.6) and (2.7) imply that, there exists $\alpha \in [0, 2\pi)$ such that

$$M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}. \tag{2.8}$$

Case - 2:
If $\det(A) = -1$, then we moreover have

$$ps - rq = -1. \tag{2.9}$$

So, conditions (2.5), (2.6) and (2.9) imply that, there exists $\beta \in [0, 2\pi)$ such that

$$M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} \cos(\beta) & \sin(\beta) \\ \sin(\beta) & -\cos(\beta) \end{pmatrix}. \tag{2.10}$$

Therefore, we have the result from (2.8) and (2.10). □

Following theorem describes the entire structure of Sym_2^0.

Theorem 2.4. *The collection* Sym_2^0 *of all trace zero symmetric matrices of size 2 is given as follows:*

$$\mathrm{Sym}_2^0 = \{\left(\begin{smallmatrix} \lambda\cos(\theta) & \lambda\sin(\theta) \\ \lambda\sin(\theta) & -\lambda\cos(\theta)\end{smallmatrix}\right) \mid \lambda\in\mathbb{R}, \theta\in[0,2\pi)\}.$$

Proof. By spectral theorem, we have that any real symmetric matrix is orthogonally diagonalisable and vice versa (cf. [3, p.347]). That is, for $n=2$, given any symmetric matrix $A = \left(\begin{smallmatrix} a & b \\ b & c\end{smallmatrix}\right)$, there exists an orthogonal matrix A_O such that

$$A_O\left(\begin{smallmatrix} a & b \\ b & c\end{smallmatrix}\right){A_O}^{-1} = \left(\begin{smallmatrix} \gamma & 0 \\ 0 & \delta\end{smallmatrix}\right). \tag{2.11}$$

Moreover, if $A\in\mathrm{Sym}_2^0$ then sum of eigenvalues of A is zero, that is, $\gamma+\delta=0$.

Case - 1:

If $\det(A_O)=1$, then by Proposition 2.3 there exists $\alpha\in[0,2\pi)$ such that $A_O = \left(\begin{smallmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha)\end{smallmatrix}\right)$. So, we also have ${A_O}^{-1} = \left(\begin{smallmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha)\end{smallmatrix}\right)$. Therefore, from (2.11) we have the following:

$$\begin{aligned} A = \begin{pmatrix} a & b \\ b & c\end{pmatrix} &= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha)\end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 0 & -\gamma\end{pmatrix}\begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha)\end{pmatrix} \\ &= \begin{pmatrix} \gamma\cos^2(\alpha)-\gamma\sin^2(\alpha) & 2\gamma\cos(\alpha)\sin(\alpha) \\ 2\gamma\cos(\alpha)\sin(\alpha) & \gamma\sin^2(\alpha)-\gamma\cos^2(\alpha)\end{pmatrix} \\ &= \begin{pmatrix} \gamma\cos(2\alpha) & \gamma\sin(2\alpha) \\ \gamma\sin(2\alpha) & -\gamma\cos(2\alpha)\end{pmatrix}. \end{aligned}$$

Case - 2:

If $\det(A_O)=-1$, then by Proposition 2.3 there exists $\beta\in[0,2\pi)$ such that $A_O = \left(\begin{smallmatrix} \cos(\beta) & \sin(\beta) \\ \sin(\beta) & -\cos(\beta)\end{smallmatrix}\right)$. So, we also have ${A_O}^{-1} = A_O = \left(\begin{smallmatrix} \cos(\beta) & \sin(\beta) \\ \sin(\beta) & -\cos(\beta)\end{smallmatrix}\right)$. Therefore, from (2.11) we have the following:

$$\begin{aligned} A = \begin{pmatrix} a & b \\ b & c\end{pmatrix} &= \begin{pmatrix} \cos(\beta) & \sin(\beta) \\ \sin(\beta) & -\cos(\beta)\end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 0 & -\gamma\end{pmatrix}\begin{pmatrix} \cos(\beta) & \sin(\beta) \\ \sin(\beta) & -\cos(\beta)\end{pmatrix} \\ &= \begin{pmatrix} \gamma\cos^2(\beta)-\gamma\sin^2(\beta) & 2\gamma\cos(\beta)\sin(\beta) \\ 2\gamma\cos(\beta)\sin(\beta) & \gamma\sin^2(\beta)-\gamma\cos^2(\beta)\end{pmatrix} \\ &= \begin{pmatrix} \gamma\cos(2\beta) & \gamma\sin(2\beta) \\ \gamma\sin(2\beta) & -\gamma\cos(2\beta)\end{pmatrix}. \end{aligned}$$

By taking $\theta=2\alpha$ if $\alpha<\pi$ & $\theta=2(\alpha-\pi)$ if $\alpha\geq\pi$ in Case - 1 and $\theta=2\beta$ if $\beta<\pi$ & $\theta=2(\beta-\pi)$ if $\beta\geq\pi$ in Case - 2, we have the theorem. ☐

We get the following obvious conclusion relating orthogonal matrices of size 2 having determinant -1 and a subset of Sym_2^0.

Corollary 2.5. *Any 2×2 orthogonal matrix A of determinant -1 is a symmetric matrix of trace zero. Moreover, given any 2×2 symmetric matrix A of trace zero, there exists a 2×2 orthogonal matrix B of determinant -1 and a scalar λ such that $A = \lambda B$. Furthermore, the set of all trace zero symmetric matrices of size 2 having eigenvalues 1 and -1 is same as the set of all size 2 orthogonal matrices having determinant -1.*

Proof. Follows directly from Proposition 2.3 and Theorem 2.4. □

We now resume the discussion about the map T_θ^λ. We did some calculations and obtained the coordinates of the point of a given point with respect to the line L_θ. Look at the same calculations from a different point of view. Consider the matrix $\mathcal{R}_\theta = \left(\begin{smallmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{smallmatrix}\right)$. Then

$$\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \cos(2\theta) + y_0 \sin(2\theta) \\ x_0 \sin(2\theta) - y_0 \cos(2\theta) \end{pmatrix}. \tag{2.12}$$

Now,

$$\begin{aligned} &x_0 \cos(2\theta) + y_0 \sin(2\theta) \\ &\qquad = x_0 \cos(2\theta) + x_0 \tan(\alpha)\sin(2\theta) (\text{as } (x_0, y_0) \text{ lies on } L_\alpha) \\ &\qquad = x_0 \sec(\alpha)(\cos(\alpha)\cos(2\theta) + \sin(\alpha)\sin(2\theta)) \\ &\qquad = x_0 \sec(\alpha)\cos(2\theta - \alpha) = x_1 \ \ (\text{by } (2.2)). \end{aligned} \tag{2.13}$$

Similarly, we have

$$\begin{aligned} &x_0 \sin(2\theta) - y_0 \cos(2\theta) \\ &\qquad = y_0 \cot(\alpha)\sin(2\theta) - y_0 \cos(2\theta) (\text{as } (x_0, y_0) \text{ lies on } L_\alpha) \\ &\qquad = y_o \operatorname{cosec}(\alpha)(\cos(\alpha)\sin(2\theta) - \cos(2\theta)\sin(\alpha)) \\ &\qquad = y_o \operatorname{cosec}(\alpha)\sin(2\theta - \alpha) = y_1 \ \ (\text{by } (2.3)). \end{aligned} \tag{2.14}$$

Therefore by (2.12), (2.13) and (2.14), we have

$$\mathcal{R}_\theta \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{2.15}$$

Recall the definition of the map T_θ^λ as defined in (2.1). Now treating points of $\mathbb{R}^2$ as column vectors, from (2.15) we can conclude that

$$T_\theta^1 = R_\theta = \mathcal{R}_\theta.$$

Moreover,

$$T_\theta^\lambda = \lambda R_\theta = \lambda \mathcal{R}_\theta. \tag{2.16}$$

Thus, (2.16) provides the geometry of the elements of Sym_0^2. Also from now on we simply use the notation R to mean both R_θ and $\mathcal{R}_\theta$ if there is there is no confusion regarding the line of reflection L_θ. So, λR and T_θ^λ denote the same map and we use them interchangeably.

We now answer the questions we posed related to the map T_θ^λ. Towards that, we have the following proposition.

Proposition 2.6. *Let n be any positive integer. Then $(\lambda R)^n = \mathbb{I}_2$ if and only if n is even and $\lambda = 1$ or -1.*

Proof. It is easy to observe that $R^2 = \mathbb{I}_2$. Therefore, if n is even and $\lambda = 1$ or -1, then $(\lambda R)^n = \lambda^n R^n = \lambda^n \mathbb{I}_2 = \mathbb{I}_2$.

Conversely, let $(\lambda R)^n = \mathbb{I}_2$. Therefore,

$$R^n = \tfrac{1}{\lambda^n}\mathbb{I}_2. \tag{2.17}$$

Recall that $R = \left(\begin{smallmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{smallmatrix}\right)$. Now if n is odd, then (2.17) implies that $\cos(2\theta) = -\cos(2\theta)$, which in turn says that $\cos(2\theta) = 0$. This contradicts (2.17). Therefore if $(\lambda R)^n = \mathbb{I}_2$ then n can't be odd. Hence n is even and moreover by (2.17) $\lambda = 1$ or -1. $\square$

Now we are in a position to find some exclusive conditions which will force the orbit $\mathrm{Or}_{(a,b)}(T_\theta^\lambda)$ to be finite. Precisely, we obtain the following.

Theorem 2.7. *Let λ be any non-zero real number and let (a, b) be any point of $\mathbb{R}^2$ other than origin. Then $\mathrm{Or}_{(a,b)}(T_\theta^\lambda)$ is finite if and only if $\lambda = 1$ or -1.*

Proof. We first show that to get the finiteness of the orbit $\mathrm{Or}_{(a,b)}(T_\theta^\lambda)$ it is sufficient to have $\lambda = 1$ or -1. For that, we proceed contra positively.

$$\begin{aligned}
\mathrm{Or}_{(a,b)}(T_\theta^\lambda) \text{ is not finite} \\
&\Rightarrow (\lambda R)^n((a,b)) \neq (\lambda R)^m((a,b)) \text{ for all non-negative } m \ \&\ n \\
&\Rightarrow (\lambda R)^{n-m}((a,b)) \neq (a,b) \text{ (assuming } n > m \text{ w.l.o.g)} \\
&\Rightarrow (\lambda R)^p \neq \mathbb{I}_2 \text{ for all even } p \\
&\Rightarrow \lambda \neq 1, -1 \text{ (by Proposition 2.6).}
\end{aligned}$$

We now prove the converse part. Let $\mathrm{Or}_{(a,b)}(T_\theta^\lambda)$ is finite. Then

$$\begin{aligned}
\mathrm{Or}_{(a,b)}(T_\theta^\lambda) \text{ is finite} &\Rightarrow (\lambda R)^n((a,b)) = (\lambda R)^m((a,b)) \\
&\quad \text{for some non-negative } m \ \&\ n \\
&\Rightarrow (\lambda R)^{n-m}((a,b)) = (a,b) \text{ (assuming } n > m \text{ w.l.o.g)} \\
&\Rightarrow (\lambda R)^p((a,b)) = (a,b), \text{ where } n - m = p \text{ (say).}
\end{aligned} \tag{2.18}$$

Now if p is even, then by (2.18) we further have the following implications.

$$\begin{aligned}
\lambda^p R((a,b)) = (a,b) &\Rightarrow \lambda^p \mathbb{I}_2((a,b)) = (a,b) \\
&\Rightarrow \lambda^p = 1 \Rightarrow \lambda = 1, -1.
\end{aligned}$$

For p odd, we further consider two mutually exclusive and exhaustive cases.

First Case - $(a,b) \in L_\theta$:

By (2.18) we have the following implications.

$$\begin{aligned}
\lambda^p R((a,b)) = (a,b) &\Rightarrow \lambda^p (a,b) = (a,b) \\
&\Rightarrow \lambda^p = 1 \Rightarrow \lambda = 1.
\end{aligned}$$

Second Case - $(a,b) \notin L_\theta$:

Let $(a,b) \in L_\alpha$ for some real number α. As p is odd, by (2.18), we have that $(\lambda R)^n((a,b)) = (\lambda R)^m((a,b))$, where one of n and m is odd and other is even. Without loss of generality, take n to be odd and m to be even. Then we have $\lambda^n R((a,b)) = \lambda^m (a,b)$. Therefore, $R((a,b))$ and (a,b) are collinear, which is a contradiction to the assumption that $(a,b) \notin L_\theta$ by Remark 2.2. So, this case is not at all feasible. □

Remark 2.8. (1) Theorem 2.7 talks about a necessary and sufficient condition for finiteness of the orbit $\mathrm{Or}_{(a,b)}(T_\theta^\lambda)$ for non-zero values of λ and for points $(a,b) \neq (0,0)$. Also, $\mathrm{Or}_{(a,b)}(T_\theta^\lambda)$ is finite if either $(a,b) = (0,0)$ or $\lambda = 0$.

(2) We now check when $\mathrm{Or}_{(a,b)}(T_\theta^\lambda)$ is either singleton or consists of two elements.
   (a) If $(a,b) = (0,0)$, then $\mathrm{Or}_{(0,0)}(T_\theta^\lambda) = \{(0,0)\}$ for all θ and λ.
   (b) If $(a,b) \neq (0,0)$, then we have the following:
      (i) If $(a,b) \in L_\theta$, then $\mathrm{Or}_{(a,b)}(T_\theta^1) = \{(a,b)\}$.
      (ii) If $(a,b) \notin L_\theta$, then $\mathrm{Or}_{(a,b)}(T_\theta^1) = \{(a,b), T_\theta^1(a,b)\}$ and $\mathrm{Or}_{(a,b)}(T_\theta^0) = \{(a,b), (0,0)\}$.
      (iii) $\mathrm{Or}_{(a,b)}(T_\theta^{-1}) = \{(a,b), T_\theta^{-1}(a,b)\}$.

   So, we discussed a few cases where the orbit $\mathrm{Or}_{(a,b)}(T_\theta^\lambda)$ is either singleton or consists of two elements only. It can be easily checked that these are all possible such orbits.

We now look at the sequence $\{(T_\theta^\lambda)^n((a,b))\}_{n\geq 0}$. We want to find some suitable $(a,b) \in \mathbb{R}^2$, θ and λ such that the sequence $\{(T_\theta^\lambda)^n((a,b))\}_{n\geq 0}$ is eventually constant. At this point, we note that whenever the sequence $\{(T_\theta^\lambda)^n((a,b))\}_{n\geq 0}$ is eventually constant, the orbit $\mathrm{Or}_{(a,b)}(T_\theta^\lambda)$ must be finite. Therefore, λ can have only three values, namely 1, -1 and 0 unless we take $(a,b) = (0,0)$ (cf. Theorem 2.7 and Remark 2.8). If we take $(a,b) = (0,0)$, then for any λ and θ, $\{(T_\theta^\lambda)^n((0,0))\}_{n\geq 0}$ is a constant sequence having all the terms equal to $(0,0)$. So look at points other than origin. Let $(a,b)(\neq (0,0))$ lies in the line L_α for some $\alpha \in \mathbb{R}$. Then clearly the sequence $\{(T_\theta^0)^n((a,b))\}_{n\geq 0}$ is eventually constant. For $\lambda = 1$, $\{(T_\theta^1)^n((a,b))\}_{n\geq 0}$ is eventually constant only when $\theta = \alpha$. Finally $\lambda = -1$ provides us eventually constant sequences only in the most simplest case, that is, when $(a,b) = (0,0)$. As in the discrete topology on $\mathbb{R}^2$, only convergent sequences are the eventually constant ones, therefore the sequence $\{(T_\theta^\lambda)^n((a,b))\}_{n\geq 0}$ is convergent in discrete topology in the following cases. The point of convergence is also given accordingly.

(1) When $(a,b) = (0,0)$, the sequence $\{(T_\theta^\lambda)^n((0,0))\}_{n\geq 0}$ converges to $(0,0)$ in discrete topology, for any λ and θ.
(2) When $\lambda = 0$, the sequence $\{(T_\theta^0)^n((a,b))\}_{n\geq 0}$ converges to $(0,0)$ in discrete topology, for any (a,b) and θ.
(3) When $\lambda = 1$, the sequence $\{(T_\theta^1)^n((a,b))\}_{n\geq 0}$ converges to (a,b) in discrete topology for any θ if $(a,b) \in L_\theta$.

As discrete topology is finer than usual topology on $\mathbb{R}^2$, the mentioned sequences are convergent in usual topology as well. We now find out whether there are some more sequences $\{(T_\theta^\lambda)^n((a,b))\}_{n\geq 0}$ that are convergent in usual topology. As the metric induced by the usual topology on $\mathbb{R}^2$ is a complete metric, to find out the convergent sequences it is enough to find the Cauchy sequences over there.

We observe that for any $(x_0, y_0) \in \mathbb{R}^2$, if for some n, $(T_\theta^\lambda)^n((x_0,y_0)) \in L_\alpha$, then $(T_\theta^\lambda)^{n+1}((x_0,y_0)) \in L_{2\theta-\alpha}$. As the lines L_α and $L_{2\theta-\alpha}$ intersect only at origin, the distance between any two consecutive terms of the sequence $\{(T_\theta^\lambda)^n((x_0,y_0))\}_{n\geq 0}$ can be made arbitrarily small, that is, the sequence $\{(T_\theta^\lambda)^n((x_0,y_0))\}_{n\geq 0}$ can be made to be a Cauchy sequence, only when the terms of the sequence approach origin. So we conclude that the only possible point of convergence of the sequence $\{(T_\theta^\lambda)^n((x_0,y_0))\}_{n\geq 0}$, with infinitely many distinct terms, is $(0,0)$. We claim that this only possibility can be attained by the sequence $\{(T_\theta^\lambda)^n((x_0,y_0))\}_{n\geq 0}$ for all non-zero λ with $|\lambda| < 1$.

Recall that $(\lambda R)^n = \lambda^n \mathbb{I}_2$ when n is even and $(\lambda R)^n = \lambda^n R$ when n is odd. Using this we calculate some upper bounds of the distances between terms of the sequence $\{(T_\theta^\lambda)^n((x_0,y_0))\}_{n\geq 0}$. Here by distance between any two points x and y of $\mathbb{R}^2$, we mean the Euclidean distance between them and denote that by $d(x,y)$.

Case 1 : When both n and m are even

$$\begin{aligned} d((T_\theta^\lambda)^n((x_0,y_0)),(T_\theta^\lambda)^m((x_0,y_0))) &= d((\lambda R)^n, (\lambda R)^m) \\ &= d((\lambda^n x_0, \lambda^n y_0), (\lambda^m x_0, \lambda^m y_0)) \\ &\leq d((\lambda^n x_0, \lambda^n y_0), (0,0)) + d((0,0), (\lambda^m x_0, \lambda^m y_0)) \\ &= (|\lambda|^n + |\lambda|^m)\sqrt{x_0^2 + y_0^2}. \end{aligned} \tag{2.19}$$

Case 2 : When both n and m are odd

$$
\begin{aligned}
d((T_\theta{}^\lambda)^n((x_0,y_0)),(T_\theta^\lambda)^m((x_0,y_0))) &= d((\lambda R)^n,(\lambda R)^m) \\
&= d((\lambda^n(x_0\cos(2\theta)+y_0\sin(2\theta)),\lambda^n(x_0\sin(2\theta)-y_0\cos(2\theta))), \\
&(\lambda^m(x_0\cos(2\theta)+y_0\sin(2\theta)),\lambda^m(x_0\sin(2\theta)-y_0\cos(2\theta)))) \\
&\le d((\lambda^n(x_0\cos(2\theta)+y_0\sin(2\theta)),\lambda^n(x_0\sin(2\theta)-y_0\cos(2\theta))),(0,0)) \\
&+ d((0,0),(\lambda^m(x_0\cos(2\theta)+y_0\sin(2\theta)),\lambda^m(x_0\sin(2\theta)-y_0\cos(2\theta)))) \\
&= (|\lambda|^n+|\lambda|^m)\sqrt{x_0^2+y_0^2}.
\end{aligned} \tag{2.20}
$$

Case 3 : When n is even and m is odd

$$
\begin{aligned}
&d((T_\theta^\lambda)^n((x_0,y_0)),(T_\theta^\lambda)^m((x_0,y_0))) = d((\lambda R)^n,(\lambda R)^m) \\
&= d((\lambda^n x_0,\lambda^n y_0),(\lambda^m(x_0\cos(2\theta)+y_0\sin(2\theta)),\lambda^m(x_0\sin(2\theta)-y_0\cos(2\theta)))) \\
&\le d((\lambda^n x_0,\lambda^n y_0),(0,0)) \\
&+ d((0,0),(\lambda^m(x_0\cos(2\theta)+y_0\sin(2\theta)),\lambda^m(x_0\sin(2\theta)-y_0\cos(2\theta)))) \\
&= (|\lambda|^n+|\lambda|^m)\sqrt{x_0^2+y_0^2}.
\end{aligned} \tag{2.21}
$$

So given any $\epsilon > 0$, by (2.19), (2.20) and (2.21), we can choose $n_0 \in \mathbb{N}$ such that for all $m, n \ge n_0$, $d((T_\theta^\lambda)^n((x_0,y_0)),(T_\theta^\lambda)^m((x_0,y_0))) \le (|\lambda|^n + |\lambda|^m)\sqrt{x_0^2+y_0^2} < \epsilon$ whenever $|\lambda| < 1$. Therefore, $\{(T_\theta^\lambda)^n((x_0,y_0))\}_{n\ge 0}$ is a Cauchy sequence for all λ with $|\lambda| < 1$ and hence convergent to $(0,0)$ in usual topology as we justified earlier that $(0,0)$ is the only possible point of convergence.

Also, it can be directly checked that the sequence $\{(T_\theta^\lambda)^n((x_0,y_0))\}_{n\ge 0}$ converges to $(0,0)$ for all λ with $|\lambda| < 1$.

Case 1 : When n is even

$$
\begin{aligned}
d((T_\theta^\lambda)^n((x_0,y_0)),(0,0)) &= d((\lambda R)^n,(0,0)) \\
&= d((\lambda^n x_0,\lambda^n y_0),(0,0)) \\
&= |\lambda|^n\sqrt{x_0^2+y_0^2}.
\end{aligned} \tag{2.22}
$$

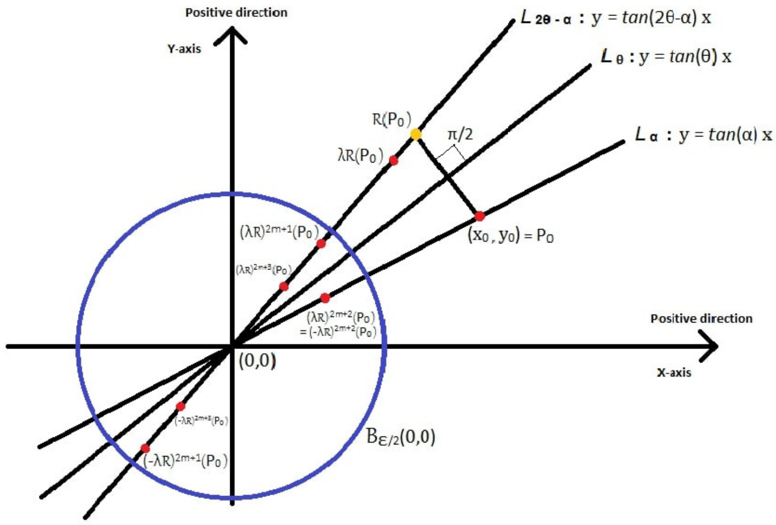


FIGURE 2. Convergence of the sequence $\{(\lambda R)^n(P_0)\}_{n\geq 0}$ for $0 <| \lambda |< 1$

Case 2 : When n is odd

$$\begin{aligned}
d((T_\theta^\lambda)^n((x_0,y_0)),(0,0)) &= d((\lambda R)^n,(0,0)) \\
&= d((\lambda^n(x_0\cos(2\theta)+y_0\sin(2\theta)),\lambda^n(x_0\sin(2\theta)-y_0\cos(2\theta))),(0,0)) \\
&= |\lambda|^n\sqrt{x_0^2+y_0^2}.
\end{aligned} \tag{2.23}$$

So given any $\epsilon > 0$, by (2.22) and (2.23), we can choose $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $d((T_\theta^\lambda)^n((x_0,y_0)),(0,0)) = |\lambda|^n\sqrt{x_0^2+y_0^2} < \frac{\epsilon}{2}$ whenever $|\lambda| < 1$. Therefore, for all λ with $|\lambda| < 1$, the terms of the sequence $\{(T_\theta^\lambda)^n((x_0,y_0))\}_{n\geq 0}$ eventually lie in the open ball $B_{\frac{\epsilon}{2}}(0,0)$ of radius $\frac{\epsilon}{2}$ and having centre at $(0,0)$ and hence convergent to $(0,0)$ in usual topology.

We want to summarize what we have discussed so far regarding the convergence of the sequence $\{(T_\theta^\lambda)^n((x_0,y_0))\}_{n\geq 0}$ from a different perspective. For that we lend some terminologies from dynamical systems and define those solely in our context.

Definition 2.9. Let X be a finite dimensional vector space over $\mathbb{R}$ and d be a metric on X. Also let $L : X \to X$ be a linear map. Then any two points x_1, x_2 of X are said to be forward asymptomatic with respect to the map L if $d(L^n(x_1), L^n(x_2)) \longrightarrow 0$ as $n \longrightarrow \infty$. For any $x \in X$, the stable set of

x with respect to L, denoted by $\mathrm{Stab}_L(x)$, is the set of all points forward asymptomatic to x with respect to L.

We now have the following theorem regarding how the stable set of any point of $\mathbb{R}^2$ with respect to any arbitrary element λR of Sym_2^0 changes all of a sudden as λ varies from $|\lambda| < 1$ to $|\lambda| \geq 1$.

Theorem 2.10. *Consider the metric space $(\mathbb{R}^2, d)$ where d denotes the usual metric. Then the following hold for any point $(a, b) \in \mathbb{R}^2$:*

(1) For any λ with $|\lambda| < 1$,

$$\mathrm{Stab}_{T_\theta^\lambda}((a, b)) = \mathbb{R}^2.$$

(2) For any λ with $|\lambda| \geq 1$,

$$\mathrm{Stab}_{T_\theta^\lambda}((a, b)) = \{(a, b)\}.$$

Proof. For any $(c, d) \in \mathbb{R}^2$ and for any odd n,

$$\begin{aligned}
&d((T_\theta^\lambda)^n((a, b)), (T_\theta^\lambda)^n((c, d))) \\
&= d((\lambda^n(a\cos(2\theta) + b\sin(2\theta)), \lambda^n(a\sin(2\theta) - b\cos(2\theta))), \\
&(\lambda^n(c\cos(2\theta) + d\sin(2\theta)), \lambda^n(c\sin(2\theta) - d\cos(2\theta)))) \\
&= |\lambda|^n\sqrt{((a-c)\cos(2\theta) + (b-d)\sin(2\theta))^2 + ((a-c)\sin(2\theta) - (b-d)\cos(2\theta))^2} \\
&= |\lambda|^n\sqrt{(a-c)^2 + (b-d)^2} \\
&= |\lambda|^n d((a, b), (c, d)).
\end{aligned} \tag{2.24}$$

Also for any $(c, d) \in \mathbb{R}^2$ and even n,

$$\begin{aligned}
d((T_\theta^\lambda)^n((a, b)), (T_\theta^\lambda)^n((c, d))) &= d((\lambda^n a, \lambda^n b), (\lambda^n c, \lambda^n d)) \\
= |\lambda|^n\sqrt{(a-c)^2 + (b-d)^2} &= |\lambda|^n d((a, b), (c, d)).
\end{aligned} \tag{2.25}$$

Therefore we have the following as λ varies.

(1) For all $(c, d) \in \mathbb{R}^2$, $d((T_\theta^\lambda)^n((a, b)), (T_\theta^\lambda)^n((c, d))) \longrightarrow 0$ as $n \longrightarrow \infty$ by (2.24) and (2.25), whenever $|\lambda| < 1$. So any point of $\mathbb{R}^2$ is forward asymptomatic to (a, b) and hence $\mathrm{Stab}_{T_\theta^\lambda}((a, b)) = \mathbb{R}^2$.

(2) By (2.24) and (2.25), for $|\lambda| \geq 1$, $d((T_\theta^\lambda)^n((a, b)), (T_\theta^\lambda)^n((c, d))) \longrightarrow 0$ as $n \longrightarrow \infty$ if and only if $d((a, b), (c, d)) = 0$ if and only if $(a, b) = (c, d)$. So the only point forward asymptomatic to (a, b) is (a, b) itself, that is to say $\mathrm{Stab}_{T_\theta^\lambda}((a, b)) = \{(a, b)\}$.

□

Remark 2.11. First part of Theorem 2.10 can be proved alternatively as follows. We first claim that $\mathrm{Stab}_{T_\theta^\lambda}((0,0)) = \mathbb{R}^2$ for any λ with $|\lambda| < 1$. When $\lambda = 0$, it is easy to see that any point of $\mathbb{R}^2$ is forward asymptomatic to $(0,0)$ with respect to T_θ^λ. Also, the same is true for any λ with $0 < |\lambda| < 1$ by (2.22) and (2.23). Hence we have the claim. Now for any point $(a,b) \neq (0,0)$,

$$\begin{aligned}
&d((T_\theta^\lambda)^n((a,b)), (T_\theta^\lambda)^n((c,d))) \\
&\quad \leq d((T_\theta^\lambda)^n((a,b)), (T_\theta^\lambda)^n((0,0))) + d((T_\theta^\lambda)^n((0,0)), (T_\theta^\lambda)^n((c,d))) \\
&\quad = d((T_\theta^\lambda)^n((a,b)), (0,0)) + d((0,0), (T_\theta^\lambda)^n((c,d)))
\end{aligned}$$

Therefore for all $(c,d) \in \mathbb{R}^2$, $d((T_\theta^\lambda)^n((a,b)), (T_\theta^\lambda)^n((c,d))) \longrightarrow 0$ as $n \longrightarrow \infty$ by (2.22) and (2.23). So any point of $\mathbb{R}^2$ is forward asymptomatic to (a,b) and hence $\mathrm{Stab}_{T_\theta^\lambda}((a,b)) = \mathbb{R}^2$.

3. Applications of the structure of Sym_2^0

In this section we talk about a couple of applications of the obtained structure of Sym_2^0. In the first subsection we obtain the structure of the set PSym_2^0. In the next subsection we reinterpret some trigonometric formulas and show how those can be used to solve purely geometric questions.

3.1. **The structure of** PSym_2^0**.** Recall that PSym_n^0 is the set of all $n \times n$ symmetric matrices, trace of whose product with any element of Sym_n^0 is zero. In this subsection we obtain the structure of PSym_2^0 in the form of following theorem.

Theorem 3.1. *The set* PSym_2^0 *is the set of all scalar matrices of size* 2.

Proof. Let $A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \in \mathrm{PSym}_2^0$ be arbitrarily chosen. Then $\mathrm{Tr}(BA) = 0$ for all $B \in \mathrm{Sym}_2^0$. Therefore by Theorem 2.4, we have:

$$\mathrm{Tr}\left(\left(\begin{smallmatrix} \gamma\cos(\theta) & \gamma\sin(\theta) \\ \gamma\sin(\theta) & -\gamma\cos(\theta) \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)\right) = 0,$$

for all $\gamma \in \mathbb{R}$ and $\theta \in [0, 2\pi)$. That is, for all $\gamma \in \mathbb{R}$ and $\theta \in [0, 2\pi)$,

$$\gamma(a\cos(\theta) + 2b\sin(\theta) - c\cos(\theta)) = 0. \tag{3.1}$$

Plugging $\gamma = 1$ and $\theta = 0$ in (3.1), we get $a = c$. Similarly, plugging $\gamma = 1$ and $\theta = \frac{\pi}{2}$ in (3.1), we get $b = 0$. Hence, the assertion follows. □

In fact, we have the following more general result. This is motivated by [4, Problem 9, Exercises VI, S2, p.190].

Theorem 3.2. *For any positive integer* $n \geq 2$, PSym_n^0 *is the set of all scalar matrices of size* n.

Proof. Consider the Frobenius inner product $\langle , \rangle$ on Sym_n given by $\langle A, B\rangle = \mathrm{Tr}(AB)$, for any $A, B \in \mathrm{Sym}_n$. Then PSym_n^0 is nothing but the orthogonal complement $(\mathrm{Sym}_n^0)^{\perp}$ of Sym_n^0 with respect to the Frobenius inner product. Now as Sym_n^0 is a subspace of Sym_n of co-dimension 1 and as $\mathbb{I}_n \in \mathrm{PSym}_n^0$, we have the theorem. □

So, the proof of Theorem 3.1 incorporates the obtained structure of Sym_2^0 and provides an alternative way of proving Theorem 3.2 for $n = 2$.

3.2. **Interpreting some trigonometric formulas.** In this subsection, we prove an interesting geometric property of $\mathbb{R}^2$ using the geometric interpretation of the elements of Sym_2^0 (cf. (2.16)). Precisely, we prove the following :

Theorem 3.3. *Given any real number* $\beta \in [0, \pi)$, *let* L_β *denotes the line passing through origin and making an angle* β *(in the anticlockwise direction) with positive direction of* x *axis. Then, given any point* (x_0, y_0) *of* $\mathbb{R}^2$, *rotating the point of reflection of* (x_0, y_0) *clockwise (respectively anticlockwise) with respect to the line* $L_{\theta/2}$ *by an angle* α *is same as reflecting* (x_0, y_0) *with respect to the line* $L_{(\theta-\alpha)/2}$ *(respectively* $L_{(\theta+\alpha)/2}$*).*

Proof. We interpret a point (x_0, y_0) of $\mathbb{R}^2$ as the 2×1 column vector $\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right)$. From (2.1), (2.16) and Theorem 2.4, we have that an element $\left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{smallmatrix}\right)$ of Sym_2^0, denoted by $R_{\theta/2}$, reflects a point (x_0, y_0) with respect to the line $L_{\theta/2}$. An orthogonal matrix $\left(\begin{smallmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{smallmatrix}\right)$ of determinant 1, denoted by O_α^1, represents clockwise rotation of the plane $\mathbb{R}^2$. Similarly, the matrix $O_{-\alpha}^1$ represents anticlockwise rotation of the plane $\mathbb{R}^2$ (cf. [3, p.348]). Hence, to prove the theorem, we need to show that $O_\alpha^1 \cdot R_\theta \left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) = R_{\theta-\alpha} \left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right)$ for clockwise case and $O^1_{-\alpha} \cdot R_\theta \binom{x_0}{y_0} = R_{\theta+\alpha}\binom{x_0}{y_0}$ for anticlockwise case. Now,

$$\begin{aligned}
O^1_\alpha \cdot R_\theta &= \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\alpha)\cos(\theta) + \sin(\alpha)\sin(\theta) & \cos(\alpha)\sin(\theta) - \sin(\alpha)\cos(\theta) \\ \cos(\alpha)\sin(\theta) - \sin(\alpha)\cos(\theta) & -\cos(\alpha)\cos(\theta) - \sin(\alpha)\sin(\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta-\alpha) & \sin(\theta-\alpha) \\ \sin(\theta-\alpha) & -\cos(\theta-\alpha) \end{pmatrix} = R_{\theta-\alpha}.
\end{aligned} \tag{3.2}$$

Similarly we have :

$$O^1_{-\alpha} \cdot R_\theta = R_{\theta+\alpha}. \tag{3.3}$$

The theorem now follows from (3.2) and (3.3) for clockwise and anticlockwise scenario respectively. □

Remark 3.4. Theorem 3.3 talks about a geometric property of the plane $\mathbb{R}^2$ in both clockwise and anticlockwise context. Let's denote that property by $P \circlearrowright$ for clockwise case and by $P \circlearrowleft$ for anticlockwise case. Now consider the following standard trigonometric equalities :

$$\begin{aligned}
\cos(\theta+\alpha) &= \cos(\alpha)\cos(\theta) - \sin(\alpha)\sin(\theta), \\
\sin(\theta+\alpha) &= \cos(\alpha)\sin(\theta) + \sin(\alpha)\cos(\theta).
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\cos(\theta-\alpha) &= \cos(\alpha)\cos(\theta) + \sin(\alpha)\sin(\theta), \\
\sin(\theta-\alpha) &= \cos(\alpha)\sin(\theta) - \sin(\alpha)\cos(\theta).
\end{aligned} \tag{3.5}$$

So, Theorem 3.3 says that $P \circlearrowright$ can be thought of as a geometric interpretation of the equalities in (3.5) (cf. (3.2)). Similarly, deciphering (3.4) geometrically, we obtain $P \circlearrowleft$ (cf. (3.3)).

4. Conclusion

It is worth mentioning that the structure of some subsets of Sym^0_n can be obtained for $n > 2$ as well by adapting the method used in Theorem 2.4, provided some conditions being suitably put on the set of eigenvalues (other than they add upto zero) of all its elements.

5. Acknowledgements

The author wishes to thank Indian Institute of Science Education and Research Tirupati (Award No. - IISER-T/Offer/PDRF/A.M./M/01/2021) for financial support. The author is grateful to the referee for several valuable comments.

References

[1] Albert, A. A. and Muckenhoupt, B., On matrices of trace zeros, *Michigan Math. J.*, **4** (1957), 1–3.

[2] Fiedler, M., Eigenvalues of nonnegative symmetric matrices, *Linear Algebra Appl.*, **9** (1974), 119–142.

[3] Herstein, I. N., Topics in algebra, 2nd ed. Lexington: Xerox College Publishing, 1975.

[4] Lang, S., Linear algebra, Inc.:Reading: Addison-Wesley Publishing Co., 1966.

[5] Loewy, R. and London, D., A note on an inverse problem for nonnegative matrices, *Linear and Multilinear Algebra*, **6** (1978), 83–90.

[6] Loewy, R., and McDonald, J. J., The symmetric nonnegative inverse eigenvalue problem for 5×5 matrices, *Linear Algebra Appl.* **393** (2004), 275–298.

[7] McDonald, J. J. and Neumann, M., The Soules approach to the inverse eigenvalues problem for nonnegative symmetric matrices of order $n \leq 5$, *Contemp. Math.*, **259** (2000), 387–407.

[8] Shoda, K., Einige Sätze über Matrizen, *Jpn. J. Math.*, **13**(3) (1937), 361–365.

[9] Spector, O., A characterization of trace zero symmetric nonnegative 5×5 matrices, *Linear Algebra Appl.*, **434**(4) (2011), 1000–1017.

Arijit Mukherjee
Department of Mathematics
Indian Institute of Science Education and Research Tirupati
Karakambadi Rd, opp. Sree Rama Engineering College,
Rami Reddy Nagar,
Mangalam, Tirupati,
Andhra Pradesh - 517 507,
India.
E-mail: mukherjee90.arijit@gmail.com

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July-December (2024), 134–146

PROPERTY (Bgw1) AND WEYL TYPE THEOREMS

NEERU KASHYAP (SHARMA)

Abstract. This paper discusses property (Bgw1), which is an extension of the property (Bgw) defined and studied in [19]. We investigate the property (Bgw1) in connection with Weyl type theorems and establish necessary and sufficient conditions for which the property (Bgw1) holds for a bounded linear operator defined on a Banach space. We study the property (Bgw1) for operators satisfying the single-valued extension property (SVEP). Certain conditions are explored on Hilbert space operators T and S so that $T \oplus S$ obeys the property (Bgw1). We also discuss the preservation of the property (Bgw) under perturbations by finite rank and nilpotent operators.

1. Introduction and Preliminaries

Let $B(X)$ denote the Banach algebra of all bounded linear operators on an infinite-dimensional complex Banach space X. For an operator $T \in B(X)$, let T^*, $N(T)$, $R(T)$, $\sigma(T)$ and $\sigma_a(T)$ denote respectively adjoint, null space, range space, spectrum and approximate spectrum of T. Let $\alpha(T)$ and $\beta(T)$ be the nullity and deficiency of T defined by $\alpha(T) = \dim N(T)$ and $\beta(T) = \operatorname{codim} R(T)$. If the range $R(T)$ of T is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is said to be an upper (resp., a lower) semi-Fredholm operator. Let $USF(X)$ denote the class of all upper semi-Fredholm operators. An operator $T \in B(X)$ is said to be semi-Fredholm if T is either an upper or a lower semi-Fredholm and the index of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$.

If $T \in B(X)$ is both upper and lower semi-Fredholm then T is said to be a Fredholm operator. An operator $T \in B(X)$ is called a Weyl operator if it is a Fredholm operator of index zero. The Weyl spectrum of T is defined by $\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$.

2020 Mathematics Subject Classification: 47A10, 47A11, 47A53
Key words and phrases: Weyl's theorem, generalized Weyl's theorem, generalized a-Browder's theorem, SVEP, property (Bgw), property (Bgw1)

Denote by $USF^{-}(X)$ the class of all upper semi B-Fredholm operators with an index less than or equal to 0. Set $\sigma_{usf^{-}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin USF^{-}(X)\}$.

For a bounded linear operator $T \in B(X)$ and a nonnegative integer n, we define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into itself (in particular $T_0 = T$). If for some integer n, the range space $R(T^n)$ is closed and T_n is an upper (resp., a lower) semi-Fredholm operator, then T is called an upper (resp., a lower) semi B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. From [8, Proposition 2.1], if T_n is a semi-Fredholm operator then T_m is also a semi-Fredholm operator for each $m \geq n$ and $\text{ind}(T_m) = \text{ind}(T_n)$. Thus, the index of a semi-B-Fredholm operator T is defined as the index of the semi-Fredholm operator T_n (see [7, 8]). An operator $T \in B(X)$ is called a B-Weyl operator if it is a B-Fredholm operator of index 0. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined as $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}$. Let $USBF^{-}(X)$ be the class of all upper semi-B-Fredholm operators with an index less than or equal to 0. The upper B-Weyl spectrum of T is defined by $\sigma_{usbf^{-}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin USBF^{-}(X)\}$.

Let $p(T) := \text{asc}(T)$ be the ascent of an operator T i.e., the smallest nonnegative integer n such that $N(T^n) = N(T^{n+1})$. If such an integer does not exist we set $\text{asc}(T) = \infty$. Analogously, let $q(T) := \text{dsc}(T)$ be the descent of an operator T i.e. the smallest nonnegative integer such that $R(T^n) = R(T^{n+1})$ and if such an integer does not exist we set $\text{dsc}(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. An operator T is called Drazin invertible if it has finite ascent and descent. The Drazin spectrum of T is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}$. We observe $\sigma_D(T) = \sigma(T) \setminus \pi(T)$, where $\pi(T)$ is the set of poles of T.

An operator $T \in B(X)$ is called an upper semi-Browder if it is an upper semi-Fredholm of finite ascent and is called Browder if it is a Fredholm of finite ascent and descent. The Browder spectrum of T is defined by $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}$. Define the set $LD(X)$ as $LD(X) = \{T \in B(X) : \alpha(T) < \infty \text{ and is } R(T^{\alpha(T)+1} \text{ closed}\}$ and $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin LD(X)\}$. An operator $T \in B(X)$ is said to be left Drazin invertible if $T \in LD(X)$. We say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I \in LD(X)$

and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda I) < \infty$.

We recall the list of all symbols and notations we use:

$E(T)$	:	eigenvalues of T that are isolated in the spectrum $\sigma(T)$,
$E_0(T)$	:	eigenvalues of T of finite multiplicity that are isolated in the spectrum $\sigma(T)$,
$E^a(T)$	:	eigenvalues of T that are isolated in approximate point spectrum $\sigma_a(T)$,
$E_0^a(T)$	:	eigenvalues of T of finite multiplicity that are isolated in $\sigma_a(T)$,
$\pi(T)$	:	poles of T,
$\pi_0(T)$	:	poles of T of finite rank,
$\pi^a(T)$	:	left poles of T,
$\pi_0^a(T)$	:	left poles of T of finite rank,
$\sigma_{BW}(T)$	:	B-Weyl Spectrum of T,
$\sigma_W(T)$	:	Weyl Spectrum of T,
$\sigma_{usbf^-}(T)$	:	upper semi-B-Weyl spectrum of T,
$\sigma_{usf^-}(T)$	:	upper semi-Weyl spectrum of T.

Following Coburn [10], we say that Weyl's theorem holds for $T \in B(X)$ if $\sigma(T) \backslash \sigma_W(T) = E_0(T)$. According to Rakočević [17], an operator $T \in B(X)$ is said to satisfy a-Weyl's theorem if $\sigma_a(T) \backslash \sigma_{usf^-}(T) = E_0^a(T)$. Following [7], we say that generalized a-Browder's theorem holds for T if $\sigma_a(T) \backslash \sigma_{usbf^-}(T) = \pi^a(T)$ and that a-Browder's theorem holds for T if $\sigma_a(T) \backslash \sigma_{usf^-}(T) = \pi_0^a(T)$. It is proved in [3, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem.

Given $T \in B(X)$, we say that generalized Browder's theorem holds for T if $\sigma(T) \backslash \sigma_{BW}(T) = \pi(T)$, and that Browder's theorem holds for T if $\sigma(T) \backslash \sigma_W(T) = \pi_0(T)$. It is proved in [3, Theorem 2.1] that generalized Browder's theorem is equivalent to Browder's theorem.

We say that T obeys generalized a-Weyl theorem if $\sigma_a(T) \backslash \sigma_{usbf^-}(T) = E^a(T)$, and that generalized Weyl's theorem holds for T if $\sigma(T) \backslash \sigma_{BW}(T) = E(T)$ [7, Definition 2.13]. Generalized a-Weyl's theorem has been studied in [3]. In [7, Theorem 3.11], it is shown that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem. Generalized Weyl's theorem has been studied in [2, 4, 5, 6, 7, 8] and the references therein. Berkani and Koliha [7] proved that generalized Weyl's theorem $\Rightarrow$ Weyl's theorem.

2. Property (Bgw1)

We will say that an operator $T \in B(X)$ has single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at $\lambda_0 \in \mathbb{C}$) if for every open disc U of λ_0 the only analytic function $f : U \to X$ which satisfies the equation $(T-\lambda I)f(\lambda) = 0$ for all $\lambda \in U$, is the function $f \equiv 0$. An operator $T \in B(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$ (see [15]). Every operator T has SVEP at an isolated point of the spectrum.

According to Duggal [12, Proposition 3.10], the following statements are equivalent.

(i) T satisfies generalized a-Browder's theorem,

(ii) T has SVEP at points $\lambda \notin \sigma_{usbf^-}(T)$.

According to [19, Definition 2.1], an operator $T \in B(X)$ is said to satisfy property (Bgw) if $\sigma_a(T) \backslash \sigma_{usbf^-}(T) = E_0(T)$. We now give a definition of property (Bgw1) for a bounded linear operator of T as an extension of property (Bgw).

Definition 2.1. $T \in B(X)$ is said to satisfy property (Bgw1) if $\sigma_a(T) \backslash \sigma_{usbf^-}(T) \subset E_0(T)$.

The property has been introduced in [16, Definition 2.10] as property (SBw_1). In this section, we establish the necessary and sufficient conditions for which the property (Bgw1) holds. We prove that T satisfies property (Bgw1) if and only if generalized a-Browder's theorem holds for T and $\pi^a(T) \subset E_0(T)$.

We start by giving a relationship between property (Bgw) and property (Bgw1):

Theorem 2.2. *Property (Bgw) holds for T if and only if T satisfies property (Bgw1) and $\sigma_{usbf^-}(T) \cap E_0(T) = \emptyset$.*

Proof. Suppose that T satisfies property (Bgw), then property (Bgw1) holds for T and $\sigma_{usbf^-}(T) \cap E_0(T) = \emptyset$. For the converse, suppose T satisfies property (Bgw1) and if $\lambda \in E_0(T)$, $\lambda \notin \sigma_{usbf^-}(T)$ since $\sigma_{usbf^-}(T) \cap E_0(T) = \emptyset$. Thus $\lambda \in \sigma_a(T) \backslash \sigma_{usbf^-}(T)$. Hence $E_0(T) \subseteq \sigma_a(T) \backslash \sigma_{usbf^-}(T)$. $\square$

From Theorem 2.2, we observe that property (Bgw) implies property (Bgw1). However, the converse of this implication does not hold in general, as we can see in the following example.

Example 2.3. Let T be the weighted unilateral shift defined by

$$T(x_1, x_2, x_3, \dots) = \left(\frac{1}{2}x_2, \frac{1}{3}x_3, \dots\right), \quad \text{for all } (x_1, x_2, \dots) \in l^2(\mathbb{N}). \text{ Then}$$

$\sigma_a(T) = \sigma_{usbf^-}(T) = \{0\}$. However, $E_0(T) = \{0\}$. Thus $\sigma_a(T)\setminus \sigma_{usbf^-}(T) = \emptyset \subset E_0(T)$, T satisfies property (Bgw1) but not property (Bgw). Furthermore, property (Bgw1) does not imply that $\sigma_{usbf^-}(T) \cap E_0(T) = \emptyset$. As in this example property (Bgw1) holds for T but $\sigma_{usbf^-}(T) \cap E_0(T) = \{0\}$.

The next result gives the relationship between property (Bgw1) and generalized a-Browder's theorem. Here $\sigma_a^{iso}(T)$ is the set of points which are isolated in $\sigma_a(T)$.

Theorem 2.4. *If $T \in B(X)$ satisfies property (Bgw1). Then generalized a-Browder's theorem holds for T and $\sigma_a(T) = \sigma_{usbf^-}(T) \cup \sigma_a^{\mathrm{iso}}(T)$.*

Proof. By [12, Proposition 3.10], it is sufficient to prove that T has SVEP at every $\lambda \notin \sigma_{usbf^-}(T)$. Let us assume that $\lambda \notin \sigma_{usbf^-}(T)$.

Case (i): If $\lambda \notin \sigma_a(T)$ then T has SVEP at λ.

Case (ii): If $\lambda \in \sigma_a(T)$ and suppose that T satisfies property (Bgw1) then

$$\lambda \in \sigma_a(T)\setminus\sigma_{usbf^-}(T) \subset E_0(T).$$

Hence, $\lambda \in \sigma^{\mathrm{iso}}(T)$, so, in this case, T has SVEP at λ.

To prove $\sigma_a(T) = \sigma_{usbf^-}(T) \cup \sigma_a^{\mathrm{iso}}(T)$, we observe that $\sigma_{usbf^-}(T) \cup \sigma_a^{\mathrm{iso}}(T) \subseteq \sigma_a(T)$ for every $T \in B(X)$. For the reverse inclusion, consider $\lambda \in \sigma_a(T)$. If $\lambda \notin \sigma_{usbf^-}(T)$ then $\lambda \in \sigma_a(T)\setminus\sigma_{usbf^-}(T)$. As T satisfies property (Bgw1), therefore $\lambda \in E_0(T)$. Thus $\lambda \in \sigma_a^{\mathrm{iso}}(T)$. Thus $\sigma_a(T) \subseteq \sigma_{usbf^-}(T) \cup \sigma_a^{\mathrm{iso}}(T)$. Therefore $\sigma_a(T) = \sigma_{usbf^-}(T) \cup \sigma_a^{\mathrm{iso}}(T)$. □

In the next theorem, we give a characterization of property (Bgw1):

Theorem 2.5. *If $T \in B(X)$, then the following statements are equivalent:*

(i) *T satisfies property (Bgw1),*

(ii) *generalized a-Browder's theorem holds for T and $\pi^a(T) \subset E_0(T)$.*

Proof. (i)⇒(ii) Assume that T satisfies property (Bgw1). By Theorem 2.4 it is sufficient to prove that $\pi^a(T) \subset E_0(T)$. Let $\lambda \in \pi^a(T) = \sigma_a(T) \backslash \sigma_{usbf^-}(T) \subset E_0(T)$.

(ii)⇒(i). If $\lambda \in \sigma_a(T) \backslash \sigma_{usbf^-}(T)$. Then generalized a-Browder's theorem implies that $\lambda \in \pi^a(T) \subset E_0(T)$. Thus $\sigma_a(T) \backslash \sigma_{usbf^-}(T) \subset E_0(T)$. □

Theorem 2.6. *Let $T \in B(X)$. If T has SVEP at points in $\sigma_a(T) \backslash \sigma_{usbf^-}(T)$, then T satisfies property (Bgw1) if and only if $\pi^a(T) \subset E_0(T)$.*

Proof. The hypothesis that T has SVEP at $\sigma_a(T) \backslash \sigma_{usbf^-}(T)$ implies that T satisfies generalized a-Browder's theorem [12, Proposition 3.10].
Hence if $\pi^a(T) \subset E_0(T)$ then $\sigma_a(T) \backslash \sigma_{usbf^-}(T) = \pi^a(T) \subset E_0(T)$. □

Operators $S, T \in B(X)$ are said to be injectively interwined, denoted, $S \prec_i T$, if there exists an injection $U \in B(X)$ such that $TU = US$. If $S \prec_i T$, then T has SVEP at a point λ implies S has SVEP at λ. To see this, let T have SVEP at λ, let U be an open neighbourhood of λ and let $f : U \to X$ be an analytic function such that $(S - \mu)f(\mu) = 0$ for every $\mu \in U$. Then $U(S - \mu)f(\mu) = (T - \mu)Uf(\mu) = 0 \Rightarrow Uf(\mu) = 0$. Since U is injective, $f(\mu) = 0$, i.e., S has SVEP at λ.

Theorem 2.7. *Let $S, T \in B(X)$. If T has SVEP and $S \prec_i T$, then property (Bgw1) holds for S if and only if $\pi^a(S) \subset E_0(S)$.*

Proof. Suppose that T has SVEP. Since $S \prec_i T$, S has SVEP. Hence the result follows from Theorem 2.6. □

Recall from [19], that if $T \in B(X)$ and $s \in \mathbb{N}$, then T has uniform decent for $n \geq s$ if $R(T) + \ker(T^n) = R(T) + \ker(T^s)$ for all $n \geq s$. If in addition $R(T) + \ker(T^s)$ is closed then T is said to have topological descent for $n \geq s$.

Also recall that $T \in B(X)$ is said to satisfy property (Bw1) if $\sigma(T) \backslash \sigma_{BW}(T) \subset E_0(T)$ [14, Definition 2.2]. The next result gives a relationship between property (Bgw1) and property (Bw1).

Theorem 2.8. *If T satisfies property (Bgw1), then it satisfies property (Bw1).*

Proof. Suppose that T satisfies property (Bgw1) and $\lambda \in \sigma(T) \backslash \sigma_{BW}(T)$. Then $T - \lambda I$ is B-Weyl and $T - \lambda I$ is upper semi-B-Fredholm with index zero. Thus $\lambda \notin \sigma_{usbf^-}(T)$.

Let $\lambda \notin \sigma_a(T)$. Since $T - \lambda I$ is an operator of topological uniform descent, then there exists $\varepsilon > 0$ such that if $0 < |\lambda - \mu| < \varepsilon$, then we have $c_n(T - \lambda I) = c_0(T - \mu I)$ and $\acute{c}_n(T - \lambda I) = \acute{c}_0(T - \mu I)$ for large enough n. Since $T - \lambda I$ is B-Weyl, $c_n(T - \lambda I) = \acute{c}_n(T - \lambda I)$. We have $\acute{c}_0(T - \lambda I) = 0$, because $\lambda \notin \sigma_a(T)$. Hence we have $c_0(T - \lambda I) = \acute{c}_0(T - \lambda I) = 0$. Consequently, $\lambda \notin \sigma(T)$ which is a contradiction. Hence $\lambda \in \sigma_a(T)$. Since T satisfies property (Bgw1), $\lambda \in E_0(T)$. Thus T satisfies property (Bw1).

Recall that an operator $T \in B(X)$ is said to satisfy property (gb) if $\sigma_a(T) \setminus \sigma_{usbf^-}(T) = \pi(T)$. The following result gives a relationship between property (Bgw1) and (gb). □

Theorem 2.9. *Let $T \in B(X)$. Then the following statements are equivalent.*

(i) *T satisfies property (Bgw1),*
(ii) *T satisfies property (gb) and $\pi(T) \subset E_0(T)$.*

Proof. (i)$\Longrightarrow$(ii): Suppose T satisfies property (Bgw1). To prove that T satisfies property (gb), by [20, Proposition 2.16] it is enough to show that T has SVEP. Let $\lambda \in \sigma_a(T) \setminus \sigma_{usbf^-}(T)$. Since T satisfies property (Bgw1), $\lambda \in E_0(T)$. Hence $\lambda \in \operatorname{iso} \sigma(T)$. Thus T has SVEP at λ.

Now we have to prove that $\pi(T) \subset E_0(T)$.

Suppose $\lambda \in \pi(T)$. Since T satisfies property (gb), $\lambda \in \sigma_a(T) \setminus \sigma_{usbf^-}(T)$. Hence $\lambda \in E_0(T)$ because T satisfies property (Bgw1).

(ii)$\Longrightarrow$(i): If $\lambda \in \sigma_a(T) \setminus \sigma_{usbf^-}(T)$, then $\lambda \in \pi(T)$ by hypothesis. Thus $\lambda \in E_0(T)$ and T satisfies property (Bgw1). □

3. Property (Bgw1) for Direct Sum

Let H and K be infinite-dimensional Hilbert spaces and T and S are two operators on H and K, respectively. In the following results, we present sufficient conditions on T and S under which property (Bgw1) will be transferred from the direct summands to the direct sum $T \oplus S$.

Theorem 3.1. *Suppose that $T \in B(H)$ and $S \in B(K)$ are such that $\sigma_p^0(T) \subset \sigma_a(S)$ and $\sigma_p^0(S) \subset \sigma_a(T)$. If T and S both possess property (Bgw1), then the following statements are equivalent.*

(i) *$T \oplus S$ possesses property (Bgw1);*
(ii) *$\sigma_{usbf^-}(T \oplus S) = \sigma_{usbf^-}(T) \cup \sigma_{usbf^-}(S)$.*

Proof. (ii)$\Longrightarrow$(i): Assume that $\sigma_{usbf^-}(T\oplus S)=\sigma_{usbf^-}(T)\cup\sigma_{usbf^-}(S)$. Since T and S both possess property (Bgw1)

$$\begin{aligned}\sigma_a(T\oplus S)\backslash\sigma_{usbf^-}(T\oplus S)&=[\sigma_a(T)\cup\sigma_a(S)]\backslash[\sigma_{usbf^-}(T)\cup\sigma_{usbf^-}(S)]\\ &\subset[E_0(T)\cap\rho_a(S)]\cup[\rho_a(T)\cap E_0(S)]\cup[E_0(T)\cap E_0(S)].\end{aligned}$$

Since by hypothesis that $\sigma_p^0(T)\subset\sigma_a(S)$ and $\sigma_p^0(S)\subset\sigma_a(T)$, then $E_0(T)\cap\rho_a(S)=\emptyset$ and $E_0(S)\cap\rho_a(T)=\emptyset$. Therefore

$$\begin{aligned}\sigma_a(T\oplus S)\backslash\sigma_{usbf^-}(T\oplus S)&=[\sigma_a(T)\cup\sigma_a(S)]\backslash[\sigma_{usbf^-}(T)\cup\sigma_{usbf^-}(S)]\\ &\subset[E_0(T)\cap E_0(S)].\end{aligned}$$

Since we know

$$\sigma_p^0(T\oplus S)=\{\lambda\in\sigma_p^0(T)\cup\sigma_p^0(S):\dim N(\lambda I-T)+\dim N(\lambda I-S)<\infty\}.$$

Then

$$\begin{aligned}E_0(T\oplus S)&=\sigma^{\text{iso}}(T\oplus S)\cap\sigma_p^0(T\oplus S)\\ &=\text{iso}[\sigma(T)\cup\sigma(S)]\cap[\sigma_p^0(T)\cup\sigma_p^0(S)],\ \text{where iso}[\sigma(T)\cup\sigma(S)]\\ &\quad\ \text{denotes the isolated points of }[\sigma(T)\cup\sigma(S)]\\ &=[E_0(T)\cap\rho(S)]\cup[\rho(T)\cap E_0(S)]\cup[E_0(T)\cap E_0(S)]\\ &=[E_0(T)\cap E_0(S)].\end{aligned}$$

Since $E_0(T)\cap\rho(S)=\emptyset$ and $E_0(S)\cap\rho(T)=\emptyset$.

(i)$\Longrightarrow$(ii): If $T\oplus S$ possesses property (Bgw1), then from Theorem 2.4, $T\oplus S$ satisfies generalized a-Browder's theorem. If T and S both possess property (Bgw1), then T and S satisfy generalized a-Browder's theorem. From [9, Theorem 2.8], we have $\sigma_{usbf^-}(T\oplus S)=\sigma_{usbf^-}(T)\cup\sigma_{usbf^-}(S)$. □

Theorem 3.2. *Suppose $T\in B(H)$ is such that $\sigma_a^{iso}(T)=\emptyset$, $\sigma(T)=\sigma_a(T)$ and $S\in B(K)$ satisfies property (Bgw1). If $\sigma_{usbf^-}(T\oplus S)=\sigma_a(T)\cup\sigma_{usbf^-}(S)$, then property (Bgw1) holds for $T\oplus S$.*

Proof. As $\sigma_a(T\oplus S)=\sigma_a(T)\cup\sigma_a(S)$ for any pair of operators, we have

$$\begin{aligned}\sigma_a(T\oplus S)\backslash\sigma_{usbf^-}(T\oplus S)&=[\sigma_a(T)\cup\sigma_a(S)]\backslash[\sigma_a(T)\cup\sigma_{usbf^-}(S)]\\ &=\sigma_a(S)\backslash[\sigma_a(T)\cup\sigma_{usbf^-}(S)]\\ &=[\sigma_a(S)\backslash\sigma_{usbf^-}(S)]\backslash\sigma_a(T)\\ &\subset E_0(S)\cap\rho_a(T),\end{aligned}$$

where $\rho_a(T)=\mathbb{C}\setminus\sigma_a(T)$.

If $\sigma_a^{iso}(T) = \emptyset$, it implies that $\sigma^{iso}(T) = \emptyset$ and $\sigma(T) = \sigma^{acc}(T)$, where $\sigma^{acc}(T) = \sigma(T)\backslash\sigma^{iso}(T)$ is the set of all accumulation points of $\sigma(T)$. Thus we have

$$\begin{aligned}\sigma^{iso}(T \oplus S) &= [\sigma^{iso}(T) \cup \sigma^{iso}(S)]\backslash[(\sigma^{iso}(T) \cap \sigma^{acc}(S)) \cup (\sigma^{acc}(T) \cap \sigma^{iso}(S))]\\ &= (\sigma^{iso}(T)\backslash\sigma^{acc}(S)) \cup (\sigma^{iso}(S)\backslash\sigma^{acc}(T))\\ &= \sigma^{iso}(S)\backslash\sigma_a(T)\\ &= \sigma^{iso}(S) \cap \rho_a(T).\end{aligned}$$

Let $\sigma_P(T)$ denote the point spectrum of T and $\sigma_{PF}(T)$ denote the set of all eigenvalues of T of finite multiplicity.

We have that $\sigma_P(T \oplus S) = \sigma_P(T) \cup \sigma_P(S)$ and $\dim N(T \oplus S) = \dim N(T) + \dim N(S)$ for every pair of operators, so that

$$\begin{aligned}&\sigma_{PF}(T \oplus S)\\ &\quad = \{\lambda \in \sigma_{PF}(T) \cup \sigma_{PF}(S) : \dim N(\lambda I - T) + \dim N(\lambda I - S) < \infty\}.\end{aligned}$$

Therefore,

$$\begin{aligned}E_0(T \oplus S) &= \sigma^{iso}(T \oplus S) \cap \sigma_{PF}(T \oplus S)\\ &= \sigma^{iso}(S) \cap \rho_a(T) \cap \sigma_{PF}(S)\\ &= E_0(S) \cap \rho_a(T).\end{aligned}$$

Thus, $\sigma_a(T \oplus S)\backslash\sigma_{usbf^-}(T \oplus S) \subseteq E_0^a(T \oplus S)$. Hence, $T \oplus S$ satisfies the property (Bgw1). □

Let $\sigma_1(T)$ denote the compliment of $\sigma_{usbf^-}(T)$ in $\sigma_a(T)$ i.e. $\sigma_1(T) = \sigma_a(T)\backslash\sigma_{usbf^-}(T)$. A straight forward application of Theorem 3.2 leads to the following corollaries.

Corollary 3.3. Suppose $T \in B(H)$ is such that $\sigma_a^{iso}(T) = \emptyset$, $\sigma(T) = \sigma_a(T)$ and $S \in B(K)$ satisfies property (Bgw1) with $\sigma^{iso}(S) \cap \sigma_{PF}(S) = \emptyset$ and $\sigma_1(T \oplus S) = \emptyset$, then $T \oplus S$ satisfies property (Bgw1).

Proof. Since S satisfies property (Bgw1), therefore given condition $\sigma^{iso}(S) \cap \sigma_{PF}(S) = \emptyset$ implies that $\sigma_a(S) = \sigma_{usbf^-}(S)$. Now $\sigma_1(T \oplus S) = \emptyset$ gives that $\sigma_a(T \oplus S) = \sigma_{usbf^-}(T \oplus S) = \sigma_a(T) \cup \sigma_{usbf^-}(S)$. Thus, from Theorem 3.2, we have that $T \oplus S$ satisfies property (Bgw1). □

Corollary 3.4. Suppose $T \in B(H)$ is such that $\sigma_1(T) \cup \sigma_a^{iso}(T) = \emptyset$, $\sigma(T) = \sigma_a(T)$ and $S \in B(K)$ satisfies property (Bgw1). If $\sigma_{usbf^-}(T \oplus S) = \sigma_{usbf^-}(T) \cup \sigma_{usbf^-}(S)$, then property (Bgw1) holds for $T \oplus S$.

4. Property (Bgw) and Perturbations

In this section, we study the preservation of property (Bgw) under perturbations by finite rank and nilpotent operators.

Theorem 4.1. *Let $T \in B(X)$. If T has property (Bgw) and F is a finite rank operator in $B(X)$ that commutes with T, then $T + F$ has property (Bgw) if and only if $\pi^a(T + F) = E_0(T + F)$.*

Proof. If $T + F$ has property (Bgw), then $\pi^a(T + F) = E_0(T + F)$. Conversely, suppose $\pi^a(T + F) = E_0(T + F)$. Since F is a finite rank operator in $B(X)$ that commutes with T, $\sigma_{usbf^-}(T) = \sigma_{usbf^-}(T + F)$ and $\sigma_{LD}(T) = \sigma_{LD}(T + F)$ [7, Theorem 4.3]. As T satisfies generalized a-Browder's theorem, $\sigma_{usbf^-}(T) = \sigma_{LD}(T)$. Now $\sigma_a(T + F) \setminus \sigma_{usbf^-}(T + F) = \sigma_a(T + F) \setminus \sigma_{LD}(T + F) = \pi^a(T + F) = E_0(T + F)$. Therefore, $T + F$ satisfies property (Bgw). □

Theorem 4.2. *Let $T \in B(X)$ and let N be a nilpotent operator commuting with T. If T satisfies property (Bgw), then $T + N$ satisfies property (Bgw) if and only if $\sigma_{usbf^-}(T + N) = \sigma_{usbf^-}(T)$.*

Proof. Assume that $T + N$ satisfies property (Bgw), then $\sigma_a(T + N) \setminus \sigma_{usbf^-}(T+N) = E_0(T+N)$. As $\sigma_a(T+N) = \sigma_a(T)$ and $E_0(T+N) = E_0(T)$. Then, $\sigma_a(T) \setminus \sigma_{usbf^-}(T + N) = E_0(T)$. Since T satisfies property (Bgw), then $\sigma_a(T) \setminus \sigma_{usbf^-}(T) = E_0(T)$. Therefore $\sigma_{usbf^-}(T + N) = \sigma_{usbf^-}(T)$. Conversely, assume that $\sigma_{usbf^-}(T + N) = \sigma_{usbf^-}(T)$, then as T satisfies property (Bgw) it follows that $T + N$ also satisfies property (Bgw). □

5. Conclusion

In conclusion, we provide a summary of the results obtained in this paper. We use the abbreviations (Bgw), (Bgw1), (Bw1), (gb), W, gW, aW, gaW to signify that an operator obeys property (Bgw), property (Bgw1), property (Bw1), property (gb), Weyl's theorem, generalized Weyl's theorem, a-Weyl's theorem and generalized a-Weyl's theorem. Similarly, the abbreviations B, aB, gB and gaB have analogous meanings concerning the Browder's theorem.

The following table summarizes the meaning of various theorems and properties:

W	$\sigma(T)\backslash\sigma_W(T) = E_0(T)$	B	$\sigma(T)\backslash\sigma_W(T) = \pi_0(T)$
aW	$\sigma_a(T)\backslash\sigma_{usf^-}(T) = E_0^a(T)$	aB	$\sigma_a(T)\backslash\sigma_{usf^-}(T) = \pi_0^a(T)$
gW	$\sigma(T)\backslash\sigma_{BW}(T) = E(T)$	gB	$\sigma(T)\backslash\sigma_{BW}(T) = \pi(T)$
gaW	$\sigma_a(T)\backslash\sigma_{usbf^-}(T) = E^a(T)$	gaB	$\sigma_a(T)\backslash\sigma_{usbf^-}(T) = \pi^a(T)$
(Bgw)	$\sigma_a(T)\backslash\sigma_{usbf^-}(T) = E_0(T)$	(gb)	$\sigma_a(T)\backslash\sigma_{usbf^-}(T) = \pi(T)$
(Bw1)	$\sigma(T)\backslash\sigma_{BW}(T) \subset E_0(T)$	(Bgw1)	$\sigma_a(T)\backslash\sigma_{usbf^-}(T) \subset E_0(T)$

In the following diagram, arrows signify implications between the properties studied in this paper and other Weyl type theorems. The numbers near the arrows are references to the results in the present paper (numbers without brackets) or to the bibliography therein (numbers in square brackets)

$$\begin{array}{ccccccc}
 & & & & \text{(gb)} & & \\
 & & & & \big\uparrow{\scriptstyle 2.10} & & \\
 & & \text{(Bgw)} & \xrightarrow{2.2} & \text{(Bgw1)} & \xrightarrow{2.9} & \text{(Bw1)} \\
 & & & & \big\downarrow{\scriptstyle 2.4} & & \\
\text{gW} & \xleftarrow{[5]} & \text{gaW} & \xrightarrow{[5]} & \text{gaB} & \xrightarrow{[5]} & \text{gB} \\
\big\downarrow{\scriptstyle [5]} & & \big\downarrow{\scriptstyle [5]} & & \big\updownarrow{\scriptstyle [3]} & & \big\updownarrow{\scriptstyle [3]} \\
\text{W} & \xleftarrow{[14]} & \text{aW} & \xrightarrow{[5]} & \text{aB} & \xrightarrow{[11]} & \text{B}
\end{array}$$

References

[1] Aiena, P., *Fredholm and Local Spectral Theory, With Applications to Multipliers*, Kluwer Acad. Publishers (2004), https://doi.org/10.1007/1-4020-2525-4

[2] Amouch, M., Weyl type theorems for operators satisfying the single valued extension property, *J. Math. Anal. Appl.* **326** (2007), 1476–1484, https://doi.org/10.1016/j.jmaa.2006.03.085

[3] Amouch, M. and Zguitti, H., On the equivalence of Browder's and generalized Browder's theorem, *Glasgow Math. J.* **48** (2006), 179–185, https://doi.org/10.1017/s0017089505002971

[4] Berkani, M., B-weyl spectrum and poles of the resolvent, *J. Math. Anal. Appl.* **272** (2002), 596–603, https://doi.org/10.1016/s0022-247x(02)00179-8

[5] Berkani, M., Index of B-Fredholm operators and generalization of a Weyl theorem, *Proc. Amer. Math. Soc.* **130** (2002), 1717–1723, https://doi.org/10.1090/s0002-9939-01-06291-8

[6] Berkani, M. and Arroud, A., Generalized Weyl's theorem and Hyponormal operators, *J. Austral. Math. Soc.* **76** (2004), 291–302, https://doi.org/10.1017/s144678870000896x

[7] Berkani, M. and Koliha, J. J., Weyl type theorems for bounded linear operators, *Acta Sci. Math. (Szeged)* **69** (2003), 359–376

[8] Berkani, M. and Sarih, M., On semi-B-Fredholm operators, *Glasgow Math. J.* **43** (3) (2001), 457–465, https://doi.org/10.1017/s0017089501030075

[9] Berkani, M. and Zariouh, H., Extended Weyl type theorems, *Math. Bohemica* **134** (4) (2009), 369–378, https://doi.org/10.21136/mb.2009.140669

[10] Coburn, L. A., Weyl's theorem for non-normal operators, *Michigan Math. J.* **13** (1966), 285–288, https://doi.org/10.1307/mmj/1031732778

[11] Djordjevic, S. V. and Han, Y. M., Browder's theorem and spectral continuity, *Glasgow Math. J.* **42** (2000), 479–486, https://doi.org/10.1017/s0017089500030147

[12] Duggal, B. P., Polaroid Operators and generalized Browder, Weyl theorems, *Math. Proc. Royal Irish Academy* **108A** (2008), 149–163, https://doi.org/10.1353/mpr.2008.0014

[13] Grabiner, S., Uniform ascent and descent of bounded operators, *J. Math. Soc. Japan* **34** (1982), 317–337, https://doi.org/10.2969/jmsj/03420317

[14] Kashyap, N., On property (Bw1), *Poincare Journal of Analysis and Applications* **10** (2) (2023), 263–270, https://doi.org/10.46753/pjaa.2023.v010i02.004

[15] Laursen, K. B. and Neumann, M. M., *An Introduction to Local Spectral Theory*, Clarendon Press, Oxford (2000)

[16] Lombarkia, F. and Zariouh, H., On Browder-type theorems, *Acta Mathematica Univ. Comenianae* **2** (2014), 281–289

[17] Oberai, K. K., On the Weyl spectrum, II, *Illinois J. Math.* **21** (1977), 84–90, https://doi.org/10.1215/ijm/1256049504

[18] Rakočević, V., Operators obeying a-Weyl's theorem, *Rev. Roumaine Math. Pures Appl.* **34** (1989), 915–919

[19] Rashid, M. H. M. and Prasad, T., Variation of Weyl type theorems, *Annal. Fuct. Anal.* **4** (1) (2013), 40–52, https://doi.org/10.15352/afa/1399899835

[20] Rashid, M. H. M., Property (gb) and perturbation, *J. Math. Anal. Appl.* **383** (2011), 82–91, https://doi.org/10.1016/j.jmaa.2011.05.001

Neeru Kashyap (Sharma)
Department of Mathematics
Bhaskaracharya College of Applied Sciences
University of Delhi
Dwarka, New Delhi 110075
INDIA.
E-mail: neeru.sharma@bcas.du.ac.in

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July - December (2024), 147–160

SOME INEQUALITIES FOR FRACTIONAL INTEGRALS VIA (η, l)-CONVEX FUNCTION

J. B. GAJERA AND R. K. JANA

(Received : 25 - 12 - 2023 ; Revised : 01 - 11 - 2024)

Abstract. The main goal of this paper is to establish a new integral equality for the k-Riemann Liouville fractional operator. We present a number of new inequalities for twice differentiable (η, l)-convex functions that are related to the Hermite-Hadamard integral inequality and generalise many previously obtained results.

1. Introduction

In the literature, the inequalities for convex functions discovered by C. Hermite and J. Hadamard [9] are crucial. The following is a statement of this inequality:

If g be a real valued convex function on interval I and $\xi_1, \xi_2 \in \mathbb{R}$ with $\xi_1 < \xi_2$, then the Hermite-Hadamard inequality defined as follow

$$g\left(\frac{\xi_1+\xi_2}{2}\right) \leq \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} g(z)dz \leq \frac{g(\xi_1)+g(\xi_2)}{2}. \tag{1.1}$$

Mathematicians have recently focused on generalizing, improving, and extending the Hermite-Hadamard inequality for various classes of convex functions. Classical convex functions have been expanded and generalized in different ways, including λ_ϕ-preinvex functions [2], s-convex [3], pseudo-convex [7], MT-convex [11] and h-convex [15].

We will use (η, l)-convex function throughout the paper. (η, l)-convex functions are a specific class of functions that exhibit convex-like properties with additional parameters η and l. Understanding the properties of (η, l)-convex functions is crucial as they offer a more nuanced perspective on the behavior of functions compared to traditional convex functions.

2010 Mathematics Subject Classification: 11A41, 16N20
Key words and phrases: prime numbers, divisibility, ring

The following is the definition of an (η, l)-convex function as introduced by Mihesan in 1993:

Definition 1.1. [8] The function g is real valued (η, l)-convex function on the interval $[0, \theta]$, if for $\xi_1, \xi_2 \in [0, \theta]$ and $\nu \in (0, 1)$, the inequality given below holds true

$$g(\nu\xi_1 + l(1-\nu)\xi_2) \leq \nu^\eta g(\xi_1) + l(1-\nu^\eta)g(\xi_2),$$

where, $\eta, l \in (0, 1]$.

Remark 1.2. From above definition:

- The (η, l)-convex function and the l-convex function coincide if and only if $\eta = 1$.
- The (η, l)-convex function and the η-convex function in the second sense coincide if and only if $l = 1$.
- The (η, l)-convex function and the convex function coincide if and only if $\eta = l = 1$.

The Hermite-Hadamard inequality (1.1) is established for conformable fractional integrals, fractional integrals, the classical integral, and more recently, generalized fractional integrals. For further information and applications, refer to [1, 4, 13, 14, 17] and the references therein.

Fractional integrals involve generalizing the concept of integration to non-integer orders, which is essential in various mathematical applications. Inequalities related to fractional integrals provide insights into the behavior of functions under fractional integration operators. We will use k-Riemann-Liouville fractional integral throughout the paper.

In 2012, Mubeen and Habibullah [12] defined the k-Riemann-Liouville fractional integral as follows :

Definition 1.3. For $k > 0$ and $g \in L^1[c, d]$, the k-Riemann-Liouville fractional integrals are introduced by Mubeen and Habibullah [12] which are as follows,

$$(I_{c^+}^{\alpha,k} g)(x) = \frac{1}{k\Gamma_k(\alpha)} \int_c^x (x-\xi)^{\frac{\alpha}{k}-1} g(\xi) d\xi, x > c$$

and

$$(I_{d^-}^{\alpha,k} g)(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^d (\xi-x)^{\frac{\alpha}{k}-1} g(\xi) d\xi, x < d.$$

The aim of this paper is to investigate Hermite-Hadamard inequalities for the k-Riemann-Liouville fractional integral using the (η, l)-convex function. This exploration of inequalities and (η, l)-convex functions serves as a foundation for advanced mathematical analysis techniques. These concepts offer a framework for examining the behavior of functions within the realm of fractional integrals, contributing significantly to the advancement of mathematical theories and their practical applications.

Now, we present essential inequalities and lemmas that are utilized throughout the paper.

Theorem 1.4. *If $g_1 \in L^\lambda[c,d]$ and $g_2 \in L^\mu[c,d]$, then $g_1 g_2 \in L^1[c,d]$ and the integral form of the Hölder inequality* [10] *can be expressed as follows:*

$$\int_c^d |g_1(\xi)g_2(\xi)|d\xi \leq \Big(\int_c^d |g_1(\xi)|^\lambda dx\Big)^{\frac{1}{\lambda}} \Big(\int_c^d |g_2(\xi)|^\mu d\xi\Big)^{\frac{1}{\mu}},$$

for $\lambda, \mu \in [1, \infty]$ such that $\frac{1}{\lambda} + \frac{1}{\mu} = 1$. The inequality become equality if $|g_1|^\lambda$ and $|g_2|^\mu$ are linearly independent in $L^1[c,d]$.

Theorem 1.5. [5] *Let $\lambda > 1$ and $\frac{1}{\lambda} + \frac{1}{\mu} = 1$. If $g_1, g_1 :\longrightarrow \mathbb{R}$ and if $|g_1|^\lambda, |g_2|^\mu$ are integrable functions on interval $[c,d]$, then Hölder İşcan Integral inequality is as follows:*

$$\begin{aligned}
&\int_c^d |g_1(\xi)g_2(\xi)|d\xi \\
&\quad \leq \frac{1}{d-c}\left\{\left(\int_c^d (d-\xi)|g_1(\xi)|^\lambda d\xi\right)^{\frac{1}{\lambda}} \left(\int_c^d (d-\xi)|g_2(\xi)|^\mu d\xi\right)^{\frac{1}{\mu}}\right. \\
&\qquad \left. + \left(\int_c^d (\xi-c)|g_1(\xi)|^\lambda d\xi\right)^{\frac{1}{\lambda}} \left(\int_c^d (\xi-c)|g_2(\xi)|^\mu d\xi\right)^{\frac{1}{\mu}}\right\}.
\end{aligned}$$

Theorem 1.6. [6] *Let $\mu \geq 1$. If g_1 and g_2 are real valued functions defined on $[c,d]$ and if $|f|, |f||g|^\mu$ integrable functions on $[c,d]$, then power-mean integral inequality is as follows:*

$$\int_c^d |g_1(\xi)g_2(\xi)|dx \leq \left(\int_c^d |g_1(\xi)|d\xi\right)^{1-\frac{1}{\mu}} \left(\int_c^d |g_1(\xi)||g_2(\xi)|^q d\xi\right)^{\frac{1}{\mu}}.$$

Lemma 1.7. [14] *Let g be a real valued differential function on I, where $c, d \in I$ with $0 \leq c \leq d$. If $g' \in L[c, d]$, then the equality given below is holds:*

$$\frac{g(c)+g(dl)}{2} - \frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I^{\theta,k}_{c^+}g(ld) + I^{\theta,k}_{dl^-}g(c)\right]$$
$$\leq \frac{k(ld-c)^2}{2(\theta+k)}\left\{\int_0^1 \left(1-(1-\xi)^{\frac{\theta}{k}+1} - \xi^{\frac{\theta}{k}+1}\right) g''(c\xi + l(1-\xi)d)d\xi\right\}.$$

2. Main Result

Theorem 2.1. *Let g be a differentiable function on I, where $c, d \in I$ with $0 \leq c \leq d$ and $g'' \in L[c, d]$. If $|g''|$ is (η, l) convex function and $\Re(\frac{\theta}{k}) > 0, \theta \neq 0$, then the below-mentioned inequality holds true:*

$$\left|\frac{g(c)+g(dl)}{2} - \frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I^{\theta,k}_{c^+}g(ld) + I^{\theta,k}_{dl^-}g(c)\right]\right|$$
$$\leq \frac{k(ld-c)^2}{2(\theta+k)}$$
$$\times\left\{|g''(c)|\left(\frac{1}{\eta+1} - B(\eta+1, \frac{\theta}{k}+2) - \frac{k}{\theta+\eta k+2k}\right)\right.$$
$$\left.+ l|g''(d)|\left(\frac{\theta}{\theta+2k} - \frac{1}{\eta+1} + B(\eta+1, \frac{\theta}{k}+2) + \frac{k}{\theta+\eta k+2k}\right)\right\}$$
$$= k_1,$$

where $\eta, l \in (0, 1]$.

Proof. By using lemma 1.7 and (η, l)-convexity of $|g''|$, we get

$$\left|\frac{g(c)+g(dl)}{2} - \frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I^{\theta,k}_{c^+}g(ld) + I^{\theta,k}_{dl^-}g(c)\right]\right|$$
$$\leq \frac{k(ld-c)^2}{2(\theta+k)}\left\{\int_0^1 \left(1-(1-\tau)^{\frac{\theta}{k}+1} - \tau^{\frac{\theta}{k}+1}\right) g''(c\tau + l(1-\tau)d)d\tau\right\}$$
$$\leq \frac{k(ld-c)^2}{2(\theta+k)}\left\{|g''(c)|\int_0^1 \left(1-(1-\tau)^{\frac{\theta}{k}+1} - \tau^{\frac{\theta}{k}+1}\right)\tau^\eta d\tau\right.$$
$$\left.+ l|g''(d)|\int_0^1 \left(1-(1-\tau)^{\frac{\theta}{k}+1} - \tau^{\frac{\theta}{k}+1}\right)(1-\tau^\eta)d\tau\right\}.$$

By evaluating integration, we get the required inequality. □

Corollary 2.2. By substituting $\eta = 1$ in Theorem 2.1, we get the result for the l-convex function:

$$\begin{aligned}
&\left|\frac{g(c)+g(dl)}{2} - \frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I_{c^+}^{\theta,k} g(ld) + I_{dl^-}^{\theta,k} g(c)\right]\right| \\
&\quad \le \frac{k(ld-c)^2}{2(\theta+k)}\left\{|g''(c)|\left(\frac{1}{2} - B(2, \frac{\theta}{k}+2) - \frac{k}{\theta+3k}\right)\right. \\
&\qquad\qquad \left. + l|g''(d)|\left(\frac{\theta}{\theta+2k} - \frac{1}{2} + B(2, \frac{\theta}{k}+2) + \frac{k}{\theta+3k}\right)\right\}.
\end{aligned}$$

Corollary 2.3. By substituting $l = 1$ in Theorem 2.1, we obtain the result for the η-convex function:

$$\begin{aligned}
&\left|\frac{g(c)+g(d)}{2} - \frac{\Gamma_k(\theta+k)}{2(d-c)^{\frac{\theta}{k}}}\left[I_{c^+}^{\theta,k} g(d) + I_{d^-}^{\theta,k} f(c)\right]\right| \\
&\quad \le \frac{k(d-c)^2}{2(\theta+k)} \\
&\qquad \times \left\{|g''(c)|\left(\frac{1}{\eta+1} - B(\eta+1, \frac{\theta}{k}+2) - \frac{k}{\theta+\eta k+2k}\right)\right. \\
&\qquad\quad \left. + |g''(d)|\left(\frac{\theta}{\theta+2k} - \frac{1}{\eta+1} + B(\eta+1, \frac{\theta}{k}+2) + \frac{k}{\theta+\eta k+2k}\right)\right\}.
\end{aligned}$$

Theorem 2.4. *Let g be a twice differentiable function on I, where $c, d \in I$ with $0 \le c \le d$ and $g'' \in L[c,d]$. If $|g''|^\mu, \lambda, \mu \ge 1, \frac{1}{\lambda}+\frac{1}{\mu} = 1$ is (η, l)-convex function and $\Re(\frac{\theta}{k}) > 0, \theta \neq 0$, then the inequality given below holds:*

$$\begin{aligned}
&\left|\frac{g(c)+g(dl)}{2} - \frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I_{c^+}^{\theta,k} g(ld) + I_{dl^-}^{\theta,k} g(c)\right]\right| \\
&\quad \le \frac{k(ld-c)^2}{2(\theta+k)}\left(1 - \frac{2k}{\lambda(\theta+k)+k}\right)^{\frac{1}{\lambda}}\left(\frac{1}{\eta+1}\right)^{\frac{1}{\mu}}\left(|g''(c)|^\mu + l\eta|g''(d)|^\mu\right)^{\frac{1}{\mu}} \\
&\quad = k_2,
\end{aligned}$$

where $(\eta, l) \in (0,1] \times (0,1]$.

Proof. From Lemma 1.7, using (η, l)-convexity of $|f''|^q$ and Hölder inequality (Theorem 1.4), we obtain

$$\begin{aligned}
&\left|\frac{g(c)+g(dl)}{2}-\frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I_{c^+}^{\theta,k}g(ld)+I_{dl^-}^{\theta,k}g(c)\right]\right|\\
&\quad\le \frac{k(ld-c)^2}{2(\theta+k)}\left\{\int_0^1\left(1-(1-\tau)^{\frac{\theta}{k}+1}-\tau^{\frac{\theta}{k}+1}\right)g''(c\tau+l(1-\tau)d)d\tau\right\}\\
&\quad\le \frac{k(ld-c)^2}{2(\theta+k)}\left(\int_0^1 1-(1-\tau)^{\lambda(\frac{\theta}{k}+1)}-\tau^{\lambda(\frac{\theta}{k}+1)}d\tau\right)^{\frac{1}{\lambda}}\\
&\qquad\qquad\times\left(\int_0^1|g''(c\tau+l(1-\tau)d)|^{\mu}d\tau\right)^{\frac{1}{\mu}}\\
&\quad\le \frac{k(ld-c)^2}{2(\theta+k)}\left(1-\frac{2k}{\lambda(\theta+k)+k}\right)^{\frac{1}{\lambda}}\left(\frac{1}{\eta+1}\right)^{\frac{1}{\mu}}\left(|g''(c)|^{\mu}+l\eta|g''(d)|^{\mu}\right)^{\frac{1}{\mu}}.
\end{aligned}$$

□

Corollary 2.5. By substituting $\eta=1$ in Theorem 2.4, we get the result for the l-convex function:

$$\begin{aligned}
&\left|\frac{g(c)+g(dl)}{2}-\frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I_{c^+}^{\theta,k}g(ld)+I_{dl^-}^{\theta,k}g(c)\right]\right|\\
&\quad\le \frac{k(ld-c)^2}{2(\theta+k)}\left(1-\frac{2k}{\lambda(\theta+k)+k}\right)^{\frac{1}{\lambda}}\left(\frac{1}{2}\right)^{\frac{1}{\mu}}\left(|g''(c)|^{\mu}+l|g''(d)|^{\mu}\right)^{\frac{1}{\mu}}.
\end{aligned}$$

Corollary 2.6. By substituting $l=1$ in Theorem 2.4, we get the result for the η-convex function:

$$\begin{aligned}
&\left|\frac{g(c)+g(d)}{2}-\frac{\Gamma_k(\theta+k)}{2(d-c)^{\frac{\theta}{k}}}\left[I_{c^+}^{\theta,k}g(d)+I_{d^-}^{\theta,k}g(c)\right]\right|\\
&\quad\le \frac{k(d-c)^2}{2(\theta+k)}\left(1-\frac{2k}{\lambda(\theta+k)+k}\right)^{\frac{1}{\lambda}}\left(\frac{1}{\eta+1}\right)^{\frac{1}{\mu}}\left(|g''(c)|^{\mu}+\eta|g''(d)|^{\mu}\right)^{\frac{1}{\mu}}.
\end{aligned}$$

Remark 2.7. By putting $k=1$ in Theorem 2.4, we get Theorem 3.4 of [16].

Theorem 2.8. *Let g be a twice differentiable function on I, where $c,d\in I$ with $0\le c\le d$ and $g''\in L[c,d]$. If $|g''|^{\mu},\lambda,\mu\ge 1,\frac{1}{\lambda}+\frac{1}{\mu}=1$ is (η,l)-convex* *function and* $\Re(\frac{\theta}{k}) > 0, \theta \neq 0$, *then the inequality given below holds:*

$$\begin{aligned}&\left|\frac{g(c)+g(dl)}{2} - \frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I_{c^+}^{\theta,k}g(ld) + I_{dl^-}^{\theta,k}g(c)\right]\right| \\ &\leq \frac{k(ld-c)^2}{2(\theta+k)}\left\{|g''(c)|^\mu I_1 + l|g''(d)|^q I_2\right\}^{\frac{1}{\mu}} \\ &= k_3,\end{aligned}$$

where $(\eta, l) \in (0,1] \times (0,1]$.

$$I_1 = \frac{1}{\eta+1} - B\left(\eta+1, \left(\frac{\theta}{k}+1\right)q+1\right) - \frac{1}{\left(\frac{\theta}{k}+1\right)q+\eta+1}$$

and

$$I_2 = \frac{\eta}{\eta+1} - \frac{2}{\left(\frac{\theta}{k}+1\right)q+1} + B\left(\eta+1, \left(\frac{\theta}{k}+1\right)\mu+1\right) + \frac{1}{\left(\frac{\theta}{k}+1\right)\mu+\eta+1}.$$

Proof. From Lemma 1.7, using (η, l)-convexity of $|f''|^\mu$ and Hölder inequality (Theorem 1.4), we obtain

$$\begin{aligned}&\left|\frac{g(c)+g(dl)}{2} - \frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I_{c^+}^{\theta,k}g(ld) + I_{dl^-}^{\theta,k}g(c)\right]\right| \\ &\leq \frac{k(ld-c)^2}{2(\theta+k)}\left(\int_0^1 1d\tau\right)^{\frac{1}{\lambda}} \\ &\quad\times\left\{\int_0^1\left(1-(1-\tau)^{\frac{\theta}{k}+1} - \tau^{\frac{\theta}{k}+1}\right)^\mu |g''(c\tau + l(1-\tau)d)|^\mu d\tau\right\}^{\frac{1}{\mu}} \\ &\leq \frac{k(ld-c)^2}{2(\theta+k)} \\ &\quad\times\left\{\int_0^1\left(1-(1-\tau)^{\left(\frac{\theta}{k}+1\right)\mu} - \tau^{\left(\frac{\theta}{k}+1\right)\mu}\right)\left(\tau^\eta|g''(c)|^\mu + l(1-\tau^\eta)|g''(d)|^\mu)\right)d\tau\right\}^{\frac{1}{\mu}} \\ &\leq \frac{k(ld-c)^2}{2(\theta+k)} \\ &\quad\times\left\{|g''(c)|^\mu\left(\int_0^1 \tau^\eta - \tau^\eta(1-\tau)^{\left(\frac{\theta}{k}+1\right)\mu} - \tau^{\left(\frac{\theta}{k}+1\right)\mu+\eta}d\tau\right)\right. \\ &\quad\left.+ l|g''(d)|^\mu\left(\int_0^1(1-\tau^\eta) - (1-\tau^\eta)(1-\tau)^{\left(\frac{\theta}{k}+1\right)\mu} - (1-\tau^\eta)\tau^{\left(\frac{\theta}{k}+1\right)\mu}d\tau\right)\right\}^{\frac{1}{\mu}} \\ &\leq \frac{k(lb-a)^2}{2(\theta+k)}\left\{|g''(c)|^\mu I_1 + l|g''(d)|^\mu I_2\right\}^{\frac{1}{\mu}}.\end{aligned}$$

□

Corollary 2.9. By substituting $\eta = 1$ in Theorem 2.8, we get the result for the l-convex function:

$$\left|\frac{g(c)+g(dl)}{2} - \frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I^{\theta,k}_{c^+}g(ld) + I^{\theta,k}_{dl^-}g(c)\right]\right|$$
$$\leq \frac{k(ld-c)^2}{2(\theta+k)}\left\{|g''(c)|^{\mu}I_3 + l|g''(d)|^{\mu}I_4\right\}^{\frac{1}{\mu}}.$$

$$I_3 = \frac{1}{2} - B\left(2, \left(\frac{\theta}{k}+1\right)\mu+1\right) - \frac{1}{\left(\frac{\theta}{k}+1\right)\mu+2}$$

and

$$I_4 = \frac{1}{2} - B\left(2, \left(\frac{\theta}{k}+1\right)\mu+1\right) - \frac{1}{\left(\frac{\theta}{k}+1\right)\mu+2}.$$

Corollary 2.10. By substituting $l = 1$ in Theorem 2.8, we get the result for the η-convex function:

$$\left|\frac{g(c)+g(d)}{2} - \frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I^{\theta,k}_{c^+}g(d) + I^{\theta,k}_{d^-}g(c)\right]\right|$$
$$\leq \frac{k(d-c)^2}{2(\theta+k)}\left\{|g''(c)|^{\mu}I_5 + |g''(d)|^{\mu}I_6\right\}^{\frac{1}{\mu}}.$$

$$I_5 = \frac{1}{2} - B\left(2, \left(\frac{\theta}{k}+1\right)\mu+1\right) - \frac{1}{\left(\frac{\theta}{k}+1\right)\mu+2}$$

and

$$I_6 = \frac{\eta}{\eta+1} - \frac{2}{\left(\frac{\theta}{k}+1\right)\mu+1} + B\left(\eta+1, \left(\frac{\theta}{k}+1\right)\mu+1\right) + \frac{1}{\left(\frac{\theta}{k}+1\right)\mu+\eta+1}.$$

Remark 2.11. By putting $k = 1$ in Theorem 2.8, we get Theorem 3.6 of [16].

Theorem 2.12. *Let g be a twice differentiable function on I, where $c, d \in I$ with $0 \leq c \leq d$ and $g'' \in L[c,d]$. If $|g''|^{\mu}, \mu > 1$ is (η, l)-convex function and $\Re(\frac{\theta}{k}) > 0, \theta \neq 0$, then the inequality given below holds:*

$$\begin{aligned}&\left|\frac{g(c)+g(dl)}{2}-\frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I_{c^+}^{\theta,k}g(ld)+I_{dl^-}^{\theta,k}g(c)\right]\right|\\&\quad\le\frac{k(ld-c)^2}{2(\theta+k)}\left(\frac{\theta}{\theta+2k}\right)^{1-\frac{1}{\mu}}\times A\\&\quad=k_4,\end{aligned}$$

where

$$\begin{aligned}A=&\left\{\left(\frac{1}{\eta+1}\right)\left(|g''(c)|^\mu+l\eta|g''(d)|^\mu\right)-\left(\frac{2k}{\theta+2k}\right)l|g''(d)|^\mu\right.\\&\left.-\left(|g''(c)|^\mu-l|g''(d)|^\mu\right)\left[B\left(\eta+1,\frac{\theta}{k}+2\right)+\frac{k}{\theta k+\eta k+2}\right]\right\}^{\frac{1}{\mu}}\end{aligned}$$

and $(\eta,l)\in(0,1]\times(0,1]$.

Proof. From Lemma 1.7, using Power mean inequality (Theorem 1.6) and (η,l)-convexity of $|f''|^\mu$, we get

$$\begin{aligned}&\left|\frac{g(c)+g(dl)}{2}-\frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I_{c^+}^{\theta,k}g(ld)+I_{dl^-}^{\theta,k}g(c)\right]\right|\\&\quad\le\frac{k(ld-c)^2}{2(\theta+k)}\left\{\int_0^1\left(1-(1-\tau)^{\frac{\theta}{k}+1}-\tau^{\frac{\theta}{k}+1}\right)g''(c\tau+l(1-\tau)d)d\tau\right\}.\end{aligned}$$

$$\begin{aligned}&\left|\frac{g(c)+g(dl)}{2}-\frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I_{c^+}^{\theta,k}g(ld)+I_{dl^-}^{\theta,k}g(c)\right]\right|\\&\quad\le\frac{k(ld-c)^2}{2(\theta+k)}\left(\int_0^1\left[1-(1-\tau)^{\frac{\theta}{k}+1}-\tau^{\frac{\theta}{k}+1}\right]dt\right)^{1-\frac{1}{\mu}}\\&\quad\times\left(\int_0^1\left(1-(1-\tau)^{\frac{\theta}{k}+1}-\tau^{\frac{\theta}{k}+1}\right)|g''(c\tau+l(1-\tau)d)|^\mu d\tau\right)^{\frac{1}{\mu}}.\end{aligned}\tag{2.1}$$

Now, by solving the first integration of the above inequality, we get

$$\int_0^1\left[1-(1-\tau)^{\frac{\theta}{k}+1}-\tau^{\frac{\theta}{k}+1}\right]d\tau=\frac{\theta}{\theta+2k}.\tag{2.2}$$

Now, by solving the second integration of the above inequality, we get

$$\begin{aligned}
&\int_0^1 \left(1-(1-\tau)^{\frac{\theta}{k}+1}-\tau^{\frac{\theta}{k}+1}\right)|g''(c\tau+l(1-\tau)d)|^{\mu}d\tau \\
&\quad=\left(\frac{1}{\eta+1}\right)\left(|g''(c)|^{\mu}+l\eta|g''(d)|^{\mu}\right)-\left(\frac{2k}{\theta+2k}\right)l|g''(d)|^{\mu} \\
&\qquad-\left(|g''(c)|^{\mu}-l|g''(d)|^{\mu}\right)\left[B\left(\eta+1,\frac{\theta}{k}+2\right)+\frac{k}{\theta k+\eta k+2}\right]. \qquad (2.3)
\end{aligned}$$

By substituting values of integration from (2.2), (2.3) in (2.1), we get the required inequality. □

Corollary 2.13. By substituting $\eta = 1$ in Theorem 2.12, we get the result for the l-convex function:

$$\begin{aligned}
&\left|\frac{g(c)+g(dl)}{2}-\frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I^{\theta,k}_{c^+}g(ld)+I^{\theta,k}_{dl^-}g(c)\right]\right| \\
&\quad\le\frac{k(ld-c)^2}{2(\theta+k)}\left(\frac{\theta}{\theta+2k}\right)^{1-\frac{1}{\mu}} \\
&\qquad\times\left\{\frac{|g''(c)|^{\mu}+l|g''(d)|^{\mu}}{2}-\left(\frac{2k}{\theta+2k}\right)l|g''(d)|^{\mu}\right. \\
&\qquad\quad\left.-\left(|g''(c)|^{\mu}-l|g''(d)|^{\mu}\right)\left[B\left(2,\frac{\theta}{k}+2\right)+\frac{k}{\theta k+k+2}\right]\right\}^{\frac{1}{\mu}}.
\end{aligned}$$

Corollary 2.14. By substituting $l = 1$ in Theorem 2.12, we get the result for the η-convex function:

$$\begin{aligned}
&\left|\frac{g(c)+g(d)}{2}-\frac{\Gamma_k(\theta+k)}{2(d-c)^{\frac{\theta}{k}}}\left[I^{\theta,k}_{c^+}g(d)+I^{\theta,k}_{d^-}g(c)\right]\right| \\
&\quad\le\frac{k(d-c)^2}{2(\theta+k)}\left(\frac{\theta}{\theta+2k}\right)^{1-\frac{1}{\mu}} \\
&\qquad\times\left\{\left(\frac{1}{\eta+1}\right)\left(|g''(c)|^{\mu}+\eta|g''(d)|^{\mu}\right)-\left(\frac{2k}{\theta+2k}\right)|f''(d)|^{\mu}\right. \\
&\qquad\quad\left.-\left(|g''(c)|^{\mu}-|g''(d)|^{\mu}\right)\left[B\left(\eta+1,\frac{\theta}{k}+2\right)+\frac{k}{\theta k+\eta k+2}\right]\right\}^{\frac{1}{\mu}}.
\end{aligned}$$

Remark 2.15. By putting $k = 1$ in Theorem 2.12, we get Theorem 3.2 of [16].

Theorem 2.16. *Let f be a twice differentiable mapping on I, where $c, d \in I$ with $0 \leq c \leq d$ and $g'' \in L[c,d]$. If $|g''|^{\mu}, \lambda > 1, \frac{1}{\lambda} + \frac{1}{\mu} = 1$ is (η, l)-convex function with $(\eta, l) \in (0,1] \times (0,1]$ and $\Re(\frac{\theta}{k}) > 0, \theta \neq 0$, then the inequality given below holds:*

$$\begin{aligned}
&\left|\frac{g(c)+g(dl)}{2} - \frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I_{c^+}^{\theta,k} g(ld) + I_{dl^-}^{\theta,k} g(c)\right]\right| \\
&\quad \leq \frac{k(ld-c)^2}{2(\theta+k)}\left(\frac{1}{2} - B\left(\lambda\left(\frac{\theta}{k}+1\right)+1, 2\right) - \frac{k}{\lambda(\theta+k)+2k}\right)^{\frac{1}{\lambda}} \\
&\qquad \times \left\{\left(|g''(c)|^{\mu} B(\eta+1,2) + |g''(dl)|^{\mu}\left[\frac{1}{2} - \frac{1}{(\eta+1)(\eta+2)}\right]\right)^{\frac{1}{\mu}}\right. \\
&\qquad\qquad \left. + \left(\left(|g''(c)|^{\mu} - l|g''(d)|^{\mu}\right)\left(\frac{1}{\eta+2}\right) + \frac{l|g''(d)|^{\mu}}{2}\right)^{\frac{1}{\mu}}\right\} \\
&\quad = k_5.
\end{aligned}$$

Proof. From Lemma 1.7, using Hölder İşcan Integral inequality (Theorem 1.5) and (η, l)-convexity of $|f''|^{\mu}$, we obtain

$$\begin{aligned}
&\left|\frac{g(c)+g(dl)}{2} - \frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I_{c^+}^{\theta,k} g(ld) + I_{bl^-}^{\theta,k} g(c)\right]\right| \\
&\leq \frac{k(ld-c)^2}{2(\theta+k)} (I_7)^{\frac{1}{\lambda}} (I_8)^{\frac{1}{\mu}} + (I_9)^{\frac{1}{\lambda}} (I_{10})^{\frac{1}{\mu}} \qquad (2.4) \\
&\leq \frac{k(ld-c)^2}{2(\theta+k)}\left(\frac{1}{2} - B\left(\lambda\left(\frac{\theta}{k}+1\right)+1, 2\right) - \frac{k}{\lambda(\theta+k)+2k}\right)^{\frac{1}{\lambda}} \\
&\quad \times \left\{\left(\int_0^1 (1-\tau)\left[\tau^{\eta}|g''(c)|^{\mu} + (1-\tau^{\eta}) l |g''(d)|^{\mu}\right] d\tau\right)^{\frac{1}{\mu}}\right. \\
&\qquad \left. + \left(\int_0^1 \tau\left[\tau^{\eta}|g''(c)|^{\mu} + (1-\tau^{\eta}) l |g''(d)|^{\mu}\right] d\tau\right)^{\frac{1}{\mu}}\right\}, \qquad (2.5)
\end{aligned}$$

where

$$I_7 = \int_0^1 (1-\tau)\left[1 - (1-\tau)^{\lambda\left(\frac{\theta}{k}+1\right)} - \tau^{\lambda\left(\frac{\theta}{k}+1\right)}\right] d\tau,$$

$$I_8 = \int_0^1 (1-\tau)|g''(c\tau + l(1-\tau)d)|^{\mu} d\tau,$$

$$I_9 = \int_0^1 \tau \left[1 - (1-\tau)^{\frac{\theta}{k}+1} - \tau^{\frac{\theta}{k}+1}\right] d\tau$$

and

$$I_{10} = \int_0^1 \tau |g''(c\tau + l(1-\tau)d)|^{\mu} d\tau.$$

Now, by evaluating integrals, we get

$$\int_0^1 (1-\tau)\left[\tau^{\eta}|\tau''(c)|^{\mu} + (1-\tau^{\eta})l|g''(d)|^{\mu}\right] d\tau$$
$$= |g''(c)|^{\mu} B(\eta+1,2) + l|g''(d)|^{\mu}\left(\frac{1}{2} - \frac{1}{(\eta+1)(\eta+2)}\right). \tag{2.6}$$

$$\int_0^1 \tau \left[\tau^{\eta}|g''(c)|^{\mu} + (1-\tau^{\eta})|g''(ld)|^{\mu}\right] d\tau$$
$$= \left(|g''(c)|^{\mu} - l|g''(d)|^{\mu}\right)\left(\frac{1}{\eta+2}\right) + \frac{l|g''(d)|^{\mu}}{2}. \tag{2.7}$$

Now, by using the values of the integrals of (2.6) and (2.7) in (2.4), we get the required inequality. □

Corollary 2.17. By substituting $\eta = 1$ in Theorem 2.16, we get the result for the l-convex function:

$$\left|\frac{g(c)+g(dl)}{2} - \frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}}\left[I_{c^+}^{\theta,k} g(ld) + I_{dl^-}^{\theta,k} g(c)\right]\right|$$
$$\leq \frac{k(ld-c)^2}{2(\theta+k)}\left(\frac{1}{2} - B\left(\lambda\left(\frac{\theta}{k}+1\right)+1,2\right) - \frac{k}{\lambda(\theta+k)+2k}\right)^{\frac{1}{\lambda}}$$
$$\times \left\{\left(\frac{|g''(c)|^{\mu}}{6} + \frac{|g''(dl)|^{\mu}}{3}\right)^{\frac{1}{\mu}} + \left(\frac{|g''(c)|^{\mu}}{3} + \frac{l|g''(d)|^{\mu}}{6}\right)^{\frac{1}{\mu}}\right\}.$$

Corollary 2.18. By substituting $l = 1$ in Theorem 2.16, we get the result for the η-convex function:

$$\left|\frac{g(c)+g(d)}{2} - \frac{\Gamma_k(\theta+k)}{2(d-c)^{\frac{\theta}{k}}}\left[I_{c^+}^{\theta,k} g(d) + I_{d^-}^{\theta,k} g(c)\right]\right|$$
$$\leq \frac{k(d-c)^2}{2(\theta+k)}\left(\frac{1}{2} - B\left(\lambda\left(\frac{\theta}{k}+1\right)+1,2\right) - \frac{k}{\lambda(\theta+k)+2k}\right)^{\frac{1}{\lambda}} \times B.$$

where

$$B = \left\{ \left(|g''(c)|^\mu B(\eta+1,2) + |g''(d)|^\mu \left[\frac{1}{2} - \frac{1}{(\eta+1)(\eta+2)} \right] \right)^{\frac{1}{\mu}} + \left((|g''(c)|^\mu - |g''(d)|^\mu) \left(\frac{1}{\eta+2} \right) + \frac{|g''(d)|^\mu}{2} \right)^{\frac{1}{\mu}} \right\}.$$

Remark 2.19. By theorems (2.1), (2.4), (2.8), (2.12) and (2.16), we obtain

$$\left| \frac{g(c)+g(dl)}{2} - \frac{\Gamma_k(\theta+k)}{2(ld-c)^{\frac{\theta}{k}}} \left[I^{\theta,k}_{c^+} g(ld) + I^{\theta,k}_{dl^-} g(c) \right] \right| \leq min\{k_1, k_2, k_3, k_4, k_5\}.$$

3. Conclusion

This study aims to prove Hermite Hadamard-type inequalities for k-Riemann-Liouville fractional integrals using a (η, l)-convex function. Several earlier results can be obtained as specific cases of our proofs.

Acknowledgement: The authors are grateful to the Editor and Reviewers for corrections and suggestions which led to the improvement of the paper. The first Author is thankful to Council of Scientific and Industrial Research(CSIR), India for financial assistance in the form of Junior Research Fellowship (File no: 09/1007(0010)/2020-EMR-I).

References

[1] Alomari, M., Darus, M., Kirmaci, U. S., Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, *Comput. Math. with Appl.*, **59**(1)(2010), 225-232.

[2] Ermeydan, S., Yildirim, H., Riemann-Liouville fractional Hermite-Hadamard inequalities for differentiable $\lambda\phi$-preinvex functions, *Malaya J. Mat*, **4**(3) (2016), 430-437.

[3] Hudzik, H., Maligranda, L., Some remarks ons-convex functions, *Aequationes Math.*, **48**(1) (1994), 100-111.

[4] İşcan, İ., Turhan, S., Generalized Hermite-Hadamard-Fejer type inequalities for GA-convex functions via Fractional integral, *MJPAA*, **2**(1) (2016), 1-13.

[5] İşcan, İ., New refinements for integral and sum forms of Hölder inequality, *J. Inequal. Appl.*, **2019**(1) (2019), 1-11.

[6] Kadakal, M., İşcan, İ., Kadakal, H., Bekar, K., On improvements of some integral inequalities. *Researchgate*, (2019) DOI, 10, 13140.

[7] Mangasarian, O. L., Pseudo-convex functions, *In Stochastic optimization models in finance* , Academic Press, (1975), 23-32.

[8] Mihesan, V. G., A generalization of the convexity. In Seminar on Functional Equations, *Approx. and Convex.*, Cluj-Napoca, Romania (1993).

[9] Mitrinović, D. S., Lacković, I. B., Hermite and convexity, *Aequ. Math.* ´ **28**(1985), 229-232.

[10] Mitrinovic, D. S., Pecaric, J., Fink A. M., *Classical and new inequalities in analysis*, **61**, Springer Science and Business Media, (2013).

[11] Mohammed, P. O., Some new Hermite-Hadamard type inequalities for MT-convex functions on differentiable coordinates, *J. King Saud Univ. Sci.*, **30**(2) (2018), 258-262.

[12] Mubeen, S., Habibullah, G. M., k-Fractional integrals and application, *Int. J. Contemp. Math. Sci.*, **7**(2)(2012), 89-94.

[13] Noor, M. A., Noor, K. I., Awan, M. U., Some quantum estimates for Hermite-Hadamard inequalities, *Comput. Appl. Math.*, **251**(2015), 675-679.

[14] Sahoo, S. K., Ahmad, H., Tariq, M., Kodamasingh, B., Aydi, H., De la Sen, M., Hermite–Hadamard type inequalities involving k-fractional operator for (h, m)-convex Functions, *Symmetry*, **13**(9)(2021), 1686.

[15] Varošanec, S., On h-convexity, *J. Math. Anal. Appl.*, **326**(1) (2007), 303-311.

[16] Wang, J., Li, X., Fe [cbreve] kan, M., Zhou, Y., Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity, *Appl. Anal.*, **92**(11)(2013), 2241-2253.

[17] Wu, Y., Qi, F., On some Hermite-Hadamard type inequalities for (s, QC)-convex functions, *SpringerPlus*, **5**(1)(2016), 1-13.

Jeet B. Gajera
Department of Mathematics,
Sardar Vallabhbhai National Institute of Technology,
395007, Surat.
E-mail: jeetgajera59@gmail.com

Ranjan Kumar Jana
Department of Mathematics,
Sardar Vallabhbhai National Institute of Technology,
395007, Surat.
E-mail: rkjana2003@yahoo.com

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July-December (2024), 161–169

A NEW PROOF FOR FRAME'S ${}_3F_2\left(-\frac{1}{4}\right)$ SERIES

NANDAN SAI DASIREDDY
(Received : 12 - 02 - 2024 ; Revised : 08 - 10 - 2024)

Abstract. The June 1963 issue of the American Mathematical Monthly included the following hypergeometric ${}_3F_2\left(-\frac{1}{4}\right)$ series, proposed by J. S. Frame.

$$S = {}_3F_2\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{array} \middle| -\frac{1}{4}\right).$$

We furnish a concise solution by establishing the following integral repersentation of S, that differs from that presented in the official journal documents.

$$S = 4\int_0^1 \frac{\ln^2(x)}{(x^2+4)^{\frac{3}{2}}}dx = \frac{3}{5}\zeta(2).$$

1. Introduction

Let's fix some nomenclature. The set of all non-negative integers is $\mathbb{N}$. The set of all positive integers is $\mathbb{Z}^+$ and the set of all complex numbers is $\mathbb{C}$. Denote the rising factorial (sometimes called also Pochhammer symbol, or shifted factorial) by

$$(a)_n = \begin{cases} a(a+1)\cdots(a+n-1), & n \in \mathbb{Z}^+, \\ 1, & n = 0, \end{cases}$$

for all complex a, the classical hypergeometric series of type for ${}_rF_s$ (see Bailey 1935) by

$${}_rF_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} \middle| \lambda\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_r)_n}{n!(b_1)_n(b_2)_n\cdots(b_s)_n}.\lambda^n,$$

2020 Mathematics Subject Classification: Primary 05A10, 33C20, 40A25; Secondary 11M06
Key words and phrases: Pochhammer symbol, hypergeometric series, dilogarithm function, Riemann Zeta function, central binomial coefficients, infinite summation formulas

which converges for $|\lambda| < 1$, $r, s \in \mathbb{N}$, for $a_1, \ldots, a_r, b_1, \ldots, b_s \in \mathbb{C}$ with $b_1, \ldots, b_s \notin \{\ldots, -3, -2, -1, 0\}$, Central binomial coefficient by $\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = 2^{2n}\frac{\left(\frac{1}{2}\right)_n}{n!} = 2^{2n}\frac{\left(\frac{1}{2}\right)_n}{(1)_n}$, for $n \geq 1$, Riemann zeta function by $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$, for $\Re(s) > 1$, hyperbolic sine by $\sinh(x) = \frac{e^x - e^{-x}}{2}$, for $-\infty < x < \infty$, inverse hyperbolic sine (see Zwillinger 1995, §6.8 for example) by $\sinh^{-1}(x) = \ln\left(x + \sqrt{1+x^2}\right)$, for $-\infty < x < \infty$, and the classical dilogarithm function by the power series $\mathrm{Li}_2(x) = \sum_{k=1}^{\infty}\frac{x^k}{k^2}$, which converges for all complex values of x with $|x| \leq 1$, with the closed-form expression for $\mathrm{Li}_2\left(-\left(\frac{1+\sqrt{5}}{2}\right)\right)$ being given in [4], [5], [6, Equation (6.8)] and [10, p. 7]:

$$\mathrm{Li}_2\left(-\left(\frac{1+\sqrt{5}}{2}\right)\right) = -\frac{3}{5}\zeta(2) - \ln^2\left(\frac{1+\sqrt{5}}{2}\right).$$

Note. In all Refs. [4], [5], [6, Equation (6.8)] and [10, p. 7], the expression for $\mathrm{Li}_2\left(-\left(\frac{1+\sqrt{5}}{2}\right)\right)$ has a typo, the correct one being
$$\mathrm{Li}_2\left(-\left(\frac{1+\sqrt{5}}{2}\right)\right) = -\frac{3}{5}\zeta(2) - \ln^2\left(\frac{1+\sqrt{5}}{2}\right).$$

The impetus for this paper comes from the alternative solution of (1.1) in the recent *Mathematics Student* article [9, pp. 117-119].

Problem 5113 in the June 1963 issue of American Mathematical Monthly journal included the following hypergeometric ${}_3F_2\left(-\frac{1}{4}\right)$ series, proposed by J. S. Frame [2, p. 672].

$$S = {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\\ \frac{3}{2}, \frac{3}{2}\end{matrix}\,\middle|\, -\frac{1}{4}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{2}\right)_n}{2^{2n}(1)_n(2n+1)^2}. \tag{1.1}$$

Subsequently, a solution due to A. Weinmann involving the use of integral formula $\int_0^{\infty} e^{-(2n+1)t}t\,dt = (2n+1)^{-2}$ appeared in the June 1964 issue of American Mathematical Monthly journal [3, pp. 691-692]. X. F. Han and

C. P. Chen [8] provided a simpler and more elegant solution of (1.1) based on the calculation of the logarithmic integral $\int_0^1 \frac{1}{x}\ln\left(\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}\right)dx$.

The purpose of our work here is to construct a new approach to (1.1), which solely relies on the calculation of the following integral:

$$I=\int_0^1 \frac{\ln^2(x)}{(x^2+4)^{\frac{3}{2}}}dx. \tag{1.2}$$

1.1. **Organization of the article.**

We proceed according to what is outlined as follows:

- In Section 2, we offer an evaluation of an integral involving $\zeta(2)$, including an especially nontrivial evaluation for the particularly difficult integral $\int_0^{\ln\left(\frac{1+\sqrt{5}}{2}\right)} \ln(\sinh u)du$;
- Finally, in Section 3, we prove an evaluation for the series $\sum_{n=0}^{\infty}\binom{2n}{n}x^n(2n+1)$ and we make use of the integral evaluated in Section 2, to rigorously prove an explicit evaluation for the difficult Frame's hypergeometric ${}_3F_2\left(-\frac{1}{4}\right)$ series.

2. A DERIVATION OF THE INTEGRAL IN (1.2)

In this section, we prove the integral in (1.2) $I=\frac{3}{20}\zeta(2)$. Based on this result, we provide a proof of the Frame's series.

Theorem 2.1. *The following integral repersentation for the constant $\zeta(2)$ holds:*

$$I=\int_0^1 \frac{\ln^2(x)}{(x^2+4)^{\frac{3}{2}}}dx=\frac{3}{20}\zeta(2).$$

Proof. For ease of understanding, we divide the proof into three steps.

Step One: First, we evaluate the indefinite integral: $\int \frac{1}{(x^2+4)^{\frac{3}{2}}}dx$.

Thanks to the substitution $x=2\tan t$ that we apply to this integral, so that $\frac{dx}{dt}=2\sec^2 t$. It follows, that

$$\int \frac{1}{(x^2+4)^{\frac{3}{2}}}dx = \frac{1}{4}\int \cos t dt = \frac{1}{4}\sin t + K = \frac{x}{4\sqrt{x^2+4}} + K.$$

where, K is the constant of integration.

Step Two: Second, we evaluate the definite integral:
$$\int_0^{\ln\left(\frac{1+\sqrt{5}}{2}\right)} \ln(\sinh u)du.$$

We substitute u by $\ln v$ to get $du = \dfrac{dv}{v}$ and rewrite the integral as:

$$\begin{aligned}
\int_0^{\ln\left(\frac{1+\sqrt{5}}{2}\right)} \ln(\sinh u)du &= \int_0^{\ln\left(\frac{1+\sqrt{5}}{2}\right)} \ln\left(e^u - e^{-u}\right)du - \ln 2\ln\left(\frac{1+\sqrt{5}}{2}\right)\\
&= \int_1^{\frac{1+\sqrt{5}}{2}} \frac{\ln\left(v^2-1\right)}{v}dv - \int_1^{\frac{1+\sqrt{5}}{2}} \frac{\ln(v)}{v}dv - \ln 2\ln\left(\frac{1+\sqrt{5}}{2}\right)\\
&= \int_1^{\frac{1+\sqrt{5}}{2}} \frac{\ln\left(v^2-1\right)}{v}dv - \int_0^{\ln\left(\frac{1+\sqrt{5}}{2}\right)} udu - \ln 2\ln\left(\frac{1+\sqrt{5}}{2}\right)\\
&= \int_1^{\frac{1+\sqrt{5}}{2}} \frac{\ln\left(v^2-1\right)}{v}dv - \frac{1}{2}\ln^2\left(\frac{1+\sqrt{5}}{2}\right) - \ln 2\ln\left(\frac{1+\sqrt{5}}{2}\right).
\end{aligned}$$

Now, in the first integral on the right side, we make the substitution: $v^2 = r$ so that $\dfrac{dr}{dv} = 2\sqrt{r}$. It follows, that

$$\int_0^{\ln\left(\frac{1+\sqrt{5}}{2}\right)} \ln(\sinh u)du = \frac{1}{2}\int_1^{\left(\frac{1+\sqrt{5}}{2}\right)^2} \frac{\ln(r-1)}{r}dr - \frac{1}{2}\ln^2\left(\frac{1+\sqrt{5}}{2}\right) - \ln 2\ln\left(\frac{1+\sqrt{5}}{2}\right). \tag{2.1}$$

Making the change of variable $r = s+1$ in the first integral on the right side of (2.1), so that $dr = ds$ and then using integration by parts, gives,

$$\int_0^{\ln\left(\frac{1+\sqrt{5}}{2}\right)} \ln(\sinh u)du = \frac{1}{2}\int_0^{\left(\frac{1+\sqrt{5}}{2}\right)^2-1} \frac{\ln s}{s+1}ds - \frac{1}{2}\ln^2\left(\frac{1+\sqrt{5}}{2}\right) - \ln 2\ln\left(\frac{1+\sqrt{5}}{2}\right)$$

$$= \frac{1}{2}\left(\left[\ln(s)\ln(s+1)\right]_0^{\left(\frac{1+\sqrt{5}}{2}\right)^2-1} - \int_0^{\left(\frac{1+\sqrt{5}}{2}\right)^2-1} \frac{\ln(s+1)}{s}ds\right) -$$

$$\frac{1}{2}\ln^2\left(\frac{1+\sqrt{5}}{2}\right) - \ln 2\ln\left(\frac{1+\sqrt{5}}{2}\right)$$

$$= \frac{1}{2}\left[\mathrm{Li}_2(-s)\right]_0^{\left(\frac{1+\sqrt{5}}{2}\right)^2-1} + \frac{1}{2}\ln^2\left(\frac{1+\sqrt{5}}{2}\right) - \ln 2\ln\left(\frac{1+\sqrt{5}}{2}\right)$$

$$= \frac{1}{2}\mathrm{Li}_2\left(-\left(\frac{1+\sqrt{5}}{2}\right)\right) + \frac{1}{2}\ln^2\left(\frac{1+\sqrt{5}}{2}\right) - \ln 2\ln\left(\frac{1+\sqrt{5}}{2}\right)$$

$$= -\frac{3}{10}\zeta(2) - \ln 2\ln\left(\frac{1+\sqrt{5}}{2}\right).$$

Step Three: Lastly, we evaluate the definite integral: $\displaystyle\int_0^1 \frac{\ln^2(x)}{(x^2+4)^{\frac{3}{2}}}dx$.

We have, using integration by parts, that

$$\int_0^1 \frac{\ln^2(x)}{(x^2+4)^{\frac{3}{2}}}dx = \left[\frac{x\ln^2(x)}{4\sqrt{x^2+4}}\right]_0^1 - \frac{1}{2}\int_0^1 \frac{\ln(x)}{\sqrt{x^2+4}}dx$$

$$= -\frac{1}{2}\int_0^1 \frac{\ln(x)}{\sqrt{x^2+4}}dx.$$

Now, in the integral $\displaystyle\int_0^1 \frac{\ln(x)}{\sqrt{x^2+4}}dx$, we make the substitution $x = 2\sinh u$ so that $\dfrac{dx}{du} = 2\cosh u$. Then,

$$-\frac{1}{2}\int_0^1 \frac{\ln(x)}{\sqrt{x^2+4}}dx = -\frac{1}{2}\int_0^{\sinh^{-1}\left(\frac{1}{2}\right)} \ln(2\sinh u)du$$

$$= -\frac{1}{2}\ln(2)\ln\left(\frac{1+\sqrt{5}}{2}\right) - \frac{1}{2}\int_0^{\ln\left(\frac{1+\sqrt{5}}{2}\right)} \ln(\sinh u)du$$

$$= -\frac{1}{2}\ln(2)\ln\left(\frac{1+\sqrt{5}}{2}\right) -$$

$$\frac{1}{2}\left(-\frac{3}{10}\zeta(2) - \ln 2\ln\left(\frac{1+\sqrt{5}}{2}\right)\right)$$

$$= \frac{3}{20}\zeta(2).$$

The proof is complete.

□

3. A proof of the Frame's ${}_3F_2\left(-\frac{1}{4}\right)$ series

Theorem 3.1. *Frame's* ${}_3F_2\left(-\frac{1}{4}\right)$ *series in* (1.1) *admits the evaluation*

$$\frac{3}{5}\zeta\left(2\right).$$

Proof. For ease of understanding, we divide the proof into three steps.

Step One: First, we evaluate the general definite integral:
$\displaystyle\int_0^1 x^{2n}\ln^2\left(x\right)dx.$

We have, using integration by parts, that

$$\begin{aligned}\int_0^1 x^{2n}\ln^2\left(x\right)dx &= \left[\frac{x^{2n+1}\ln^2\left(x\right)}{2n+1}\right]_0^1 - \frac{2}{2n+1}\int_0^1 \ln(x)x^{2n}dx\\ &= -\frac{2}{2n+1}\int_0^1 \ln(x)x^{2n}dx.\end{aligned}$$

Applying integration by parts for the second time, we get,

$$\begin{aligned}\int_0^1 x^{2n}\ln^2\left(x\right)dx &= -\frac{2}{2n+1}\left(\left[\frac{x^{2n+1}\ln\left(x\right)}{2n+1}\right]_0^1 - \frac{1}{2n+1}\int_0^1 x^{2n}dx\right)\\ &= -\frac{2}{2n+1}\left(-\frac{1}{2n+1}\left[\frac{x^{2n+1}}{2n+1}\right]_0^1\right)\\ &= \frac{2}{\left(2n+1\right)^3}.\end{aligned}$$

Remark. The above general definite integral formula
$\displaystyle\int_0^1 x^{2n}\ln^2\left(x\right)dx = \frac{2}{\left(2n+1\right)^3}$ holds only for $n > -\dfrac{1}{2}$.

Step Two: Second, we evaluate the infinite series involving the central binomial coefficient : $\displaystyle\sum_{n=0}^{\infty}\binom{2n}{n}x^n\left(2n+1\right).$

Invoking the well-known generating function of the central binomial coefficient, $\displaystyle\sum_{n=0}^{\infty}\binom{2n}{n}k^n = \frac{1}{\sqrt{1-4k}}, \quad |k| < \frac{1}{4}.$ we have,

$$\sum_{n=0}^{\infty}\binom{2n}{n}x^{2n}=\frac{1}{\sqrt{1-4x^2}}. \tag{3.1}$$

Multiplying by x on both sides of (3.1) gives,

$$\sum_{n=0}^{\infty}\binom{2n}{n}x^{2n+1}=\frac{x}{\sqrt{1-4x^2}}. \tag{3.2}$$

Differentiating both sides of (3.2) with respect to x by means of the power rule, we obtain,

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty}\binom{2n}{n}x^{2n+1}\right)=\frac{d}{dx}\left(\frac{x}{\sqrt{1-4x^2}}\right) \quad \text{or}$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}x^{2n}\left(2n+1\right)=\frac{\sqrt{1-4x^2}+\dfrac{4x^2}{\sqrt{1-4x^2}}}{1-4x^2}=\frac{1}{\left(1-4x^2\right)^{\frac{3}{2}}}. \tag{3.3}$$

Replacing x with $\sqrt{x}$ on both sides of (3.3) gives,

$$\sum_{n=0}^{\infty}\binom{2n}{n}x^{n}\left(2n+1\right)=\frac{1}{\left(1-4x\right)^{\frac{3}{2}}}.$$

Step Three: Lastly, we evaluate the sum: $S=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{2}\right)_n}{2^{2n}\left(1\right)_n\left(2n+1\right)^2}$.

$$\begin{aligned}
S=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{2}\right)_n}{2^{2n}\left(1\right)_n\left(2n+1\right)^2}&=\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}}{2^{4n}\left(2n+1\right)^2}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}\left(2n+1\right)}{2^{4n}\left(2n+1\right)^3}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}\left(2n+1\right)}{2^{4n}}\left(\frac{1}{2}\int_0^1 x^{2n}\ln^2(x)\,dx\right)\\
&=\frac{1}{2}\left[\int_0^1\ln^2(x)\left(\sum_{n=0}^{\infty}\binom{2n}{n}\left(2n+1\right)\left(\frac{-x^2}{16}\right)^n\right)dx\right]\\
&=4\int_0^1\frac{\ln^2(x)}{\left(x^2+4\right)^{\frac{3}{2}}}dx\\
&=4\left(\frac{3}{20}\zeta(2)\right)=\frac{3}{5}\zeta(2).
\end{aligned}$$

Here, in Step Three, Since the partial sums of the series are bounded in absolute value by 1, the dominated convergence theorem justifies interchanging the order of summation and integration.

The proof is complete.

□

Acknowledgement: The author wishes to express his sincerest thanks to the referee of the Mathematics Student Journal for a very careful and meticulous proofreading of a previous version of this article and for a number of insightful comments concerning the English corrections. In particular, the author appreciates the suggestion of an alternative title for the paper.

References

[1] Bailey, W. N., Generalized Hypergeometric Series, Cambridge Tracts in Mathematics and Mathematical Physics. *Cambridge Univ. Press,* Cambridge, 1935. Zbl: 0011.02303.

[2] Frame, J. S., Advanced problem 5113, *Amer. Math. Monthly* **70**(6) (1963), 672–679. doi: 10.1080/00029890.1963.11992084.

[3] Weinmann, A., Sum of an Infinite Series, Solution to problem 5113, *Amer. Math. Monthly* **71**(6) (1964), 689–696. doi: 10.1080/00029890.1964.11992300.

[4] Zagier, D., The remarkable dilogarithm, *J. Math. Phys. Sci.* **22**(1) (1988), 131–145. MR: 0940391. Zbl: 0669.33001.

[5] Zagier, D., The dilogarithm function in geometry and number theory, Number Theory and Related Topics, Papers presented at the Ramanujan Colloquium, (Bombay, 1988), Tata Inst. Fund. Res. Stud. Math., Vol. 12, *Tata Inst. Fund. Res. and Oxford University Press,* 1989, pp. 231-249. Available from: https://www.math.tifr.res.in/~publ/studies/Number-Theory-And-Related-Topics.pdf. Zbl: 0744.33011.

[6] Maximon, L. C., The dilogarithm function for complex argument, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **459**(2039) (2003), 2807–2819. doi: 10.1098/rspa.2003.1156. MR: 2015991. Zbl: 1050.33002.

[7] Zwillinger, D. (ed.), CRC Standard Mathematical Tables and Formulae, 30th Ed., *CRC Press,* Boca Raton, FL, 1995. Zbl: 1379.00008.

[8] Han, X. F., and Chen, C. P., Some properties of the remainders in certain series representations for the constant π, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **116**(3) (2022), Paper No. 123, 20 pp. doi: 10.1007/s13398-022-01271-z. Zbl: 1496.11141.

[9] Nimbran, A. S., and Levrie, P., Some series involving the central binomial coefficient with sums containing $\pi^2/5$, *Math. Student* **88**(1-2) (2019), 117–124. Available from: https://indianmathsoc.org/ms/mathstudent-part-1-2019.pdf.

[10] Zagier, D., The Dilogarithm Function, Frontiers in Number Theory, Physics, and Geometry II-On Conformal Field Theories, Discrete Groups and Renormalization, Springer, Berlin–Heidelberg, 2007, pp. 3–65. doi: 10.1007/978-3-540-30308-4_1. MR: 2290758. Zbl: 1176.11026.

Nandan Sai Dasireddy
House No. 3-11-363, Pavani Plaza, Flat No:401,
Road No. 2, Shivaganga Colony, Siris Road,
L.B. Nagar, Hyderabad, Telangana, India.
E-mail: dasireddy.1818@gmail.com

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July-December (2024), 170–178

SOME SERIES AND INTEGRAL REPRESENTATIONS FOR THE COMPUTATION OF THE MITTAG-LEFFLER FUNCTION E_α(Z)

SUJIT K BOSE

(Received : 17 - 02 - 2024 ; Revised : 14 - 06 - 2024)

Abstract. Motivated by a recent finding (Bose [2]) that delay in fiber-optic data transmission can be modeled in terms of the Mittag-Leffler function $E_\alpha(x)$, where $\alpha(< 1)$ and x is real and positive, the problem of computation of the function is studied here by representing it as simple integrals, when $0 < \alpha < 2$. Without loss of applicability, the method is developed for complex z, instead of real x. In this process, some representations of $E_\alpha(z)$ are obtained in terms of the incomplete Gamma functions, convergent series, and asymptotic series for $z \to \infty$. The alternate case $\alpha \geq 1$ presents no difficulty as the analytic series defining $E_\alpha(z)$ can be directly employed for the computation. The method presented here is simpler compared to that of Gorenflo et. al. [9], who treated it under greater generality.

1. Introduction

The special function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \ \ z \in C \tag{1}$$

where C denotes the set of complex numbers is called the Mittag-Leffler function (Erdélyi et. al. [6], section 18.1, p.206). The function was introduced by Mittag-Leffler [15] in connection with his method of summation of divergent series. Certain properties of the function were also investigated by him in the next few years including a contour integral representation of the function. The function was found to possess some basic properties

2010 Mathematics Subject Classification: 33E12, 33B20
Key words and phrases: Mittag-Leffler function, integral representations, incomplete Gamma functions, series expansion,;asymptotic series

in the category of entire functions, holomorphic in the entire complex z-plane. The function was subsequently generalized by Wiman [17] in which the Gamma function in the series (1) is replaced by $\Gamma(\alpha n + \beta)$.

Interest in the functions of Mittag-Leffler type has grown in the recent past because of its close connection to the solution of fractional differential equations and integral equations of Abel type (Hille and Tamarkin [12], Samko et. al. [16], Gorenflo and Mainardi [7]). Such equations crop up in modeling a large variety of physical processes appearing in different applications (Gorenflo and Mainardi [8], Mainardi [14], Hilfer [11]). A recent survey article by Houbold, Mathai and Saxena [13] lists exhaustive references related to these topics and much more.

A different kind of application of the Mittag-Leffler function was found by Conway and Maxwell [5] in the theory of queues. A queue consists of persons arriving randomly at a service station for some service. When the operating discipline of "first come first serve" (FCFS), it is known that the arrivals have a Poisson probability distribution, and servicing an exponential distribution (Bose [3]). However, if there is some overcrowding in servicing, Conway and Maxwell [5] show that the state dependent servicing distribution can be modeled in terms of the function $E_\alpha(x)$, where x is real. In a similar manner, Bose [2] has shown that "packet" transmission of data in fiber-optic cables can be delayed following the same probability distribution. Chakraborty and Ong [4] have devoted a paper on such Mittag-Leffler function distribution (MLFD).

Due to these developments, the question of computing $E_\alpha(x)$, $x > 0$, or more generally $E_\alpha(z)$ ($z \in C$), is treated in this paper. The computational problem has been treated by Gorenflo et. al. [9] in a more general manner, but it is shown in this paper that with the limited objective of computing $E_\alpha(z)$, the procedure is quite simple. Firstly it is noted that the series (1) can be easily summed if $\alpha \geq 1$. There arises some difficulty of lack of convergence of the series when $0 < \alpha < 1$, especially for small values of α. The latter case is treated by expressing the contour integral representation of the function in a suitable integral form that can be easily computed. The applicability of the integral form in fact is valid for a longer range

$0 < \alpha < 2$. In addition the integral form leads to a new convergent series and asymptotic series for $z \to \infty$.

2. Some special cases

It is possible to arrive at simple closed form representation of $E_\alpha(z)$ for $\alpha = 0, 1, 2, 3, 4$. According to Houbold et. al. [13] for $z \in C$

$$E_0(z) = \frac{1}{1-z}, \quad |z| < 1 \tag{2}$$

$$E_1(z) = e^z \tag{3}$$

$$E_2(z) = \cosh(\sqrt{z}) \tag{4}$$

$$E_3(z) = \frac{1}{2}\left[e^{z^{1/3}} + 2\,e^{-\frac{1}{2}z^{1/3}} \cos\left(\frac{\sqrt{3}}{2}\,z^{1/3}\right)\right] \tag{5}$$

$$E_4(z) = \frac{1}{2}\left[\cos(z^{1/4}) + \cosh(z^{1/4})\right] \tag{6}$$

The computation of the function for the above special cases from Eqs. (2)-(6) is thus quite simple. For other values of α, the infinite series (1) can be employed for computing $E_\alpha(z)$ for all values of z in the finite part of the complex plane provided that $\alpha > 1$. In the contrary case $\alpha < 1$, an integral representation of $E_\alpha(z)$ given in the next section is useful.

3. Integral representation of $E_\alpha(z)$, $\alpha < 2$

Mittag-Leffler's integral representation states that (Erdélyi et. al. [6], p. 206)

$$E_\alpha(z) = \frac{1}{2\pi i} \int_L \frac{t^{\alpha-1}\,e^t}{t^\alpha - z}\,dt \tag{7}$$

where the path of integration is a loop L which starts and ends at $-\infty$, encircling the branch point at $t = 0$ and the poles given by $t^\alpha = z$ in the positive sense $-\pi < \arg(t) < \pi$ of the cut complex t-plane along the negative real axis. Segregating the poles, the path L can be deformed to run parallel to two sides of the negative real axis of the t-plane with an indentation of the branch point $t = 0$ by a circle of small radius ϵ. When $\alpha < 2$, there is only one pole for $t^\alpha = z$. Setting $\tau = t^\alpha$, the residue at the pole $\tau = z^{1/\alpha}$ in the complex τ-plane is $\frac{1}{\alpha}\,e^{z^{1/\alpha}}$, where the principal value of $z^{1/\alpha}$ is taken. The contribution of the infinitesimally small circle $t = \epsilon\,e^{i\theta}$, $-\pi < \theta < \pi$ to Eq. (7) is

$$\frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{\epsilon^{\alpha-1}\,e^{i(\alpha-1)\theta}\,e^{\epsilon(\cos\theta + i\sin\theta)}}{\epsilon^\alpha e^{i\alpha\theta} + z} \times \epsilon\, i\theta\, e^{i\theta} d\theta \to 0, \quad \epsilon \to 0 \text{ if } z \neq 0 \tag{8}$$

On the two sides of the branch cut along the negative real axis of the t-plane, one can write $t = \xi e^{-i\pi}$ for the lower side of the contour L, and $t = \xi e^{i\pi}$ for the upper side, where ξ is a real variable. The contribution of the two integrals along these lines to Eq. (7) is

$$\frac{1}{2\pi i}\int_{\infty}^{0}\frac{\xi^{\alpha-1}e^{-i\pi(\alpha-1)}e^{-\xi}}{\xi^{\alpha}e^{-i\pi\alpha}-z}\times(-d\xi)+\frac{1}{2\pi i}\int_{0}^{\infty}\frac{\xi^{\alpha-1}e^{i\pi(\alpha-1)}e^{-\xi}}{\xi^{\alpha}e^{i\pi\alpha}-z}\times(-d\xi)$$

$$= -\frac{z\,\sin\alpha}{\pi}\int_0^\infty \frac{\xi^{\alpha-1}e^{-\xi}\,d\xi}{\xi^{2\alpha}-2z\,\cos\pi\alpha\,\xi^{\alpha}+z^2} \tag{9}$$

Collecting the contributions of Eqs. (8), (9) and that of the pole, one gets

$$E_\alpha(z) = \frac{1}{\alpha}\,e^{z^{1/\alpha}} -$$

$$\frac{z\,\sin\pi\alpha}{\pi}\int_0^\infty \frac{\xi^{\alpha-1}e^{-\xi}\,d\xi}{\xi^{2\alpha}-2z\,\cos\pi\alpha\,\xi^{\alpha}+z^2}\,,\quad 0<\alpha<2,\ z\neq 0,\ |\arg(z)|<\pi \tag{10}$$

Setting $\xi^\alpha = \eta$, one also gets

$$E_\alpha(z) = \frac{1}{\alpha}\,e^{z^{1/\alpha}} - \frac{z\,\sin\pi\alpha}{\pi\alpha}\int_0^\infty \frac{e^{-\eta^{1/\alpha}}d\eta}{\eta^2-2z\,\cos\pi\alpha\,\eta+z^2}\,,\quad 0<\alpha<2,\ z\neq 0,$$
$$|\arg(z)|<\pi \tag{11}$$

If the pole z in Eq. (7) is on the negative real axis, that is to say on the branch cut itself, it does not contribute a residue to Eq. (7) being outside the domain of integration. For this case writing $z = -x$, $x > 0$, as in Eq. (11) one gets

$$E_\alpha(-x) = \frac{x\,\sin\pi\alpha}{\pi\alpha}\int_0^\infty \frac{e^{-\eta^{1/\alpha}}d\eta}{\eta^2+2x\,\cos\pi\alpha\,\eta+x^2}\,,\quad x>0 \tag{12}$$

Eq. (12) is equivalent to the form quoted by Houbold et. al. [13] if one sets $\eta = x\zeta^\alpha$, that is to say

$$E_\alpha(-x) = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\zeta^{\alpha-1}e^{-\zeta x^{1/\alpha}}d\zeta}{\zeta^{2\alpha}+2x\,\cos\pi\alpha\,\zeta^{\alpha}+1}\,,\quad x>0 \tag{13}$$

4. Representation in terms of incomplete Gamma functions

According to Abramowitz and Stegun [1], formula 22.9.3, p. 783, the generating function of the Gegenbauer polynomials $C_n^1(x)$ is

$$\frac{1}{z^2-2xz+1} = \sum_{n=0}^{\infty} C_n^1(x)\,z^n,\quad \text{if } |z|<1 \tag{14}$$

For $x = \cos\pi\alpha$, one has for the the $C_n^1(\cdot)$ poynomial the formula (Gradshteyn and Ryzhik [10], formula 8.973, p. 1031)

$$C_n^1(\cos\pi\alpha) = \frac{\sin(n+1)\pi\alpha}{\sin\pi\alpha} \tag{15}$$

Hence Eq. (14) yields

$$\frac{1}{z^2 - 2\cos\pi\alpha\, z + 1} = \sum_{n=0}^{\infty} \frac{\sin(n+1)\pi\alpha}{\sin\pi\alpha}\, z^n\,, \quad \text{if } |z| < 1 \tag{16}$$

The expansion (16) is employed in Eq. (10) by splitting the integral in to two parts over the intervals $(0,\, |z|^{1/\alpha})$ and $(|z|^{1/\alpha},\, \infty)$, obtaining

$$E_\alpha(z) = \frac{1}{\alpha}\, e^{z^{1/\alpha}} - \frac{1}{\pi}\sum_{n=0}^{\infty} \sin(n+1)\pi\alpha \Big[\frac{1}{z^{n+1}} \int_0^{|z|^{1/\alpha}} \xi^{(n+1)\alpha-1} e^{-\xi}\, d\xi$$

$$+ z^{n+1} \int_{|z|^{1/\alpha}}^{\infty} \xi^{-(n+1)\alpha-1} e^{-\xi}\, d\xi\Big]$$

$$= \frac{1}{\alpha}\, e^{z^{1/\alpha}} - \frac{1}{\pi}\sum_{n=0}^{\infty} \sin(n+1)\pi\alpha \Big[\frac{1}{z^{n+1}}\, \gamma\{(n+1)\alpha, |z|^{1/\alpha}\} + z^{n+1}\, \Gamma\{-(n+1)\alpha, |z|^{1/\alpha}\}\Big] \tag{17}$$

where $\gamma\{\cdot,\cdot\}$ and $\Gamma\{\cdot,\cdot\}$ are the incomplete Gamma functions.

5. A power series representation for $\alpha \neq 1$

The power series representation of the function $\gamma(a,\, x)$ is (Gradshteyn and Ryzhik [10], formula 8.534, p. 941)

$$\gamma(a,\, x) = \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{a+m}}{m!\,(a+m)} \tag{18}$$

The formula for $\Gamma(a,\, x)$ follows from the relation $\Gamma(a,\, x) = \Gamma(a) - \gamma(a,\, x)$. Using Eq. (18) and the series for $\Gamma(a,\, x)$ in Eq. (17), one has

$$E_\alpha(z) =$$

$$\frac{1}{\alpha}\, e^{z^{1/\alpha}} - \frac{1}{\pi}\sum_{n=0}^{\infty} \sin(n+1)\pi\alpha\, \Big[\Big(\frac{|z|}{z}\Big)^{n+1} \sum_{m=0}^{\infty} \frac{(-1)^m\, |z|^{m/\alpha}}{m!\,[(n+1)\alpha + m]} + \Gamma\{-(n+1)\alpha\}\, z^{n+1}$$

$$-\Big(\frac{z}{|z|}\Big)^{n+1} \sum_{m=0}^{\infty} \frac{(-1)^m\, |z|^{m/\alpha}}{m!\,[-(n+1)\alpha + m]}\Big]$$

or,

$$E_\alpha(z) = \frac{1}{\alpha}\, e^{z^{1/\alpha}} - \frac{1}{\pi}\sum_{n=0}^{\infty} \sin(n+1)\pi\alpha\, \Big[\Gamma\{-(n+1)\alpha\}\, z^{n+1}$$

$$+2\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{(n+1)\alpha\cos(n+1)\theta+im\sin(n+1)\theta}{(n+1)^2\alpha^2-m^2}\,|z|^{m/\alpha}\Big] \tag{19}$$

where $i=\sqrt{-1}$, and θ =arg(z). For $\Gamma\{-(n+1)\alpha\}$ to be finite it is required that $\alpha\neq 1$. Evidently when z is real equal to $x>0$, $\theta=0$ and the formula simplifies to

$$E_\alpha(x)=\frac{1}{\alpha}\,e^{x^{1/\alpha}}-$$
$$\frac{1}{\pi}\sum_{n=0}^{\infty}\sin(n+1)\pi\alpha\Big[\Gamma\{-(n+1)\alpha\}\,x^{n+1}+2(n+1)\alpha\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\,\frac{x^{m/\alpha}}{(n+1)^2\alpha^2-m^2}\Big] \tag{20}$$

6. Asymptotic series expansions for $|z|\to\infty$

It is known that (Abramowitz and Stegun [1], formula 6.5.32, p. 263) for real x

$$\Gamma(a,\,x)\sim x^{a-1}e^{-x}\left[1+\frac{a-1}{x}+\frac{(a-1)(a-2)}{x^2}+\cdots\right],\quad x\to\infty \tag{21}$$

Hence, writing Eq. (17) as

$$E_\alpha(z)=\frac{1}{\alpha}\,e^{z/\alpha}-\frac{1}{\pi}\sum_{n=0}^{\infty}\sin(n+1)\pi\alpha\left[\frac{1}{z^{n+1}}\Big\{\Gamma\{(n+1)\alpha\}-\Gamma\{(n+1)\alpha,\,|z|^{1/\alpha}\Big\}\right.$$
$$\left.+z^{n+1}\,\Gamma\{-(n+1)\alpha,\,|z|^{1/\alpha}\}\right]$$
$$\sim\frac{1}{\alpha}\,e^{z/\alpha}-\frac{1}{\pi}\sum_{n=0}^{\infty}\sin(n+1)\pi\alpha\left[\frac{\Gamma\{(n+1)\alpha\}}{z^{n+1}}\right.$$
$$-\Big(\frac{|z|}{z}\Big)^{n+1}|z|^{-1/\alpha}e^{-|z|^{1/\alpha}}\Big\{1+\frac{(n+1)\alpha-1}{|z|^{1/\alpha}}+\frac{((n+1)\alpha-1)((n+1)\alpha-2)}{|z|^{2/\alpha}}+\cdots\Big\}$$
$$+\Big(\frac{z}{|z|}\Big)^{n+1}|z|^{-1/\alpha}e^{-|z|^{1/\alpha}}\Big\{1-\frac{(n+1)\alpha+1}{|z|^{1/\alpha}}+\frac{((n+1)\alpha+1)((n+1)\alpha+2)}{|z|^{2/\alpha}}-\cdots\Big\},$$
$$|z|\to\infty \tag{22}$$

The last two terms in Eq. (22) tend to 0 as $|z|\to\infty$. Hence one gets

$$E_\alpha(z)\sim\frac{1}{\alpha}\,e^{z/\alpha}-\frac{1}{\pi}\sum_{n=0}^{\infty}\sin(n+1)\pi\alpha\,\frac{\Gamma\{(n+1)\alpha\}}{z^{n+1}},\quad |z|\to\infty \tag{23}$$

Eq. (23) may be compared with the asymptotic expansion given in Erdéyli et. al. [6], section 18.1, formula (10), p. 208.

7. Computation of $\mathbf{E}_\alpha(\mathbf{z})$

As noted earlier it is easy to compute the function $E_\alpha(z)$ for $\alpha > 1$ from the defining Eq. (1). For $0 < \alpha < 1$, one may use any one of the Eqs. (11), (19), or (23); the last one when $|z|$ is large. It is apparent that Eq.(11) is easiest to apply for computing of the function by applying the simple Simpson's rule of numerical integration to the integral term as the integrand decreases exponentially as η increases to infinity. As an illustration, let z be real, say x, and α take up the three values 0.2, 0.5, 0.8. For graphical presentation of computed results, factoring $e^{x/\alpha}$ and taking logarithm, Eq. (11) is written as

$$\ln E_\alpha(x) = \frac{x}{\alpha} + \ln\left[\frac{1}{\alpha} - e^{-x^{1/\alpha}}\,\frac{x\,\sin\pi\alpha}{\pi}\int_0^\infty \frac{e^{-\eta^{1/\alpha}}\,d\eta}{\eta^2 - 2x\,\cos\pi\alpha\,\eta + x^2}\right] \quad (24)$$

The computed results are shown in figure 1. It is observed from the figure

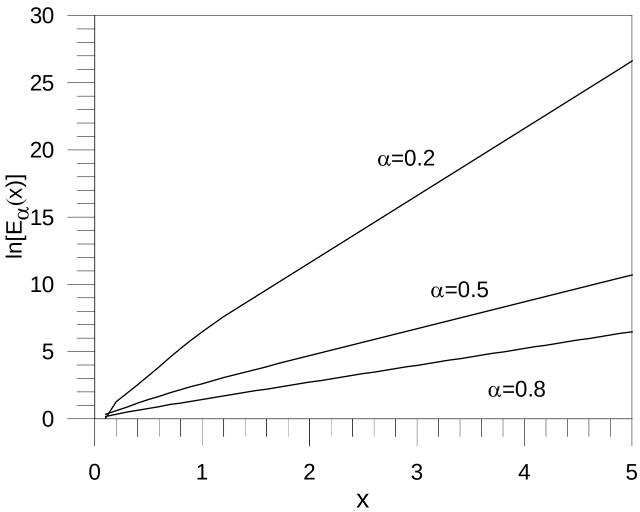


Figure 1. $\ln[E_\alpha(x)]$ versus x for $\alpha = 0.2,\ 0.5,\ 0.8$.

that the smaller α is the steeper is the rise of the function $\ln[E_\alpha(x)]$ and therefore of the function $E_\alpha(x)$.

Concluding comments

The Mittag-Leffer function $E_\alpha(z)$, where $\alpha > 0$ and z in general complex, has been found to have some interesting applications in fractional differential equations and Abel type integral equations that appear in different physical scenarios. A noteworthy application has also been found in recent times in queuing models that are state dependent (Conway and Maxwell [5]). Bose [2] by similar arguments found an application in the transmission of packets of data in fiber-optic communication channels.

Though the function theoretic properties of the function have been investigated to some extent, its computational aspect has only recently received attention (Gorenflo et. al. [9]), in greater generality of the function.

In this paper, it is pointed out that the function can be directly computed from its defining power series definition Eq. (1) when $\alpha \geq 1$. For $0 < \alpha < 1$ on the other hand the function can be easily computed from the integral representation Eq. (11).

In this new finding, some additional representations of the function in convergent series form, or as asymptotic series for $z \to \infty$ have also been found.

Acknowledgement:

The author is thankful to the referees for critically pointing out the lacunae in the presentation of this article.

References

[1] M. Abramowitz, I. Stegun, Handbook of Mathematical Functions, Dover Publications, New York, 1968.

[2] S. K. Bose, On the transmission of data packets through fiber-optic cables of uniform index, *J. Opt. Comm.* (2011), 1-5.

[3] S. K. Bose, Operations Research Methods, Alpha, Science, 2012.

[4] S. Chakraborty, S. H. Ong, SMittag-Leffler function distribution function, *Stat. Distrib. Appl.* **4** (2017), 1-17.

[5] R. W. Conway, W. L. Maxwell, A queuing model with state dependent rates, *J. of Inust. Engineering*, **12** (1962), 132-136.

[6] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, Higher Transcendental Functions, Vol. 3, McGraw-Hill, New York, 1955.

[7] R. Gorenflo, F. Mainardi, The Mittag-Leffler function in the Riemann-Liouville fractional calculus, In: A. A. Kilbas (Eds.), boundary value problems, special functions and fractional calculus, Minsk , 1996, 215-225.

[8] R. Gorenflo, F. Mainardi, Fractional oscillations and Mittag-Leffler functions, Preprint No. A-14/96, Free University of Berlin, 1994.

[9] R. Gorenflo, J. Loutchko, Y. Luchko, Computation of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ and its derivative, Fract. Calc. Appl. Anal. Cite Seer, (2002), 1-26.

[10] I.S. Gradshteyn, I. M. Ryzhik, Tables of Integrals, Series and Products, Academic Press, New York, 1965.

[11] R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[12] E. Hille, J. D. Tamarkin, On the theory of linear integral equations, Annals of Mathematics, 31 (1930), 479-528.

[13] H. J. Houbold, A. M. Mathai, R. K. Saxena, Mittag-Leffler functions and their applications, *J. App. Math.* Article Id:298628 (2011), 1-51.

[14] F. Mainardi, Fractiona; Calculus: Some Basic Problems in Continuum and Statistical Mechanics, In: Fractals and Fractional Calculus in Continuum Mechanics (Ed. A. Carpinteri, F. Mainardi), Springer Verlag, Wien, 1997.

[15] G. M. Mittag-Leffler, Sur la funzione $E_{\alpha}(x)$, *C.R. Acad. Sci. Paris*, **137** (1903), 554-558.

[16] S. G. Samko, A. A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, New York, 1993.

[17] A. Wiman, Über den Fundamental satz in der Theorie der Functionen $E_{\alpha}(x)$, *Acta Mathematica*, **29** (1905), 191-201.

Sujit K Bose
S.N. Bose National Center for Basic Scineces
Salt Lake City, Kolkata 700106, West Bengal, India.
E-mail: sujitkbose1@gmail.com

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July-December (2024), 179–182

A TOPOLOGICAL PROOF OF THE FUNDAMENTAL THEOREM OF ARITHMETIC

JHIXON MACÍAS

(Received : 23 - 05 - 2024 ; Revised : 25 - 07 - 2024)

Abstract. In this note, we present a topological proof of the Fundamental Theorem of Arithmetic. To the best of our knowledge, it is the first of its kind to be presented.

- To my beloved wife, Alenka Calderón.

1. Introduction and Preliminaries

The idea of using topology to prove basic theorems of number theory is not new. For instance, in 1955, Furstenberg showed using an arithmetic progression topology on the integers the infinitude of primes, see [1]. Furthermore, J. Macías recently presented a new topological proof of the infinitude of prime numbers using a novel topology; see [2]. The aim of this brief note is to present a topological proof of the fundamental theorem of arithmetic.

First recall that in topology, an *Alexandrov topology* is a topology in which the intersection of every family of open sets is open and a topological space is said to be *ultraconnected* if no two nonempty closed sets are disjoint.

Now, consider the topological space $X = (\mathbf{N_2}, \tau)$ where τ is the topology generated by the base $\beta := \{\mathcal{O}_n : n \geq 2\}$, $\mathcal{O}_n := \{d \in \mathbf{N_2} : \ d|n\}$ and $\mathbf{N_2} := \mathbb{N} \setminus \{1\}$. The topology τ is known as the *divisor topology*, see [3, Example 57]. Most noted, there are two properties unrelated to the fundamental theorem of arithmetic that will be useful are the following: τ is an Alexandrov topology, and for every $x \in X$, the closure of $\{x\}$ in X (denoted by, $\mathbf{Cl}_X(\{x\})$) is the set $x\mathbb{N}$ (the multiples of x). This last property allows us to guarantee that X is ultraconnected, and therefore connected. All the

2010 Mathematics Subject Classification: 11A41, 54G05, 54H11
Key words and phrases: Prime numbers, topology, fundamental theorem of arithmetic

previously mentioned properties of the divisor topology can be found in [3, pp. 79-80].

Remark 1.1. Every ultraconnected space Y is path-connected. Indeed, if a and b are two points Y and c is a point of $\mathbf{Cl}(\{a\}) \cap \mathbf{Cl}(\{b\})$, the function $f : [0,1] \to Y$ defined by

$$f(t) = \begin{cases} a & if \quad 0 \leq t < \frac{1}{2} \\ \\ c & if \quad\quad t = \frac{1}{2} \\ \\ b & if \quad \frac{1}{2} < t \leq 1 \end{cases}$$

is a continuous path between a and b, see [3, p. 29]. All path-connected space is connected.

2. The Topological Proof

Let $\mathbb{P}$ be the set of prime numbers. It is well known (and clear) that, if the fundamental theorem of arithmetic holds, then $\mathbb{P}$ is dense in X. So, it is natural to ask whether the converse also holds. The answer to this question is affirmative.

Theorem 2.1. *If $\mathbb{P}$ is dense in X, then the fundamental theorem of arithmetic holds.*

Proof. Sindce τ is Alexandrov, the closure operator of X distributes over arbitrary unions of subsets. Hence,

$$\mathbf{N_2} = \mathbf{Cl}_X(\mathbb{P}) = \mathbf{Cl}_X\left(\bigcup_{p\in\mathbb{P}}\{p\}\right) = \bigcup_{p\in\mathbb{P}} \mathbf{Cl}_X(\{p\}) = \bigcup_{p\in\mathbb{P}} p\mathbb{N}. \tag{2.1}$$

Let $x \in \mathbf{N_2}$. Then, by Equation (2.1), $x = p_1 \cdot k_1$, where p_1 is a prime and k_1 is either 1 or $k_1 \in \mathbf{N_2}$. If $k_1 = 1$, then x is prime. Otherwise, $k_1 \in \mathbf{N_2}$ and $1 < k_1 < x$, then $k_1 = p_2 \cdot k_2$ with p_2 prime and either $k_2 = 1$, or $k_2 \in \mathbf{N_2}$ and $1 < k_2 < k_1$. Repeating this process, we have $x = p_1 \cdot p_2 \cdots p_n$, where each p_i is prime (not necessarily distinct). Hence, x is the product of primes.

So far, we have demonstrated that every $x \in \mathbf{N_2}$ is either prime or a product of primes. Now, let's prove that this product of primes is unique for all $x \in \mathbf{N_2}$. Suppose the contrary and let x be the smallest positive integer having more than one representation as the product of primes, say

$$x = p_1 \cdot p_2 \cdots p_r = q_1 \cdot q_2 \cdots q_s. \tag{2.2}$$

It is clear that r and s are greater than 1. Now, the primes $p_1, p_2, \ldots, p_r$ have no members in common with $q_1, q_2, \ldots, q_s$ because if, for example, p_1 were a common prime, then we could divide it out of both sides of Equation 2.2 to get two distinct factorings of $\frac{x}{p_1}$. But this would contradict our assumption that all integers smaller than x are uniquely factorable. Now, consider the open set $\mathcal{O}_x = \mathcal{O}_{p_1 \cdot p_2 \cdots p_r} = \mathcal{O}_{q_1 \cdot q_2 \cdots q_s}$. Since $\mathbb{P}$ is dense in X, there exists a prime p such that $p \in \mathcal{O}_x$. Equivalently, $p \mid x$. Thus, $p \mid p_1 \cdot p_2 \cdots p_r$ and $p \mid q_1 \cdot q_2 \cdots q_s$. Note that necessarily p would be a common prime among $p_1, p_2, \ldots, p_r$ and $q_1, q_2, \ldots, q_s$, which we have already seen is absurd. Hence, the density of $\mathbb{P}$ in X implies the Fundamental Theorem of Arithmetic. □

Remark 2.2. Note that we have obtained a proof of the uniqueness of prime factorization without invoking Euclid's Lemma. This is achieved by assuming that $\mathbb{P}$ is dense in X.

Finally, let us prove that $\mathbb{P}$ is dense in X (utilizing topological tools), and thus, by virtue of Theorem 2.1, obtain the topological proof of the Fundamental Theorem of Arithmetic.

Proof. Suppose that $\mathbb{P}$ is not dense in X. Then there exists $n \in \mathbf{N}_2$ such that $\mathcal{O}_n \cap \mathbb{P} = \emptyset$. Thus n is not prime and cannot be factored into primes. Likewise, neither do the divisors of n. Let $\Delta := \{n \in \mathbf{N}_2 : \mathcal{O}_n \cap \mathbb{P} = \emptyset\}$. It is clear that $\mathbf{N}_2 \setminus \Delta$ is the set of all $m \in \mathbf{N}_2$ such that m is prime or can be factored into primes. Therefore, for every $n \in \Delta$ and $m \in \mathbf{N}_2 \setminus \Delta$, we have $\mathcal{O}_n \cap \mathcal{O}_m = \emptyset$. Now, let $A = \bigcup_{n \in \Delta} \mathcal{O}_n$ and $B = \bigcup_{m \in \mathbf{N}_2 \setminus \Delta} \mathcal{O}_m$. Then A and B are disjoint open sets in X, such that $\mathbf{N}_2 = A \cup B$, in other words, A and B form a separation of X, which is absurd since X is connected.

□

Acknowledgement: I am grateful to the referee and my current advisor, Reyes Ortiz, for the comments which improved the quality of the paper.

References

[1] H. Furstenberg, On the infinitude of primes, *American Mathematical Monthly* **62** (1955), 353.

[2] J. Macías, Another topological proof of the infinitude of prime Numbers, *Integers* **24** (2024), A 47.

[3] L. A. Steen and J. A. Seebach, Counterexamples in topology, Springer, 1978.

Jhixon Macías
Department of Mathematics
Faculty of Arts and Sciences
University of Puerto Rico at Mayaguez,
00681-9000, Mayaguez, Puerto Rico.
E-mail: jhixon.macias@upr.edu

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July-December (2024), 183–190

AN ALGEBRAIC IDENTITY WITH AN APPLICATION IN PROBABILITY

SREEHARI MADDIPATLA

(Received : 12 - 07 - 2024 ; Revised : 16 - 10 - 2024)

Abstract. We prove the algebraic identity given at (1.1) and illustrate an application to occupancy problem in probability.

1. Introduction and main result

We prove an algebraic identity involving real numbers. In Section 2 we discuss an application of the algebraic identity to an occupancy problem in probability.

Theorem 1. *Let* $a_i > 0, i = 1, 2, \ldots, k;\ k \geq 2$. *Consider*

$$A(n) = \sum_{i=1}^{k} a_i^n - \sum_{1\leq i_1<i_2\leq k} (a_{i_1} + a_{i_2})^n + \sum_{1\leq i_1<i_2<i_3\leq k} (a_{i_1} + a_{i_2} + a_{i_3})^n - \ldots + (-1)^{k-1}(a_1 + a_2 + \ldots + a_k)^n. \tag{1.1}$$

Then $A(n) = 0$ *for* $1 \leq n < k$.

Proof. Let $n < k$. Let us look at the coefficient of a_i^n for fixed $i \leq k$ from each of the k terms in the right side of (1.1). The contribution of successive terms to this coefficient are

$$1,\ -\frac{(k-1)!}{1!\,(k-2)!},\ (-1)^2\,\frac{(k-1)!}{2!\,(k-3)!}, \ldots, (-1)^{(k-1)}\frac{(k-1)!}{(k-1)!\,0!}.$$

Thus the total contribution, i.e., the coefficient of a_i^n in $A(n)$ is
$\sum_{m=0}^{k-1}(-1)^m\,\frac{(k-1)!}{m!\,(k-1-m)!} = (1-1)^{k-1} = 0.$
Next, let us consider the terms involving $\prod_{j=1}^{r} a_{i_j}^{y_j}$ for fixed $1 \leq i_1 < i_2 <$

2010 Mathematics Subject Classification: 13J30, 60C99
Key words and phrases: Multinomial theorem, Occupancy problem in probability.

$\ldots < i_r \le k, 1 < r \le k, \sum_1^r y_j = n$ in the right side of (1.1). There will be $k-r+1$ terms involving $\prod_{j=1}^r a_{i_j}^{y_j}$, namely,

$$(-1)^{r-1} \sum_{1 \le i_1 < i_2 < \ldots < i_{r+1} \le k} (a_{i_1} + a_{i_2} + \ldots + a_{i_r})^n,$$

$$(-1)^r \sum_{1 \le i_1 < i_2 < \ldots < i_{r+1} \le k} (a_{i_1} + a_{i_2} + \ldots + a_{i_r} + a_{i_{r+1}})^n, \ldots,$$

$$(-1)^{k-1}(a_{i_1} + a_{i_2} + \ldots + a_{i_k})^n.$$

They contribute, successively,

$$(-1)^{r-1+m} \binom{k-r}{m} \binom{n}{y_1, \ldots, y_r}, m = 0, 1, 2, \ldots, k-r,$$

where

$$\binom{n}{y_1, \ldots, y_r} = \frac{n!}{\prod_{j=1}^r y_j!}.$$

Hence the coefficient of $\prod_{j=1}^r a_{i_j}^{y_j}$ is

$$\sum_{m=0}^{k-r} (-1)^{r-1+m} \binom{k-r}{m} \binom{n}{y_1, \ldots, y_r} =$$

$$(-1)^{r-1} \binom{n}{y_1, \ldots, y_r} \sum_{m=0}^{k-r} (-1)^m \binom{k-r}{m} = 0$$

implying $A(n) = 0$.

Remark 1.1. The above result may not hold for $n = k$. This is easily seen by taking $k = n = 2$.

We now prove

Corollary 1.2. *Consider*

$$T_k(s) = \sum_{r=1}^{k} (-1)^{r-1} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le k} (1 - p_{i_1} - \ldots - p_{i_r})^s, \quad 1 \le s < k, k \ge 2 \tag{1.2}$$

where all $p_j > 0$ *and* $\sum_1^k p_j < 1$. *Then* $T_k(s) = 1$ *for* $1 \le s < k$.

Proof. Write

$$T_k(s) = \sum_{n=0}^{s} \frac{s!}{n!\,(s-n)!} (-1)^n \sum_{r=1}^{k} (-1)^{r-1} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le k} (p_{i_1} + \ldots + p_{i_r})^n. \tag{1.3}$$

In (1.3) the term corresponding to $n = 0$ is

$$\sum_{r=1}^{k}(-1)^{r-1}\ \frac{k!}{r!\ (k-r)!} = -[\sum_{r=1}^{k}(-1)^{r}\ \frac{k!}{r!\ (k-r)!}]$$

$$= -[\sum_{r=0}^{k}(-1)^{r}\ \frac{k!}{r!\ (k-r)!} - 1] = -(1-1)^{k} + 1 = 1. \qquad (1.4)$$

Hence

$$T_k(s) = 1+\sum_{n=1}^{s}\frac{s!}{n!\ (s-n)!}(-1)^{n}\sum_{r=1}^{k}(-1)^{r-1}\sum_{1\le i_1<i_2<\ldots<i_r\le k}(p_{i_1}+\ldots+p_{i_r})^{n}. \qquad (1.5)$$

Recall that, by Theorem 1, for $1 \le n \le s < k$, $\sum_{r=1}^{k}(-1)^{r-1}\sum_{1\le i_1<i_2<\ldots<i_r\le k}(p_{i_1}+\ldots+p_{i_r})^{n} = 0$. Hence the claim follows from (1.4) and (1.5).

2. Application to an occupancy Problem.

Consider a system with $k+1$ counters into which balls are successively thrown at random. We shall, for convenience, rename the $(k+1)$st counter as 0th counter. Let $p_j > 0$ denote the probability that a ball thrown at random falls into the jth counter. $\sum_0^k p_j = 1$. Suppose we observe the system till each of the counters numbered $1, 2, \ldots, k$ receives at least one ball. For $i = 1, 2, \ldots, k$ let Y_i be the number of balls thrown by the time the ith counter receives a ball for the first time. Then $Y_1, Y_2, \ldots, Y_k$ are independent and identically distributed random variables. Suppose the system is considered full (or the throwing of the balls is suspended) as soon as all these k counters receive at least one ball. Let $V = \max\{Y_1, Y_2, \ldots, Y_k\}$. Then $V \ge k$ with probability 1. Let $y_1, y_2, \ldots, y_k$ take non-negative integer values and let $y_j \le k$. Then we can split $\{0 \le y_j \le k, j = 1, 2, \ldots, k\}$ and observe that

$$\begin{aligned}\sum_{0\le y_j\le k, j=1,2,\ldots,k} = &\sum_{1\le y_j\le k, j=1,2,\ldots,k} + \sum_{1\le i_1\le k}\ \sum_{y_{i_1}=0, 0\le y_j\le k, j\ne i_1} - \\ &- \sum_{1\le i_1<i_2\le k}\ \sum_{y_{i_1}=0=y_{i_2}, 0\le y_j\le k, j\ne i_1, i_2} + \ldots + \\ +(-1)^{k-2} &\sum_{1\le i_1<\ldots<i_{k-1}\le k}\ \sum_{y_{i_s}=0, 1\le s\le k-1; 0\le y_j\le k, j\ne i_1,\ldots i_{k-1}} + \\ &+(-1)^{k-1}\sum_{y_{i_j}=0, j=1,2,\ldots,k}\end{aligned} \qquad (2.1)$$

where the summands are same for all the sums involved.
In [1] it is proved that for $v \geq k$

$$P_k(V=v) = \sum_{r=1}^{k} \sum_{\substack{x_{1r}\geq 1,\ldots,x_{kr}\geq 1, x_{rr}=1,\\ \sum_{j=0;j\neq r}^{k} x_{jr}=v-1}} \frac{(v-1)!}{\prod_{j=0}^{k} x_{jr}!} \prod_{j=0}^{k} p_j^{x_{jr}}. \tag{2.2}$$

It is also obtained that

$$P_k(V=v) = \sum_{r=1}^{k} p_r(1-p_r)^{v-1} - \sum_{1\leq i_1<i_2\leq k} (p_{i_1}+p_{i_2})(1-p_{i_1}-p_{i_2})^{v-1}$$
$$+ \ldots + (-1)^{k-1}(1-p_1-\ldots-p_k)^{v-1} (\sum_{j=1}^{k} p_j). \tag{2.3}$$

We prove that the two expressions at (2.2) and (2.3) are equivalent and also prove that the expression at (2.3) is a proper probability distribution.
We shall use (2.1) with y_j replaced by x_{jr}, j taking values $0, 2, \ldots, k$ instead of $1, \ldots, k$. For convenience let us denote $A_k = \{0, 2, 3, \ldots, k\}$. Further let the summands be the multinomial probabilities

$$\frac{(v-1)!}{\prod_{j\in A_k} x_{jr}!} \prod_{j\in A_k} p_j^{x_{jr}}.$$

We have from (2.1), after some rearrangement,

$$\sum_{\substack{1\leq x_{jr}\leq k, j\in A_k,\\ \sum_{j\in A_k} x_{jr}=v-1}} \frac{(v-1)!}{\prod_{j\in A_k} x_{jr}!} \prod_{j\in A_k} p_j^{x_{jr}} = \sum_{\substack{0\leq x_{jr}\leq k, j\in A_k,\\ \sum_{j\in A_k} x_{jr}=v-1}} \frac{(v-1)!}{\prod_{j\in A_k} x_{jr}!} \prod_{j\in A_k} p_j^{x_{jr}}$$
$$- \sum_{i_1\in A_k} \sum_{\substack{x_{i_1 r}=0, 0\leq x_{jr}\leq k,, j\neq i_1\\ \sum_{j\in A_k} x_{jr}=v-1}} \frac{(v-1)!}{\prod_{j\in A_k} x_{jr}!} \prod_{j\in A_k} p_j^{x_{jr}}$$
$$+ \sum_{i_1,i_2\in A_k} \sum_{\substack{x_{i_1 r}=0=x_{i_2 r}, 0\leq x_{jr}\leq k, j\neq i_1, i_2,\\ \sum_{j\in A_k} x_{jr}=v-1}} \frac{(v-1)!}{\prod_{j\in A_k} x_{jr}!} \prod_{j\in A_k} p_j^{x_{jr}} - \ldots$$
$$+(-1)^{k-1} \sum_{i_1,\ldots,i_{k-1}\in A_k} \sum_{\substack{x_{i_s,r}=0, s=1,2,\ldots,k-1,\\ 0\leq x_{jr}\leq k, j\neq i_1,i_2,\ldots,i_{k-1}, \sum x_{jr}=v-1}} \frac{(v-1)!}{\prod_{j\in A_k} x_{jr}!} \prod_{j\in A_k} p_j^{x_{jr}} +$$
$$+(-1)^{k} \sum_{\substack{x_{jr}=0, j\in A_k\\ \sum_{j\in A_k} x_{jr}=v-1}} \frac{(v-1)!}{\prod_{j\in A_k} x_{jr}!} \prod_{j\in A_k} p_j^{x_{jr}}. \tag{2.4}$$

For notational convenience set $x_{rr} = 1$. Let us fix $r = 1$ in the right side of (2.2) and consider the corresponding sum. Then we have from (2.4) and the multinomial theorem

$$\sum_{\substack{x_{11}=1\le x_{01}\le k,1\le x_{21}\le k,\dots,1\le x_{k1}\le k\\ \sum_{j=0;j\neq 1}^{k} x_{j1}=v-1}} \frac{(v-1)!}{\prod_{j=0}^{k} x_{j1}!}\prod_{j=0}^{k} p_j^{x_{j1}}$$

$$= p_1 \sum_{\substack{1\le x_{01}\le k,1\le x_{21}\le k,\dots,1\le x_{k1}\le k\\ \sum_{j=0;j\neq 1}^{k} x_{j1}=v-1}} \frac{(v-1)!}{\prod_{j=0}^{k} x_{j1}!}\prod_{j=0,j\neq 1}^{k} p_j^{x_{j1}}$$

$$= p_1\Big[(\sum_{j=0}^{k} p_j - p_1)^{v-1} - \sum_{i_1=2}^{k}(\sum_{j=0}^{k} p_j - p_1 - p_{i_1})^{v-1}$$

$$+ \sum_{2\le i_1<i_2\le k}\left(\sum_{j=0}^{k} p_j - p_1 - p_{i_1} - p_{i_2}\right)^{v-1} - \dots + (-1)^{k-1}p_0^{v-1}\Big]$$

$$= p_1\Big[(1-p_1)^{v-1} - \sum_{i_1=2}^{k}(1-p_1-p_{i_1})^{v-1}$$

$$+ \sum_{2\le i_1<i_2\le k}(1-p_1-p_{i_1}-p_{i_2})^{v-1} - \dots + (-1)^{k-1}(1-\sum_{j=1}^{k} p_j)^{v-1}\Big]. \quad (2.5)$$

Similarly for $1 \le r \le k$ we get

$$\sum^{k}_{\substack{x_{rr}=1\le x_{0r}\le k,\dots,1\le x_{r-1,r}\le k.1\le x_{r+1,r}\le k,\dots,1\le x_{kr}\le k,\\ \sum_{j=0;j\neq r} x_{jr}=v-1}} \frac{(v-1)!}{\prod_{j=0}^{k} x_{jr}!}\prod_{j=0}^{k} p_j^{x_{jr}}$$

$$= p_r\Big[(1-p_r)^{v-1} - \sum_{i_1\neq r}(1-p_r-p_{i_1})^{v-1} + \sum_{i_1\neq r\neq i_2}(1-p_r-p_{i_1}-p_{i_2})^{v-1}$$

$$- \dots + (-1)^{k-1}(1-\sum_{j=1}^{k} p_j)^{v-1}\Big]. \quad (2.6)$$

From (2.2), (2.5) and (2.6) we observe that the right sides of (2.2) and (2.3) are equivalent. From (2.3) we have

$$\sum_{v=k}^{\infty} P_k(V=v) = \sum_{r=1}^{k}(1-p_r)^{k-1} - \sum_{1\le i_1<i_2\le k}(1-p_{i_1}-p_{i_2})^{k-1} + \dots$$

$$+(-1)^{k-1}(1-p_1-\dots-p_k)^{k-1}$$

$$= \sum_{r=1}^{k} (-1)^{r-1} \sum_{1 \le i_1 < \ldots < i_r \le k} (1 - p_{i_1} - p_{i_2} - \ldots - p_{i_r})^{k-1}. \qquad (2.7)$$

Our aim is to provide a direct proof of the fact that V is a proper random variable.

Theorem 2.

$$\sum_{v=k}^{\infty} P_k(V = v) = 1$$

for all $k \geq 2$.

Proof. Denote $S(k) = \sum_{v=k}^{\infty} P_k(V = v)$ and note that

$$S(2) = \sum_{r=1}^{2} (1 - p_r) - (1 - p_1 - p_2) = 1.$$

Further

$$S(3) = \sum_{r=1}^{3} (1 - p_r)^2 - (1 - p_1 - p_2)^2 - (1 - p_1 - p_3)^2$$
$$-(1 - p_2 - p_3)^2 + (1 - p_1 - p_2 - p_3)^2.$$

Observe that

$$S(2) - S(3) = \sum_{r=1}^{2} (1 - p_r)\, p_r - (1 - p_3)^2 - p_3(p_1 + p_2)$$
$$+(1 - p_1 - p_3)^2 + (1 - p_2 - p_3)^2 - (1 - \sum_{1}^{3} p_j)^2$$
$$= -(1 - p_3)^2 + 2p_1p_2 + (1 - p_1 - p_3)^2 + (1 - p_2 - p_3)^2$$
$$= 2p_1p_2 - p_1(2 - 2p_3 - p_1) + p_1(1 - p_2 - p_3) = 0$$

implying $S(3) = 1$. Next we prove that for all k, $S(k) - S(k+1) = 0$ which establishes that $S(k) = 1$ for all $k \geq 2$. Consider

$$S(k) - S(k+1) =$$

$$\sum_{1}^{k} (1 - p_i)^{k-1}\, p_i - (1 - p_{k+1})^k - \sum_{1 \le i_1 < i_2 \le k} (1 - p_{i_1} - p_{i_2})^{k-1}\, (p_{i_1} + p_{i_2})$$
$$+ \sum_{1 \le i_1 \le k} (1 - p_{i_1} - p_{k+1})^k + \ldots + (-1)^{k-1} (1 - p_1 - \ldots - p_{k+1})^k$$

$$= \sum_{r=1}^{k}(-1)^{r-1} \sum_{1\le i_1<i_2<\ldots<i_r\le k} (1-p_{i_1}-\ldots-p_{i_r})^{k-1}(p_{i_1}+\ldots+p_{i_r})$$

$$+\sum_{r=1}^{k}(-1)^{r-1} \sum_{1\le i_1<i_2<\ldots<i_r\le k} (1-p_{i_1}-\ldots-p_{i_r}-p_{k+1})^{k}-(1-p_{k+1})^{k}. \quad (2.8)$$

We break the RHS into 3 parts- Part 1 consisting of terms free from p_{k+1}, Part 2 consisting of coefficients of $(-p_{k+1})^s$ for $s = 1, 2, \ldots, (k-1)$ and Part 3 consisting of coefficient of $(-p_{k+1})^k$.

$$Part\ 1 = \sum_{1}^{k}(1-p_i)^{k-1}\ p_i - 1 +$$

$$+\sum_{r=2}^{k}(-1)^{r-1} \sum_{1\le i_1<i_2<\ldots<i_r\le k} (1-p_{i_1}-\ldots-p_{i_r})^{k-1}(p_{i_1}+\ldots+p_{i_r})$$

$$+\sum_{r=1}^{k}(-1)^{r-1} \sum_{1\le i_1<i_2<\ldots<i_r\le k} (1-p_{i_1}-\ldots-p_{i_r})^{k}$$

$$= \sum_{r=1}^{k}(-1)^{r-1} \sum_{1\le i_1<i_2<\ldots<i_r\le k} (1-p_{i_1}-\ldots-p_{i_r})^{k-1}(p_{i_1}+\ldots+p_{i_r})$$

$$+\sum_{r=1}^{k}(-1)^{r-1} \sum_{1\le i_1<i_2<\ldots<i_r\le k} (1-p_{i_1}-\ldots-p_{i_r})^{k} - 1$$

$$= \sum_{r=1}^{k}(-1)^{r-1} \sum_{1\le i_1<i_2<\ldots<i_r\le k} (1-p_{i_1}-\ldots-p_{i_r})^{k-1} - 1. \quad (2.9)$$

Next Part 2 (coefficient of $(-p_{k+1})^s$)=

$$= \sum_{r=1}^{k}(-1)^{r-1}\frac{k!}{s!\ (k-s)!} \sum_{1\le i_1<i_2<\ldots<i_r\le k} (1-p_{i_1}-\ldots-p_{i_r})^{k-s} - \frac{k!}{s!\ (k-s)!}$$

$$= \frac{k!}{s!\ (k-s)!}[\sum_{r=1}^{k}(-1)^{r-1} \sum_{1\le i_1<i_2<\ldots<i_r\le k} (1-p_{i_1}-\ldots-p_{i_r})^{k-s} - 1]. \quad (2.10)$$

Finally Part 3 (Coefficient of $(-p_{k+1})^k$) $= \sum_{r=1}^{k}(-1)^{r-1}\frac{k!}{r!\ (k-r)!} - 1 = \sum_{r=0}^{k}(-1)^{r-1}\frac{k!}{r!\ (k-r)!} = -(1-1)^k = 0$.

Thus the result that $S(k) = 1$ for all k follows now from (2.8), (2.9), (2.10), Corollary 1.2 and the fact that $S(2) = 1$.

References

[1] Rattihalli, R. N. Multivariate Geometric Distribution, 2023. (Private communication)

Sreehari Maddipatla
6-B, Vrundavan Park,
New Sama Road, Vadodara-390024
Email: msreehari03@yahoo.co.uk.

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July-December (2024), 191–203

MEASURABLE SUBGROUPS OF REAL NUMBERS

S. G. DANI
(Received : 10 - 08 - 2024 ; Revised : 08 - 10 - 2024)

Abstract. We give an explicit elementary construction of Borel measurable subgroups of $\mathbb{R}$ which are uncountable and have Lebesgue measure zero, and exhibit a family $\{G_t\}_{t\in\mathbb{R}}$ of these subgroups such that for any $s, t \in \mathbb{R}$ with $s < t$, G_s is a subgroup of G_t and G_t/G_s is uncountable. We recall other known examples of uncountable subgroups of measure 0, and discuss some distinguishing features of the examples that we introduce. We also discuss the class of non-measurable subsets of $\mathbb{R}$ from a group-theoretic perspective.

1. Introduction

One of the points that is normally emphasized in a first course on Lebesgue measures is that apart from countable subsets, which necessarily have measure zero, there are also uncountable subsets of real numbers, like the Cantor set, which have measure 0; here and in the sequel in this paper, by a measure we mean the Lebesgue measure, even when the name is not mentioned. In this context one may ask whether there exist uncountable subsets of measure 0 which are also subgroups.*

The answer to this happens to be in the affirmative. Examples of such subgroups are known, and in particular have been described in connection with notions of fractal geometry, in [1], § 12.4 (see § 4 below for some details). The present author also constructed, before finding that reference, a class of such subgroups directly, in terms of binary expansions of numbers. The construction is analogous in spirit to that of the Cantor set and the examples are substantially different from those described in [1]. The purpose of this note is to present the construction, as it may be of independent

2010 Mathematics Subject Classification: 28A05, 28A78, 28A99, 20K20, 20K27
Key words and phrases: real line, Cantor set, uncountable measurable subgroups, binary expansions, non-measurable subsets

* The question was actually asked to me recently by Shameek Paul.

interest to a general reader. From the construction we also produce examples of strictly ordered uncountable families of such subgroups, and vector subspaces over the field of rational numbers, parametrized by real numbers (see Corollary 3.3).

Another point that is emphasized while introducing the Lebesgue measure is that there exist subsets which are not measurable, if we may assume the axiom of choice; the typical example of such a set, which goes back to G. Vitali (1905), is produced using the equivalence relation identifying two real numbers if they differ by a rational number, and forming a set, using the axiom of choice, which has exactly one element from each equivalence class. In the last section we discuss non-measurable subsets of $\mathbb{R}$, generalizing this from a group-theoretic perspective.

2. Uncountable measurable subgroups

In the sequel for any measurable subset we denote by $\ell(E)$ the Lebesgue measure of E. For any subset E of $\mathbb{R}$ we denote by $E - E$ the set of differences, viz. $\{s - t \mid s, t \in E\}$, and by $E + t$, where $t \in \mathbb{R}$, the translate of E by t, viz. $\{s + t \mid s \in E\}$.

Before going to the construction we recall the following about measurable subsets and subgroups. It is a well-known result, and the first part may be found in particular in [4], Lemma 4.27; in the hypothesis there, E is assumed to be of finite measure, but that condition is redundant, since to get the desired conclusion it suffices to prove the statement for a subset of finite positive measure. We shall however present here a proof, which is a variant of the argument in [1]; this will also serve for a ready reference.

Proposition 2.1. *Let E be a measurable subset of $\mathbb{R}$ with $\ell(E) > 0$. Then $E - E$ contains an interval of positive length. Consequently, every measurable subgroup of positive measure is the whole of $\mathbb{R}$.*

Proof. We begin by noting the following: Let E is a measurable subset of (a, b), with $a, b \in \mathbb{R}$, $a < b$, such that $\ell(E) > \frac{1}{2}(b-a)$ and let $\mu = \ell(E)$. Now for $t > 0$ the sets E and $E+t$ are of measure μ and contained $(a, b+t)$. Hence if $2\mu > (b-a)+t$, namely if $t < 2\mu - (b-a)$, then $E \cap (E+t)$ is of positive measure, and in particular nonempty, so we get that $t \in E - E$. Thus $(0, \delta)$ is contained in $E - E$, for $\delta = 2\mu - (b-a) > 0$. The Proposition can now be deduced, for any measurable set E with $0 < \ell(E) < \infty$, using the regularity property of the Lebesgue measure, as follows. By that property there exists, in particular, an open set O containing E for which $\ell(O) < 2\ell(E)$. Now O is a union of a pairwise disjoint sequence of open intervals $\{I_k\}$ and since $\ell(E) > \frac{1}{2}\ell(O)$ we get that there exists k such that $\ell(E \cap I_k) > \frac{1}{2}\ell(I_k)$. Now, by the above argument there exists $\delta > 0$ such that $(0, \delta) \subset (E \cap I_k) - (E \cap I_k) \subset E - E$.

The second assertion is a consequence of the first, since by the first part E contains an interval of positive length, which by the group property readily implies that it is the whole. □

We shall now describe the intended uncountable subgroups in terms of the binary expansions of numbers. Recall that any $t \in [0, 1)$ has a binary expansion of the form $t = 0.d_1 d_2 \cdots d_n \cdots$ with $d_n = 0$ or 1; the expansion is unique except for binary rational numbers in which case there are two expansions possible, one which has all digits 1 after some stage, and another with 0s after a stage; in our considerations below the ambiguity will not matter.

We consider sequences $\sigma = \{\sigma_n\}_{n=1}^{\infty}$, where $\sigma_n \in \mathbb{N}$, $\sigma_{n+1} \geq \sigma_n$ for all $n \in \mathbb{N}$, and $(\sigma_{2n} - \sigma_{2n-1}) \to \infty$ as $n \to \infty$. Let Σ denote the collection of all such sequences. Starting with any sequence $\{\alpha_k\}_{k=1}^{\infty}$ with $\alpha_k \in \mathbb{N}$ such that $\{\alpha_{k+1} - \alpha_k \mid k \in \mathbb{N}\}$ is unbounded, we can get a sequence $\sigma \in \Sigma$ as follows: choosing a sequence $\{k_n\}$ such that $\alpha_{k_n+1} - \alpha_{k_n} \to \infty$ we define $\sigma_n = \alpha_{k_{2n-1}}$ for all $n \in \mathbb{N}$.

We shall construct for each $\sigma \in \Sigma$ a Borel measurable subset G_σ which is a subgroup. For the purpose of giving examples of such subgroups it would indeed suffice to consider specific sequences, e.g. $\sigma_n = n^2$ for all n; see § 3 for more examples of specific kind. However, from a broader perspective, and perhaps some future uses, it would be convenient to work with sequences satisfying the general conditions as above.

Let $\sigma \in \Sigma$ be given. For each $k \in \mathbb{N}$ let T_k be the subset defined by

$$T_k = \{t = 0.d_1 d_2 \ldots \in [0, 1] \mid \forall n \in \mathbb{N}, d_i = d_j \ \forall i, j \in [\sigma_{2n-1} + 1, \sigma_{2n} - k]\};$$

in other words, T_k consists of the numbers in $t \in [0, 1)$ with the property that in their binary expansions the digits occurring at places from each of the intervals $[\sigma_{2n-1} + 1, \sigma_{2n} - k]$ are all equal, namely either all are 0 or all are 1 (it may be observed that the ambiguity in the binary expansions in the case of binary rationals does not matter, as the latter belong to T_k

with respect to either of the expansions); whether the digits are 0 or 1 can vary with n. It may also be clarified, that the interval $[\sigma_{2n-1}+1, \sigma_{2n}-k]$ can be empty for some n (when k is large) in which case the condition is understood to be vacuously true; note however that for any k, it being independent of n, the intervals as above are nonempty for all large n, and their size tends to infinity.

We note in particular that T_k, $k \in \mathbb{N}$, form a monotonically increasing sequence of Borel subsets of $[0,1]$. Let $T = \bigcup_{k=0}^{\infty} T_k$; note that T_k, $k \in \mathbb{N}$, and T depend on σ, though we have chosen not to incorporate σ as a part of the notation for them. Now let

$$G_\sigma = \mathbb{Z} + T = \{m + t \mid m \in \mathbb{Z} \text{ and } t \in T\}.$$

Then each G_σ is a Borel subset of $\mathbb{R}$; we shall show that each is, in fact, a subgroup; see Theorem 2.4 below. We first note the following.

Proposition 2.2. *For any $\sigma \in \Sigma$, a rational number belongs to G_σ if and only if it is a binary rational (a number of the form $k/2^n$ for some $n \in \mathbb{N}$). In particular, G_σ is a proper subset of $\mathbb{R}$.*

Proof. It is clear that all binary rationals belong to G_σ for all $\sigma \in \Sigma$. Also, it suffices to prove the converse statement for rational numbers r contained in $(0,1)$. The binary expansion $0.d_1d_2...$ of such a number is eventually periodic, that is, there exist natural numbers n_0 and p such that for $n \geq n_0$, $d_{n+p} = d_n$. Such a number can belong to G_σ, for a $\sigma \in \Sigma$, only if either $d_n = 0$ for all large n or $d_n = 1$ for all large n, and in either case it is a binary rational. □

Remark 2.3. In the spirit of the usual geometric construction of the Cantor set, the sets T_k, $k \in \mathbb{N}$ may be described as subsets as follows. Since by Proposition 2.2 all binary rationals are contained in T, we may restrict our attention to numbers other than binary rationals. Let I denote the complement of binary rationals in $[0,1]$, which also we shall still view as an interval, ignoring the absence of the binary rationals. Before any deletion begins the interval I is divided into 2^{σ_1} equal intervals. Next consider the subdivision of I into $2^{\sigma_2 - k}$ equal subintervals; in this subdivision each of the previous intervals gets divided into $2^{\sigma_2 - k - \sigma_1}$ equal subintervals, if $\sigma_2 - k - \sigma_1 > 0$; if not, no subdivision takes place. Two of these intervals at the two ends are retained and the rest are deleted; if there are only two then there is no deletion. This procedure is then repeated, considering the intervals from the subdivision into 2^{σ_3} intervals that are contained in the intervals which are retained and from each of the intervals only those that lie at the end of the subdivision into $2^{\sigma_4 - k}$ equal intervals are retained, and so on.

Theorem 2.4. *For every $\sigma \in \Sigma$ the subset G_σ defined as above is an uncountable subgroup of $\mathbb{R}$ with Lebesgue measure 0.*

Remark 2.5. Before going to the proof it would be worthwhile to note the following point. Given $t, t' \in [0, 1)$ with binary expansions $t = 0.d_1d_2\dots$ and $t' = 0.d'_1d'_2\dots$ determining the binary expansion of $t + t'$ involves some technical issues; when one of the binary expansions is terminating (the digits are 0 for all large n) then we can add the numbers in the usual way of familiar arithmetic, with 'carry overs'. However, when both the expansions are non-terminating, determination of the binary expansion of the sum is in general problematic; in this case t and t' are defined as real numbers, through the binary expansions, as limits of sequences (or series) and $t + t'$ is defined as a real number, as a limit - however, the digits in the binary expansion can not be determined by taking limits. It is nevertheless possible get around this difficulty, in confirming the relations as required in proving the theorem, by considering some possible cases, without having to know the specific binary expansions of the sums or differences of numbers. The author originally arrived at the result through such a procedure. However, for ready communicability of the proof a different route is adopted here. The reader is encouraged to reconstruct such a proof through arithmetic with binary representations, involving "carry overs" etc..

For the calculations in the proof of the main theorem we note following.

Remark 2.6. Let x be a positive number, expressed as $m + 0.d_1d_2\dots$, where $0.d_1d_2\dots$ is the binary expansion of $x - m$. Let $p, q \in \mathbb{Z}^+$, with $p < q$, and consider the block (finite sequence) of digits $[d_{p+1}, \dots, d_q]$. This block is the same as the block of digits in a $(q - p)$-digit binary integer produced as follows. Consider $[2^q x]$, the integer part of $2^q x$. It can be expressed, uniquely, as $[2^q x] = 2^{q-p}\alpha + \beta$, where $\alpha, \beta \in \mathbb{Z}^+$ and $\beta \leq 2^{q-p} - 1$. Then β has a (unique) $(q - p)$-digit binary representation; note that the digits on the left can be zero, and the number itself can be 0. It is straightforward to see that the block of digits in β is precisely the block $[d_{p+1}, \dots, d_q]$. In particular we note that for t from T_k as above, for any $n \geq 0$, if $p = \sigma_{2n}$ and $q = \sigma_{2n+1} - k$, then $[2^q t] = 2^{q-p}\alpha + \beta$, with $\alpha \in \mathbb{Z}^+$ and β either 0 or $2^{q-p} - 1$, the latter being the $(q-p)$-digit number for which all digits are 1.

Proof of Theorem 2.4: Let $\sigma = \{\sigma_n\}_{n=0}^{\infty}$ be a given sequence in Σ, and G_σ be the subset defined as above. It is clear that G_σ is uncountable since for $n \in \mathbb{N}$, for the σ_nth digit in the binary expansions of elements of G_σ, both 0 or 1 are admissible, independently of the digits at other places.

We now prove that G_σ is closed under addition. Let $x, x' \in G_\sigma$. Then we have $x = m + t$ and $x' = m' + t'$, where $m, m' \in \mathbb{Z}$ and $t, t' \in T$. Hence $x + x' = (m + m' + [t + t']) + \langle t + t' \rangle$, where $[t + t']$ denotes the integral part of $t + t'$ (equal to 0 or 1) and $\langle t + t' \rangle$ is the remaining fractional part. Thus it suffices to prove that $\langle t + t' \rangle \in T$.

Now consider $k \in \mathbb{N}$ such that $t, t' \in T_k$. Let $n \geq 0$ be arbitrary. For notational simplicity in the calculations below we put

$$p = \sigma_{2n-1} \text{ and } q = \sigma_{2n} - k.$$

Let $\tau = [2^q t]$, $\tau' = [2^q t']$, $\rho = \langle 2^q t \rangle$ and $\rho' = \langle 2^q t' \rangle$. As $t, t' \in T_k$, by Remark 2.6 it follows that τ and τ' are of the form $2^{q-p}\alpha + \beta$ and $2^{q-p}\alpha' + \beta'$, where $\alpha, \alpha' \in \mathbb{Z}^+$ and β and β' are either 0 or $2^{q-p} - 1$. We therefore have

$$2^q(t + t') = 2^q t + 2^q t' = \tau + \rho + \tau' + \rho' = 2^{q-p}(\alpha + \alpha') + (\beta + \beta') + (\rho + \rho').$$

We note that since β and β' are either 0 or $2^{q-p} - 1$, the only possibilities for the sum $\beta + \beta'$ are 0, $2^{q-p} - 1$, or $2(2^{q-p} - 1) = 2^{q-p} + (2^{q-p} - 2)$. Also, since $\rho, \rho' \in [0, 1)$, $[\rho + \rho']$ is either 0 or 1. This shows that the only possible values for $[(\beta + \beta') + (\rho + \rho')]$ are

$$0, 1, 2^{q-p} - 1, 2^{q-p}, 2^{q-p} + (2^{q-p} - 2) \text{ and } 2^{q-p} + (2^{q-p} - 1).$$

Thus we get that $[2^q(t + t')]$ has the form $2^{q-p}(\alpha + \alpha' + \delta) + \theta$, where $\delta = 0$ or 1, and $\theta \in \{0, 1, 2^{q-p} - 1, 2^{q-p} - 2\}$. If $\theta = \delta_1\delta_2 \ldots \delta_{q-p}$ is the binary representation of θ, we see that for each of the possibilities for θ, we have $\delta_i = \delta_j$ for all $i, j \in [1, q - p - 1]$; only δ_{q-p} can differ from the others. By Remark 2.6 this means that if $0.d_1d_2 \ldots$ is the binary expansion of $\langle t + t' \rangle$, then the possibilities for the block of digits $d_{p+1}, \ldots, d_q$ are all such that $d_i = d_j$ for all $i, j \in [p + 1, q - 1]$. Substituting for p and q we see that $d_i = d_j$ for all $i, j \in [\sigma_{2n-1} + 1, \sigma_{2n} - k - 1]$. Since this holds for all n we get that $\langle t + t' \rangle \in T_{k+1} \subset T$, as sought to be shown.

We next show that G_σ contains the negatives of all its elements. Let $x = m + t \in G_\sigma$, where $m \in \mathbb{Z}$ and $t \in T$. We have $-x = -m - t = -(m+1) + (1-t)$, so it suffices to show that for all $t \in T$, $1 - t \in T$. Consider any $t \in T_k$, for some $k \in \mathbb{N}$. Let $n \geq 0$ be arbitrary and, as before, let $p = \sigma_{2n}$ and $q = \sigma_{2n+1} - k$. Let $\tau = [2^q t]$ and $\rho = \{2^q t\}$. Then we have $\tau = 2^{q-p}\alpha + \beta$, with $\alpha \in \mathbb{Z}^+$ and $\beta \in \{0, 2^{q-p} - 1\}$. Thus

$$\begin{aligned} 2^q(1-t) = 2^q - (\tau + \rho) &= 2^q - 2^{q-p}\alpha - \beta - \rho \\ &= 2^{q-p}(2^p - \alpha - 1) + (2^{q-p} - \beta - 1) + (1 - \rho). \end{aligned}$$

Now $[1-\rho]$ is either 0 or 1 (the latter holds when $\rho = 0$). Since β is either 0 or $2^{q-p} - 1$ we get that $[(2^{q-p} - \beta - 1) + (1 - \rho)]$ is one of $0, 1, 2^{q-p} - 1$ and 2^{q-p}. It follows that $[2^q(1-t)] = 2^{q-p}\alpha' + \beta'$, where $\alpha' = 2^p - \alpha - 1 + \delta$, where δ is either 0 or 1, and β' is one of $0, 1$, or $2^{q-p} - 1$. If $\delta_1 \ldots \delta_{q-p}$ is the binary representation of β', in all the possible cases we see that $\delta_i = \delta_j$ for all $i, j \in [1, q-p-1]$; only δ_{q-p} can differ from the others. By Remark 2.6 this shows that if $0.d_1d_2\ldots$ is the binary representation of $1-t$ then $d_i = d_j$ for all $i, j \in [p+1, q-1] = [\sigma_{2n-1} + 1, \sigma_{2n} - k - 1]$. Since this holds for all n we get that $1 - t \in T_{k+1} \subset T$. This completes the proof that G_σ contains the negatives of all its elements, and in turn that G_σ is a subgroup.

Finally, we note that G_σ is a Borel subset and by Proposition 2.2 is a proper subset of $\mathbb{R}$. Hence by Proposition 2.1, it follows that $\ell(G_\sigma) = 0$. This completes the proof of the theorem. □

Remark 2.7. One can also check directly that $\ell(G_\sigma) = 0$, without recourse to Propositions 2.1 and 2.2. Since $G_\sigma = \mathbb{Z} + T = \cup_{m\in\mathbb{Z}}(m + T)$ and $\ell(m + T) = \ell(T)$ for all $m \in \mathbb{Z}$, it suffices to show that $\ell(T) = 0$. Since $T = \cup_{k\in\mathbb{N}} T_k$, in turn it suffices to show that $\ell(T_k) = 0$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Recall that $(\sigma_{2n+1} - \sigma_{2n}) \to \infty$, as $n \to \infty$. Let n_0 such that $\sigma_{2n+1} - \sigma_{2n} \geq 2$ for all $n \geq n_0$. Then T_k is contained in the set

$$E := \{t = 0.d_1d_2\ldots \mid d_{\sigma_{2n}+1} = d_{\sigma_{2n}+2} \text{ for all } n \geq n_0\}.$$

We note that for any $j \in \mathbb{N}$, $\ell(\{t = 0.d_1d_2\cdots \mid d_j = d_{j+1}\}) = \frac{1}{2}$. It follows that $\ell(E) = 0$, as it is the intersection of infinitely many subsets of the latter kind which are pairwise independent of each other. Hence $\ell(T_k) = 0$, and in turn $\ell(G_\sigma) = 0$.

From the above construction we can also get uncountable vector subspaces of $\mathbb{R}$ over the field $\mathbb{Q}$ of rational numbers, with Lebesgue measure 0.

Corollary 2.8. *Let $\sigma \in \Sigma$ and G_σ be the subgroup as above. Let*

$$V_\sigma = \left\{ \frac{1}{q} x \mid q \in \mathbb{N} \text{ and } x \in G_\sigma \right\}.$$

Then V_σ is an uncountable vector subspace of $\mathbb{R}$ over $\mathbb{Q}$, which is a Borel set of measure 0.

Proof. We note that for $x_1, x_2 \in G_\sigma$ and $q_1, q_2 \in \mathbb{N}$, $\frac{1}{q_1}x_1 + \frac{1}{q_2}x_2 = \frac{1}{q_1 q_2}(q_2 x_1 + q_1 x_2) \in V_\sigma$, since $q_2 x_1 + q_1 x_2 \in G_\sigma$ and $q_1 q_2 \in \mathbb{N}$. Also for any $r = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b > 0$, and $v = \frac{1}{q}x \in V_\sigma$, where $q \in \mathbb{N}$ and $x \in G_\sigma$, we have $rv = \frac{a}{b}(\frac{1}{q}x) = \frac{1}{bq}(ax) \in V_\sigma$, since $bq \in \mathbb{N}$ and $ax \in G_\sigma$. Thus V_σ is a vector subspace of $\mathbb{R}$ over $\mathbb{Q}$. It is clearly uncountable, as it contains G_σ. Also, as G_σ is a Borel set of measure 0, so is $\frac{1}{q}G_\sigma$ for all $q \in \mathbb{N}$, and since V_σ is a (countable) union of these it follows that V_σ is a Borel subset and $\ell(V_\sigma) = 0$. □

Remark 2.9. We note that for any $\sigma \in \Sigma$, any vector subspace W of V_σ over $\mathbb{Q}$ is a Lebesgue measurable subspace, since $\ell(V_\sigma) = 0$. The cardinality of such subspaces is the same as the cardinality of $\mathcal{P}(\mathbb{R})$, viz. the class of all subsets of $\mathbb{R}$.

3. Comparison of the measurable subgroups

It is possible to compare the measurable subgroups constructed in the last section, starting with a class of sequences of natural numbers, and get a sense of the vastness of the collection. In this section we discuss various results in this respect.

Proposition 3.1. *Let $\sigma = \{\sigma_n\}, \sigma' = \{\sigma'_n\}$ be two sequences in Σ. Suppose that there exist $n_0 \in \mathbb{N}$ and $M \in \mathbb{N}$ such that for all $n \geq n_0$ we have*

i) $\sigma_{2n-1} \leq \sigma'_{2n-1}$ and

ii) $\sigma'_{2n} \leq \sigma_{2n} + M$.

Then G_σ is a subgroup of $G_{\sigma'}$; if, moreover, $\sigma_{2n-1} < \sigma'_{2n-1}$ for infinitely many $n \in \mathbb{N}$, then G_σ is a subgroup of $G_{\sigma'}$ and $G_{\sigma'}/G_\sigma$ is uncountable.

Proof. For $k \in \mathbb{N}$ let T_k and T'_k be the subsets of $[0, 1]$ associated with the sequences σ and σ' in the construction the subgroups G_σ and $G_{\sigma'}$, as above. Consider $t \in T_k$ and let $t = 0.d_1 d_2 \ldots$ be its binary representation. Then we have $d_i = d_j$ for all $i, j \in [\sigma_{2n-1} + 1, \sigma_{2n} - k]$ for all $n \in \mathbb{N}$. For $n \geq n_0$ we have $\sigma'_{2n-1} \geq \sigma_{2n-1}$ and $\sigma'_{2n} \leq \sigma_{2n} + M$, and hence we get in particular that $d_i = d_j$ for all $i, j \in [\sigma'_{2n-1}+1, \sigma'_{2n}-(k+M)]$. When k is large enough so that $\sigma'_{2m} - (k+M) \leq \sigma'_{2m-1}$ for all $m = 1, \ldots, n_0$ we get that $d_i = d_j$ for all $i, j \in [\sigma'_{2n-1}+1, \sigma'_{2n}-(k+M)]$ for all $n \in \mathbb{N}$. Hence $t \in T'_{k+M}$ and in particular $t \in G_{\sigma'}$. Since $G_\sigma = \mathbb{Z} + \bigcup T_k$ it follows that G_σ is contained in $G_{\sigma'}$.

Now suppose that $\sigma_{2n-1} < \sigma'_{2n-1}$ for infinitely many n; let $\{n_k\}$ be an increasing sequence of n's for which this holds. We first observe that G_σ is a proper subgroup of $G_{\sigma'}$. Let $t \in [0,1]$ be an element such that $t = 0.d_1d_2\ldots$, with $d_i = 1$ for $i = 2\sigma'_{n_k} - 1$ and $d_i = 0$ for $i \in [\sigma'_{2n_k-1}+1, \sigma_{2n}]$ for all $n \in \mathbb{N}$. Then we see that t is an element of $G_{\sigma'}$ which is not contained in G_σ.

Now let $\{n_k\}$ be the sequence as above and for each $r \in \mathbb{N}$ define a sequence $\sigma^{(r)} = \{\sigma_n^{(r)}\}$ by $\sigma_n^{(r)} = \sigma_n + 1$ if $n = 2n_{2^r k} - 1$ for some $k \in \mathbb{N}$ and $\sigma_n < \sigma_{n+1}$ (this holds for all but finitely k's), and $\sigma_n^{(r)} = \sigma_n$ otherwise. Then we see that each $\{\sigma_n^{(r)}\}$ is an element of Σ. Applying the conclusions noted above, we compare the subgroups corresponding to various pairs, and get that $G_\sigma \subset G_{\sigma^{(r)}} \subset G_{\sigma'}$ and also, for any $r, r' \in \mathbb{N}$ if $r > r'$ then $G_{\sigma^{(r)}}$ is a proper subgroup of $G_{\sigma^{(r')}}$. We note that for each $r \in \mathbb{N}$, $G_{\sigma^{(r)}}$ is in one-one correspondence (as a set) with the cartesian product of $G_{\sigma^{(r)}}/G_{\sigma^{(r+1)}}$ and $G_{\sigma^{(r+1)}}$, and hence $G_{\sigma^{(1)}}/\cap_{r=1}^{\infty} G_\sigma^{(r)}$ is in one-one correspondence with $\prod_{r=1}^{\infty}(G_{\sigma^{(r)}}/G_{\sigma^{(r+1)}})$. As the cardinality of each $G_{\sigma^{(r)}}/G_{\sigma^{(r+1)}}$ is at least two, and G_σ is contained in $\cap_{r=1}^{\infty} G_\sigma^{(r)}$, this shows that $G_{\sigma^{(1)}}/G_\sigma$ is uncountable. Therefore $G_{\sigma'}/G_\sigma$ is uncountable. □

Example 3.2. For any $x \in \mathbb{R}$ let $\sigma^{(x)}$ denote the sequence $\{\sigma_n^{(x)}\}$ defined as follows: as before we denote by $[\,t\,]$ the integer part for $t \in \mathbb{R}$ and for all $n \in N$ let

$$\sigma_{2n-1}^{(x)} = (2n-1)^2 - [e^{-x}\log(n+1)], \text{ and}$$
$$\sigma_{2n}^{(x)} = \min\{(2n)^2, (2n+1)^2 - [e^{-x}\log(n+2)]\};$$

note that the second term in parenthesis is the same as $\sigma_{2n+1}^{(x)}$. Then $\sigma^{(x)}$ is a monotonically increasing sequence. We note that for n such that $e^{-x}\log(n+2) \leq 4n+1$, which for any x holds for all but finitely many n's, we have $\sigma_{2n}^{(x)} = (2n)^2$. We see from this that $\sigma^{(x)} \in \Sigma$ for all $x \in \mathbb{R}$. Now let $x, x' \in \mathbb{R}$, with $x < x'$, and let $\sigma = \{\sigma_n\}$ and $\sigma' = \{\sigma'_n\}$ be the sequences in Σ corresponding to x and x' respectively; for convenience we suppress x and x' from the notation. Then we see that $\sigma_{2n} = \sigma'_{2n} = (2n)^2$ for all large n, and

$\sigma'_{2n-1} - \sigma_{2n-1}$, which equals $[e^{-x}\log(n+1)] - [e^{-x'}\log(n+1)]$, is positive for all large n, since $e^{-x}\log(n+1) - e^{-x'}\log(n+1) = (e^{-x} - e^{-x'})\log(n+1) \geq 1$ for all large n. Hence by Proposition 3.1 we get that $G_{\sigma(x)}$ is a subgroup of $G_{\sigma(x')}$ and $G_{\sigma(x')}/G_{\sigma(x)}$ is uncountable.

Corollary 3.3. *There exists a family $\{W_x\}_{x\in\mathbb{R}}$ of vector subspaces of $\mathbb{R}$ over the field $\mathbb{Q}$, which are Borel measurable subsets with measure 0, and for any $x, x' \in \mathbb{R}$, W_x is a vector subspace of $W_{x'}$ and the dimension of $W_{x'}/W_x$ over $\mathbb{Q}$ is uncountable.*

Proof. For any $x \in \mathbb{R}$ let $G_{\sigma(x)}$ be the subgroup as in in Examples 3.2 and let W_x be the vector subspace of $\mathbb{R}$ over $\mathbb{Q}$ spanned by $G_{\sigma(x)}$. Then as seen in Corollary 2.8 each W_x is a Borel subset of measure 0. Let $x, x' \in \mathbb{R}$. Then as noted above $G_{\sigma(x)}$ is a subgroup of $G_{\sigma(x')}$ and $G_{\sigma(x')}/G_{\sigma(x)}$ is uncountable. This implies that $W_x \subset W_{x'}$ and $W_{x'}/W_x$ is uncountable, and hence of uncountable dimension over $\mathbb{Q}$.

4. Other constructions of uncountable subgroups of measure zero

As mentioned in the Introduction, there have been other examples of uncountable subgroups of measure zero, found in [1]. Here we briefly recall the examples and also formulate a general construction inspired by them. We shall however not go into the details of fractal dimensions that are involved, for which the reader is referred to [1].

Let $\varphi = \{n_k\}$ be an increasing sequence of natural numbers and ψ be a positive real-valued function on $\mathbb{N}$ such that $n\psi(n) \to 0$ $n \to \infty$. Define, for all $r \in \mathbb{N}$,

$$G_{\varphi,\psi}(r) = \left\{ t \in \mathbb{R} \mid \forall k \in \mathbb{N},\ \exists\, p \in \mathbb{Z} \text{ such that } |t - \frac{p}{n_k}| \leq r\psi(n_k) \right\}, \text{ and}$$

$$G_{\varphi,\psi} = \bigcup_{r\in\mathbb{N}} G_{\varphi,\psi}(r).$$

We note that if $t \in G_{\varphi,\psi}(r)$ and $t \in G_{\varphi,\psi}(r')$, where $r, r' \in \mathbb{N}$, and, for a $k \in \mathbb{N}$, $p, p' \in \mathbb{Z}$ are such that $|t - \frac{p}{n_k}| \leq r\psi(n_k)$ and $|t' - \frac{p'}{n_k}| \leq r'\psi(n_k)$, then we have $|(t - t') - \frac{p-p'}{n_k}| \leq (r + r')\psi(n_k)$; this shows that $t - t' \in G_{\varphi,\psi}(r + r')$, and in turn that $G_{\varphi,\psi}$ is a subgroup. We see that for any $m \in \mathbb{Z}$, $\ell(G_{\varphi,\psi}(r) \cap [m, m+1]) \leq 2(n_k + 1)r\psi(n_k)$, for each k, and since $n_k\psi(n_k) \to 0$ as $k \to \infty$ we get that $\ell(G_{\varphi,\psi}(r) \cap [m, m+1]) = 0$; since this holds for each m we get that $\ell(G_{\varphi,\psi}(r)) = 0$ for each r, and in turn that $\ell(G_{\varphi,\psi}) = 0$. We note also that each $G_{\varphi,\psi}(r)$ is a Borel measurable set. For a general choice of φ and ψ as above, $G_{\varphi,\psi}$ being uncountable, or even a nontrivial subgroup, does not follow, without some further conditions.

It is proved in [1] that if $s \in (0,1)$, and $\varphi = \{n_k\}$ is such that $n_{k+1} \geq \max\{n_k^k, 4n_k^{1/s}\}$ for each k, and $\psi(n) = n^{-1/s}$ for all $n \in \mathbb{N}$, then $G_{\varphi,\psi}$ is of Hausdorff dimension s; this implies in particular that each of them is an uncountable subgroup with Lebesgue measure 0. This provides in particular an uncountable family of such subgroups, they being of distinct Hausdorff dimensions, one for each $s \in (0,1)$. The inclusion relations between these are not clear however, and a stratification analogous to Proposition 3.1 may not be possible for examples drawn from them.

Remark 4.1. We note that our examples G_σ, described in the preceding sections, are not only of measure 0, but their Hausdorff dimension is also zero; in fact for any G_σ, $\sigma \in \Sigma$, it can be readily seen that the lower box-dimension (see [1], § 3.1, for definition) of $G_\sigma \cap I$ is 0 for any bounded interval; (it may be borne in mind here that the box dimension is defined only for bounded metric spaces, calling for consideration of only intersections with bounded intervals). We note that the lower box dimension of a set majorizes its Hausdorff dimension (see [1], (3.17)), and since the Hausdorff dimension of G_σ is the same as that of $G_\sigma \cap I$ for any interval I, this gives a stronger statement that for any $\sigma \in \Sigma$ the Hausdorff dimension of G_σ is 0. Thus the subgroups G_σ, $\sigma \in \Sigma$, are "smaller", or "thinner", than the examples from [1] recalled above, while still being uncountable – in particular they are distinct from the latter class of examples.

Remark 4.2. It can be seen that if $\varphi = \{n_k\}$ is a sequence such that n_k divides n_{k+1} for all k, then $G_{\varphi,\psi}$ is uncountable for any ψ such that $n\psi(n) \to 0$ as $n \to \infty$; in this case $G_{\varphi,\psi} \cap [0,1]$ is seen to be the intersection of a decreasing sequence of uncountable compact sets, not having any isolated points, and hence uncountable. Similar conclusion can also be drawn from rapid growth of $\{n_k\}$, in place of divisibility, considerably weaker than the condition involved in the examples from [1] recalled above. Thus the family $G_{\varphi,\psi}$ also provides more examples of uncountable subgroups of measure 0, other than those recalled from [1] as above – we shall however not go into further details of this here.

5. Non-measurable subsets

The usual construction of non-measurable sets, due to Vitali, may also be viewed in a group-theoretic perspective, involving in place of the rationals any countable subgroup which is not cyclic; we note that a subgroup of $\mathbb{R}$ is either cyclic or dense in $\mathbb{R}$ (according to whether it admits a smallest positive number or not).

Proposition 5.1. *Let G be a countable dense subgroup of $\mathbb{R}$ and let $\approx$ be the equivalence relation defined by $s \approx t$ if $s - t \in G$. Let E be a subset of R such that the intersection with each equivalence class with respect to $\approx$ is a singleton set. Then E is not measurable.*

Proof. Suppose E is measurable. For each $t \in \mathbb{R}$ we denote by $t + E$ the subset $\{t + s \mid s \in E\}$. Since the intersection of E with every equivalence class is nonempty it follows that $\bigcup_{g \in G}(g + E) = \mathbb{R}$. Each $g + E$ is a measurable subset and by translation invariance of the Lebesgue measure $\ell(g+E) = \ell(E)$. The preceding observation therefore implies that $\ell(E) > 0$. Hence by Proposition 2.1 the set of differences $E - E$ contains an interval of positive length. Now let $g \in G$ and suppose that $g \in E - E$. Then there exist $e, f \in E$ such that $g = e - f$. This means however that e and f belong to the same equivalence class with respect to the relation $\approx$. Since E contains only one representative from any equivalence class we get that $e = f$ and hence $g = 0$. Thus $G \cap (E - E) = \{0\}$. But this is a contradiction since G is dense (by hypothesis) and $E - E$ is proved to contain an interval of positive length. Hence E is not measurable. □

Remark 5.2. The proof of Proposition 5.1 can be completed without involving Proposition 2.1, along the lines of the usual proof in the case of rationals, as follows. After deducing that $\ell(E) > 0$ we may proceed as follows. There exists $m \in \mathbb{N}$ such that $\ell(E \cap [-m, m]) > 0$. Let $E_m = E \cap [-m, m]$. Since G is dense it contains a sequence $\{g_i\}$ of distinct elements such that $|g_i| < 1$ for all i. Then $\{g_i + E_m\}$ is a sequence of pairwise disjoint measurable subsets, and since $\ell(E_m) > 0$, by translation invariance we get that $\ell(\bigcup_{i=1}^{\infty} g_i + E_m) = \infty$. However this is a contradiction since the set is contained in $[-m - 1, m + 1]$, and must have Lebesgue measure at most $2(m + 1)$. Hence E is not a measurable subset.

Existence of a set E satisfying the condition in the hypothesis, that it intersects each equivalence class in a singleton set, involves, and is assured by the axiom of choice. When G is a vector space over $\mathbb{Q}$, of countable dimension, such a set may be "visualized" in terms of the vector space structure of $\mathbb{R}$ over $\mathbb{Q}$. A vector subspace E over $\mathbb{Q}$ complementary to G, namely such that $G \oplus E = \mathbb{R}$ serves as a set satisfying the condition in the proposition in this instance. Note however that existence of such complements, in the case of vector spaces of uncountable dimension as in this instance, also depends on the axiom of choice; in fact it is known that existence of non-measurable sets can not be proved without involving the axiom of choice (see for instance [4], p. 93). It may also be observed that in this case we get a non-measurable subset which is a vector space over $\mathbb{Q}$ (in particular a subgroup).

Acknowledgement: The author is thankful to C.S. Aravinda and Shreyasi Datta for going over an earlier version of this article and offering useful comments.

References

[1] Kenneth Falconer, Fractal geometry. Mathematical foundations and applications. Second edition. John Wiley & Sons, Inc., Hoboken, NJ, 2003. xxviii+337 pp.

[2] H. L. Royden, Real analysis. Third edition. Macmillan Publishing Company, New York, 1988. xx+444 pp.

[3] Terence Tao, An introduction to measure theory, Graduate Studies in Mathematics, 126. American Mathematical Society, Providence, RI, 2011. xvi+206 pp.

[4] William P. Ziemer, Modern real analysis, Second edition. With contributions by Monica Torres, Graduate Texts in Mathematics 278, Springer, Cham, 2017. xi+382 pp.

S. G. Dani
UM-DAE Center for Excellence in Basic Sciences
University of Mumbai, Santacruz
Mumbai 400098
Email: shrigodani@cbs.ac.in

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos. 3-4, July - December(2024), 204–213

PROBLEM SECTION

In volume 93(1-2) 2024 of the the Mathematics Student, we had invited solutions for a set of eight new problems.

We have received solutions for problems 4, 6, 8 and part (1) of problem 2 of MS 93(1-2) 2024. There were some inadvertent errors in the statements of problems 5 and 7. We are sorry for the errors and the inconvenience that might have caused, and we include the corrected versions along with a set of eight new problems for the current issue MS 93(3-4) 2024. Dr. R. Mohan from Azim Premji University, Bengaluru, India has provided solutions for problems 2 part 1, 4 and 6. Dr. Henry Ricardo, Westchester Area Math Circle, New York, USA also has provided solution for problem 4. We received two solutions for problem 8 from Mr. Arka Prabha Roy, IIT Kharagpur, India and Dr. Hari Kishan, D. N. College, Meerut, India. Problems 1, 2 part (2), 3 and 5 have remained unsolved. We encourage the readers to provide solutions for the problems; and propose new problems for the problem section. Below we present the solutions, so far received, based on the recommendations of the proposers and the experts. We appreciate the contributions from the proposers and sincerely acknowledge all solutions received from the readers.

First we present new problems for this volume. We invite solutions for these problems and for problems 1, 2 part (2), 3, 5 of MS 93 (1-2) 2024 from the readers till March 30, 2025. Correct solutions received by this date will be published in volume 94 (1-2) 2025 of The Mathematics Student, which is scheduled to be published in April 2025.

New Problems

MS 93(3-4) 2024 : Problem 1 (proposed by **Dr. Henry Ricardo**, Westchester Area Math Circle, New York, USA).

If an $n \times n$ matrix A commutes with all $n \times n$ nilpotent matrices, must A be nilpotent? Determine the whole class of these matrices. (Recall that a square matrix M is said to be nilpotent whenever $M^k = O$ for some positive integer k.)

MS 93 (3-4) 2024 : Problem 2 (proposed by **Mr. Himadri Lal Das**, IIT Kharagpur, India).

Let $\{F_n\}_{n\geq 0}$ be the sequence of *Fibonacci numbers*, where $F_0 = 1$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$, $\forall\, n \in \mathbb{N}$. Evaluate the following limit, if it exists

$$\lim_{n\to\infty}\left(\sum_{\substack{k_1+k_2+\cdots+k_n=n,\\ 1\leq i\leq n}} \frac{k_i F_{i-1}}{k_1!k_2!\ldots k_n!}\right)^{\frac{1}{n}}.$$

MS 93(3-4) 2024 : Problem 3 (proposed by **Dr. Andrés Ventas**, Santiago de Compostela, Spain).

Let (x_0, y_0) denote the fundamental solution to the negative Pell's equation $x^2 - Dy^2 = -1$ and let (x, y) denote the fundamental solution to the positive Pell's equation $x^2 - Dy^2 = +1$, both of them for the same D. Prove that all the infinite solutions to the negative Pell's equation, $x^2 - Dy^2 = -1$ satisfy the following recurrences,

$$x_{n+2} = 2xx_{n+1} - x_n;\ \text{with } x_{-1} = -x_0 \text{ and } x_0 = x_0,$$

$$y_{n+2} = 2xy_{n+1} - y_n;\ \text{with } y_{-1} = y_0,\ \text{and } y_0 = y_0.$$

Problems 4-6 (proposed by **Dr. B. Sury**, ISI, Bengaluru, India).

MS 93(3-4) 2024 : Problem 4.

Let Q denote the set of all positive integers that are one less than the perfect powers > 1. That is,

$$Q = \{3, 7, 8, 15, 24, 26, 31, \cdots\} = \{4-1, 8-1, 9-1, 16-1, 25-1, \cdots\}.$$

Evaluate $\sum_{q\in Q}\frac{1}{q}$.

MS 93(3-4) 2024 : Problem 5.

For a function f from the set $\mathbb{N}$ of positive integers to itself, define $f^{n+1} = f \circ f^n$ for all $n \geq 1$. For a positive integers r, determine all positive integers d such that there exists a function f satisfying

$f^r(n) = n + d$ for all $n \in \mathbb{N}$. For such r, d determine the number of such functions.

MS 93(3-4) 2024 : Problems 6.

Consider the series $\sum_{n\geq 1} \frac{1}{n^3 \sin^2(n)}$. Show that the sequence of partial sums s_n satisfies $s_{354} < 5$ while s_{355} is close to 30. Note that π is close to 355/113. Consider the infimum $irr(\pi)$ of all constants $\alpha > 0$ such that there are only finitely many solutions of $|\pi - p/q| \leq \frac{1}{q^\alpha}$. If $irr(\pi) \geq 2.5$, show that the series diverges. It is expected (but unknown) that $irr(\pi) = 2$; if this is so, show that the series converges.

Problems 7-8 (proposed by **Dr. Chudamani Pranesachar Anil Kumar**, KREA University, India).

MS 93(3-4) 2024 : Problem 7.

Let $\mathbb{F}$ be a field. Let $A, B \subset \mathbb{F}$ be two finite subsets. For any $\xi \in \mathbb{F}^*$, We say $\{(a_1, b_1), (a_2, b_2)\} \subset A \times B$ is a ξ-pair in $A \times B$ if $\frac{b_1 - b_2}{a_1 - a_2} = \xi$. Let F_ξ be the set of all ξ pairs in $A \times B$. Let $B - A\xi = \{b - a\xi \mid a \in A, b \in B\}$. A line $L \subset \mathbb{F}^2$ is said to be k-rich if L contains exactly k points of the set $A \times B$.

(1) Show that if $\mathbb{F}$ is a finite field then there exists $\xi \in \mathbb{F}^*$ such that
$$| F_\xi | \geq \frac{2}{| \mathbb{F}^* |} \binom{| A |}{2} \binom{| B |}{2}.$$
(2) (Here $\mathbb{F}$ need not be finite). If there are no k-rich lines in $A \times B$ with slope ξ for any $k \geq 4$ then show that the number of points in $A \times B$ is bounded below and above as:
$$\min\left(\frac{3}{2} | B - A\xi | + \frac{1}{2} | F_\xi |, | B - A\xi | + | F_\xi | \right) \geq | A || B | \geq | B - A\xi | + \frac{2}{3} | F_\xi | .$$

MS 93(3-4) 2024 : Problems 8.

Let n be a positive integer and $S = \{1, 2, \cdots, n\}$. We say a partition of S say $\mathcal{X} = \{S_1, S_2, \cdots, S_k\}$ is split adjacent to a partition $\mathcal{Y} = \{T_1, T_2, \cdots, T_l\}$ of S if the following condition holds. There exist i, r, s such that $1 \leq i \leq l, 1 \leq r \neq s \leq k$ and $T_i = S_r \cup S_s$ and $\mathcal{Y} \backslash \{T_i\} = \mathcal{X} \backslash \{S_r, S_s\}$, that is, exactly some two parts of $\mathcal{S}$ combine together to give a part of $\mathcal{T}$ and the remaining parts are the same. We denote this by

$$\mathcal{Y} \longrightarrow \mathcal{X}.$$

A path from the partition $\{S\}$ of S where there is only one part to the finest partition $\{\{1\},\{2\},\{3\},\cdots,\{n\}\}$ of S where there are n parts is a sequence of split adjacent partitions $\mathcal{Y}_i$ of S of the form

$$\mathcal{Y}_1 = \{S\} \longrightarrow \mathcal{Y}_2 \longrightarrow \cdots \longrightarrow \mathcal{Y}_{t-1} \longrightarrow \mathcal{Y}_t = \{\{1\},\{2\},\{3\},\cdots,\{n\}\}.$$

(1) Enumerate the number of paths from $\{S\}$ to $\{\{1\},\{2\},\{3\},\cdots,\{n\}\}$.
(2) Let $\mathcal{U} = \{A_1, A_2, \cdots, A_m\}$ be a fixed partition of S. Enumerate the number of paths from $\{S\}$ to $\{\{1\},\{2\},\{3\},\cdots,\{n\}\}$ passing through $\mathcal{U}$.

Revised Problem 5 of MS 93(1-2) 2024.

Find all triangles with vertices $A = (0,0)$, $B = (4,3)$ and $C = (u,v)$ where u, v are integers and AC, BC have integer lengths.

Revised Problem 7 of MS 93(1-2) 2024.

Let α, β and γ be any real numbers satisfying $\alpha^2(\beta+\gamma) + \beta^2(\gamma+\alpha) + \gamma^2(\alpha+\beta) = -2\alpha\beta\gamma$. Let

$$A := \sin^4\alpha + \sin^4\beta + 16\cos^4\left(\frac{\alpha+\beta+2\gamma}{2}\right)\,\sin^4\left(\frac{\alpha+\beta}{2}\right),$$

$$B := \sin^3\alpha + \sin^3\beta - 8\cos^3\left(\frac{\alpha+\beta+2\gamma}{2}\right)\,\sin^3\left(\frac{\alpha+\beta}{2}\right)$$

and

$$C := \sin^7\alpha + \sin^7\beta - 128\cos^7\left(\frac{\alpha+\beta+2\gamma}{2}\right)\,\sin^7\left(\frac{\alpha+\beta}{2}\right).$$

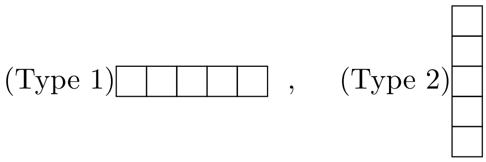
Prove that $\dfrac{AB}{C} = \dfrac{6}{7}$.

Solutions to the New Problems

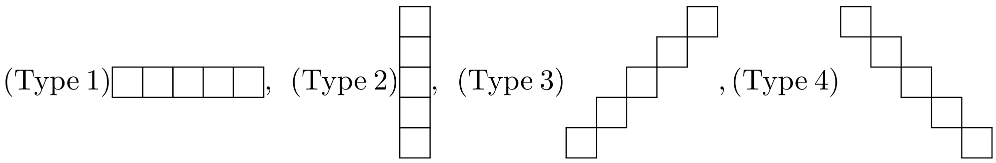
MS 93 (1-2) 2024 : Problem 2 (proposed by **Dr. Chudamani Pranesachar Anil Kumar**, KREA University, India).

In a 2023×2023 chess-board (not the usual 8×8), the four corner squares are removed.

(1) Can the rest be covered by a combination of 5×1 dominoes, (that is, 5 square boxes) by putting them horizontally, vertically on the board?



(2) Can the rest be covered by a combination of 5×1 dominoes, (that is, 5 square boxes) by putting them horizontally, vertically or diagonally in both ways (as shown in the figure) on the board?



Solution for Part 1: (by **Dr. R. Mohan**, Azim Premji University, Bengaluru, India).

Let $n = 5k + 3$ for some positive integer k. Each unit square on an $n \times n$ chessboard is indexed by coordinates (i, j), where $0 \leq i, j \leq n - 1$. The lower-left corner is indexed by $(0, 0)$, and the upper-right corner by $(n - 1, n - 1)$. Therefore, the corner squares of the chessboard are located at $(0, 0)$, $(0, n - 1)$, $(n - 1, 0)$, and $(n - 1, n - 1)$.
Let $f(x, y)$ be a polynomial in the variables x and y defined as follows:

$$\begin{aligned} f(x, y) &= \left(1 + x + x^2 + \cdots + x^{n-1}\right)\left(1 + y + y^2 + \cdots + y^{n-1}\right) \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} x^i y^j. \end{aligned} \tag{1}$$

Then each term $x^i y^j$ represents the unit square indexed by (i, j).

(1) Let if possible assume that we can cover the $n \times n$ chessboard with corners removed by type 1 and type 2 pentaminoes. A type 1 pentomino occupies 5 horizontal units squares on the board, say

from (a, b) to $(a+4, b)$. So placing a type 1 on the chess board may be represented as

$$x^a y^b + x^{a+1}y^b + x^{a+2}y^b + x^{a+3}y^b + x^{a+4}y^b$$
$$= x^a y^b(1 + x + x^2 + x^3 + x^4).$$

Similarly, placing a type 2 on the chess board may be represented as

$$x^a y^b + x^a y^{b+1} + x^a y^{b+2} + x^a y^{b+3} + x^a y^{b+4}$$
$$= x^a y^b(1 + y + y^2 + y^3 + y^4).$$

Then we can write

$$\begin{aligned} f(x,y) &= (1 + x + x^2 + x^3 + x^4)g(x,y) \\ &\quad + (1 + y + y +^2 + y^3 + y^4)h(x,y) \\ &\quad + \left(1 + x^{n-1} + y^{n-1} + x^{n-1}y^{n-1}\right). \end{aligned} \tag{2}$$

Now, let ω be the 5^{th} primitive complex root of unity. Then for $t \equiv r \pmod 5$ we have $\omega^p = \omega^r$. We also have

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0.$$

Substituting $x = y = \omega$ in equation 1 gives us

$$f(\omega, \omega) = (1 + \omega + \omega^2 + \cdots + \omega^{n-1})^2.$$

However, we can write

$$\begin{aligned} 1 + \omega + \omega^2 + \cdots + \omega^{n-1} &= (1 + \omega + \cdots + \omega^4) \\ &\quad + \omega^5(1 + \omega + \cdots + \omega^4) \\ &\quad\ \vdots \\ &\quad + \omega^{5(k-1)}(1 + \omega + \cdots + \omega^4) \\ &\quad + \omega^{5k} + \omega^{5k+1} + \omega^{5k+2}. \end{aligned}$$

Substituting $(1 + \omega + \cdots + \omega^4) = 0$, we get

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 1 + \omega + \omega^2.$$

So

$$\begin{aligned} f(\omega, \omega) &= (1 + \omega + \omega^2)^2 \\ &= (-(\omega^3 + \omega^4))^2 \\ &= \omega^6 + 2\omega^7 + \omega^8 \end{aligned}$$

$$= \omega + 2\omega^2 + \omega^3. \tag{3}$$

From equation 2, we get

$$\begin{aligned} f(\omega,\omega) &= 1 + \omega^{n-1} + \omega^{n-1} + \omega^{2(n-1)} \\ &= 1 + 2\omega^{5k+2} + \omega^{10k+4} \\ &= 1 + 2\omega^2 + \omega^4. \end{aligned} \tag{4}$$

Comparing equations 3 and 4, we get $\omega + 2\omega^2 + \omega^3 = 1 + 2\omega^2 + \omega^4$ or equivalently,

$$\omega + \omega^3 = 1 + \omega^4$$

which is not true. Hence if $n \equiv 3 \pmod 5$ (in particular for $n = 2023$), then we cannot cover the $n \times n$ chessboard with corners removed by type 1 and type 2 pentominoes.

MS 93 (1-2) 2024 : Problem 4 (proposed by **Dr. B. Sury**, ISI, Bengaluru, India).

Let $f\colon [0,1] \to \mathbb{R}$ be differentiable. Suppose f, f' have no common zeroes. Prove that the zero set of $f\colon \{x \in [0,1] \mid f(x) = 0\}$ must be finite.

Solution: (by **Dr. Henry Ricardo,** Westchester Area Math Circle, New York, USA).

Suppose that f has infinitely many zeros $\{x_n : n \in \mathbb{N}\} \subset [0,1]$. Since $[0,1]$ is compact, these zeros must have a limit point $\hat{x}$ in the interval. The continuity of f implies that this limit point is also a zero of f:
$0 = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(\hat{x})$. Furthermore, we have

$$f'(\hat{x}) = \lim_{n\to\infty} \frac{f(x_n) - f(\hat{x})}{x_n - \hat{x}} = \lim_{n\to\infty} \frac{0-0}{x_n - \hat{x}} = 0.$$

This contradiction establishes that the zero set of f must be finite.

---

93 (1-2) 2024 Problem 6 (proposed by **Dr. B. Sury**, ISI, Bengaluru, India).

Let G be a group on which the m-th power map and n-th power map, $m, n \in \mathbb{N} - \{1\}$, are both homomorphisms. If $m(m-1)/2$ and $n(n-1)/2$ are relatively prime, prove that G must be abelian.
Conversely, if the greatest common divisor of $m(m-1)/2$ and $n(n-1)/2$ is greater than 1, show there exist non-abelian groups G on which the m-th power map and the n-th power maps are homomorphism.

Solution: (by **Dr. R. Mohan,** Azim Premji University, Bengaluru, India).

A complete solution to this problem can be found in Gallian, J. A., & Reid, M. (1993). Abelian Forcing Sets. *The American Mathematical Monthly, 100(6), 580 - 582.* For the sake of completeness we add the solution here: Let

$$E(G) := \{k \in \mathbb{Z} \mid (xy)^k = x^k y^k \text{ for all } x, y \in G\}.$$

We are given that $m, n \in E(G)$. If we show that $2 \in E(G)$, we are done. We note that

$$\gcd\left(\frac{n(n-1)}{2}, \frac{m(m-1)}{2}\right) = 1 \iff \gcd\left(n(n-1), m(m-1)\right) = 2.$$

It is not necessary that $E(G)$ is an additive subgroup of $\mathbb{Z}$. We shall define a subset of $E(G)$ that is also an additive subgroup of $\mathbb{Z}$.
Suppose that $n, m \in E(G)$ and $x, y \in G$. Then on the one hand we have

$$(xy)^{mn} = (xy)^m (xy)^n = x^m y^m x^n y^n,$$

and on the other hand we have

$$x^{m+n} y^{m+n} = x^m x^n y^m y^n.$$

So if for all $x \in G$, $x^n \in Z(G)$, the center of G. Then we should have for any $m \in E(G)$, and $x, y \in G$ we should have

$$(xy)^{m+n} = x^{m+n} y^{m+n}.$$

So $m + n \in E(G)$.
To make use of the above property, let us define

$$A(G) := \{k \in E(G) \mid x^k \in Z(G) \text{ for all } x \in G\}.$$

It is not difficult to check that $A(G)$ is an additive subgroup of $\mathbb{Z}$.
Now we claim that if $n \in E(G)$ then $n(n-1) \in A(G)$. Observe that if $m, n \in E(G)$, then $mn \in E(G)$. Also, if $n \in E(G)$ then $(1-n) \in E(G)$ as

well. So if $n \in E(G)$ then $n(1-n) \in E(G)$. It is easy to check that n-th powers commute with $(1-n)$-th powers. That is, for any $x, y \in G$, $x^n y^{1-n} = y^{1-n} x^n$. In particular, since $x^{n(1-n)}$ is both an n-th power and a $(1-n)$-th power, $x^{n(1-n)}$ should commute with y^n and y^{1-n}, for any arbitrary $y \in G$, and hence should commute with $y = y^n y^{1-n}$. Thus, if $n \in E(G)$, then $n(1-n) \in A(G)$, and since $A(G)$ is a subgroup, $n(n-1) \in A(G)$.

Finally, since $n, m \in E(G)$, we have $n(n-1), m(m-1) \in A(G)$. So their integer linear combination $2 \in A(G)$, and hence G is abelian.

For the converse, let p be a prime number and consider the *Heisenberg* group modulo p, defined as follows:

$$G_p := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z}/p\mathbb{Z} \right\}.$$

Taking

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

we may check that $xy \neq yx$ in G_p, and hence G_p is not abelian. Let is consider the cases where $p = 2$ and p is odd separately. Note that G_2 is isomorphic to D_4, the group of symmetries of a square. Also, for any $x \in G_2$, $x^4 = e$. This implies that if n is any multiple of 4, then for all $x, y \in G_2$, $(xy)^n = x^n y^n$. This means that for any integer k, we should have

$$(xy)^{4k} = x^{4k} y^{4k} \text{ and } (xy)^{4k+1} = x^{4k+1} y^{4k+1}.$$

Now define the set

$$S_2 := \{4k, 4k+1 \mid k \in \mathbb{Z}\} = \{n \in \mathbb{Z} \mid n(n-1) \equiv 0 \pmod 4\}.$$

Similarly, if p is odd then $x^p = e$, and for any multiple n of p, we have $(xy)^n = x^n y^n$ for any $x, y \in G_p$. So for any integer k, we should have

$$(xy)^{pk} = x^{pk} y^{pk} \text{ and } (xy)^{pk+1} = x^{pk+1} y^{pk+1}.$$

Define the set

$$S_p := \{pk, pk+1 \mid k \in \mathbb{Z}\} = \{n \in \mathbb{Z} \mid n(n-1) \equiv 0 \pmod p\}.$$

Let $d := \gcd\left(\frac{n(n-1)}{2}, \frac{m(m-1)}{2}\right)$ and $d > 1$. Let q be a prime that divides d. Then q divides both $n(n-1)/2$ and $m(m-1)/2$. Thus $2q$ divides

$n(n-1)$ and $m(m-1)$. If $q=2$, then $n, m \in S_2$ and hence G_2 is the required non-abelian group. If q is odd, since q divides $2q$, then $n, m \in S_q$ and G_q is the required non-abelian group.

---

93 (1-2) 2024 Problem 8 (proposed by **Dr. Andrés Ventas**, Santiago de Compostela, Spain).

Given the following two constants defined by continued fractions with all their coefficients repeated, one with a positive sign, the Golden Section, $\varphi = [1, 1, 1, 1, \cdots]$, and the other with a negative sign, the Pena Trevinca constant, τ,

$$\tau = 3 - \cfrac{1}{3 - \cfrac{1}{3 - \cfrac{1}{3 - \cdots}}}$$

Prove that $\tau = \varphi + 1$.

Solution: (by **Dr. Hari Kishan**, Department of Mathematics, D. N. College, Meerut, India).

We have

$\varphi = [1,\ 1,\ 1,\ 1\ldots] = 1 + \frac{1}{1+\frac{1}{1+\ldots}} = 1 + \dfrac{1}{\varphi}.$

This gives $\varphi^2 - \varphi - 1 = 0$ and therefore,

$$\varphi = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2},\ \frac{1-\sqrt{5}}{2} \text{ (rejected as } \varphi > 0.) \qquad (1)$$

Further,

$\tau = 3 - \frac{1}{3-\frac{1}{3-\ldots}} = 3 - \dfrac{1}{\tau}$; and hence we obtain $\tau^2 - 3\tau + 1 = 0$.

Solving we get

$$\tau = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3+\sqrt{5}}{2},\ \frac{3-\sqrt{5}}{2} \text{ (rejected as } \tau > 0).$$

So $\tau = 1 + \dfrac{1+\sqrt{5}}{2} = 1 + \varphi$ by (1); as desired.

FORM IV

1.	Place of Publication:	PUNE
2.	Periodicity of publication:	QUARTERLY
3.	Printer's Name: Nationality: Address:	DINESH BARVE INDIAN PARASURAM PROCESS 38/8, ERANDWANE PUNE-411 004, INDIA
4.	Publisher's Name: Nationality: Address:	M. M. SHIKARE INDIAN GENERAL SECRETARY THE INDIAN MATHEMATICAL SOCIETY EMERITUS PROFESSOR DEPT OF MATHEMATICS JSPM UNIVERSITY PUNE 412207, MAHARASHTRA, INDIA
5.	Editor's Name: Nationality: Address:	G. P. YOUVARAJ INDIAN RAMANUJAN INSTITUTE FOR ADVANCED STUDY IN MATHEMATICS, UNIVERSITY OF MADRAS, CHENNAI 600005, TN, INDIA
6.	Names and addresses of individuals who own the newspaper and partners or shareholders holding more than 1% of the total capital:	THE INDIAN MATHEMATICAL SOCIETY

I, M. M. Shikare, the General Secretary of the IMS, hereby declare that the particulars given above are true to the best of my knowledge and belief.

M. M. Shikare
Dated: November 15, 2024 Signature of the Publisher

Published by Prof. M. M. Shikare for the Indian Mathematical Society, type set by Prof. G. P. Youvaraj, Former Director and Head, Ramanujan Institute for Advanced Study in Mathematics, Chepauk, Chennai - 600005, India and printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No. 129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune-411041 (India)

Printed in India

The Mathematics Student ISSN: 0025-5742
Vol. 93, Nos.3-4, July - December (2024)

EDITORIAL BOARD

Back volumes of our periodicals, except for a few numbers out of stock, are available.

Edited by G. P. Youvaraj and published by M. M. Shikare, the General Secretary of the IMS, for the Indian Mathematical Society.

Typeset by Prof. G. P. Youvaraj, Former Director and Head, Ramanujan Institute for Advanced Study in Mathematics, Chepauk, Chennai - 600005, India and printed by Dinesh Barve at Parashuram Process, Shed No. 1246/3, S. No.129/5/2, Dalviwadi Road, Barangani Mala, Wadgaon Dhayari, Pune - 411041, Maharashtra, India. Printed in India