# IDENTITIES FOR POWERS OF FIBONACCI NUMBERS

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Abstract. We prove an identity for powers of Fibonacci numbers whose indices are in arithmetic sequence. The identity generalizes Jarden's identity for powers of consecutive Fibonacci numbers. The methods described here are applied to obtain analogous identities for Fibonacci and Lucas polynomials.

## 1. Introduction

Consider a generalized Fibonacci sequence $G_n$, that is a sequence that satisfies the recurrence relation $G_n = G_{n-1}+G_{n-2}$. Jarden [1] proved that powers of consecutive terms in a generalized Fibonacci sequence satisfy the identity

$$\sum_{j=0}^{k+1}(-1)^{j(j+1)/2}\binom{k+1}{j}_1 G_{n-j}^k = 0,$$

for any positive integer $k$, where $\binom{k}{j}_1$ is the Fibonomial number defined as

$$\binom{k}{j}_1 = \frac{F_k}{F_1}\cdot\frac{F_{k-1}}{F_2}\cdot\ldots\cdot\frac{F_{k-j+1}}{F_j},$$

and $\binom{k}{0}_1 = 1$. In this paper we extend this identity to the case in which the terms are not necessarily consecutive, but follow the pattern of an arithmetic sequence. Specifically, we prove that any generalized Fibonacci sequence $G_n$ satisfies the identity

$$\sum_{j=0}^{k+1}(-1)^{j+jkd-j(j-1)d/2}\binom{k+1}{j}_d G_{n+jd}^k = 0, \tag{1.1}$$

for all integers $n$, $d$, and $k$ such that $d, k \geq 1$, where the numbers

$$\binom{k}{j}_d = \frac{F_{kd}}{F_d}\cdot\frac{F_{(k-1)d}}{F_{2d}}\cdot\frac{F_{(k-2)d}}{F_{3d}}\cdot\ldots\cdot\frac{F_{(k-j+1)d}}{F_{jd}},$$

for $j \geq 1$, are the $d$-Fibonomials and $\binom{k}{0}_d = 1$. The proof of identity (1.1) leads to the construction of a family of square-free polynomials in one variable. We study divisibility properties among members of this family, and conclude other identities among Fibonacci and Lucas numbers that include $d$-Fibonomials.

This paper is organized as follows. Section 2 is used to construct a polynomial $P_{k,d}$ that is needed in order to prove equation (1.1). In section 3 we prove factorization and divisibility properties of $P_{k,d}$, and deduce an explicit formula for $P_{k,d}$ in Theorem 3.7 that is used to deduce identity (1.1) in Theorem 3.8. In section 4 we give some applications of the formulas found in section 3 and prove some identities for Fibonacci and Lucas numbers. Section 5 is devoted to dicussing how the identities in previous sections are generalized to Fibonacci Polynomials.

## 2. Construction of a Family of Polynomials

Let $\mathcal{S}$ be the space of all sequences of complex numbers. We define a shift operator $S : \mathcal{S} \to \mathcal{S}$. Given a sequence $x_n$, $S(x)$ is the sequence

$$S(x)_n = x_{n+1}.$$

Given a polynomial $P(x) = \sum_{k=0}^{m} a_k x^k$ with complex coefficients $a_k$, we define the operator $P(S) : \mathcal{S} \to \mathcal{S}$ by

$$P(S) = \sum_{k=0}^{m} a_k S^k,$$

where $S^k$ is the composition of the shift operator with itself $k$ times, so it is the operator that shifts a sequence by $k$. For example, if $P(x) = x^2 - x - 1$, then $P(S)(F)_n = F_{n+2} - F_{n+1} - F_n = 0$, if $F$ is the Fibonacci sequence. In order to simplify our notation, we will write $P(S)x_n$ instead of the more proper $P(S)(x)_n$.

Given a generalized Fibonacci sequence $G$ and a positive integer $k$, we would like to find a polynomial $P$ such that $P(S)G^k = 0$. Binet's formula implies that

$$G_n = a\varphi^n + b(-1/\varphi)^n,$$

for some numbers $a$ and $b$, where $\varphi$ is the Golden Ratio. Therefore, by the Binomial Theorem,

$$P(S)G_n^k = \sum_{j=0}^{k} P((-1)^j \varphi^{k-2j}) \binom{k}{j} a^{k-j} b^j (-1)^{nj} \varphi^{nk-2nj}. \tag{2.1}$$

If $P$ is chosen so that $P((-1)^j \varphi^{k-2j}) = 0$, for all $j = 0, \ldots, k$, then $P(S)G^k = 0$. The minimal polynomial that satisfies these equations is

$$P_{k,1}(x) = \prod_{j=0}^{k} (x - (-1)^j \varphi^{k-2j}). \tag{2.2}$$

Later on we will find explicit formulas for $P_{k,1}$, and we will see that it contains $k+2$ non-zero terms, and solves the problem for $d = 1$. In order to solve this problem for $d > 1$, we notice that we are looking for a polynomial of the form

$$P(x) = \sum_{j=0}^{m} C_j x^{jd}, \tag{2.3}$$

for some $m$, such that $P(S)G^k = 0$ for any generalized Fibonacci sequence. In order to find a formula for $P$, notice first that if $\omega$ is a complex number such that $\omega^d = 1$, and $P$ is given by equation (2.3) then $P(\omega x) = P(x)$.

Conversely, if $P$ is a polynomial such that $P(\omega x) = P(x)$ for every complex number $\omega$ such that $\omega^d = 1$, then $P$ is given by an equation of the form (2.3), because in this case the $k$-th derivative of $P$ at 0 satisfies $P^{(k)}(0) = \omega^k P^{(k)}(0)$, so that $P^{(k)}(0) = 0$, if $\omega$ is chosen as a primitive $d$ root of unity and $k$ is not a multiple of $d$. Therefore, by Maclaurin formula $P(x) = \sum_{k=0}^{m} a_k x^{dk}$, for some $m$.

**Definition 2.1.** *Let $\omega$ be a primitive $d$ root of unity. We define the polynomial $P_{k,d}$, called the $d$ symmetrization of $P_{k,1}$, by*

$$P_{k,d}(x) = \prod_{i=1}^{d} P_{k,1}(\omega^i x) = \prod_{i=1}^{d} \prod_{j=0}^{k} (\omega^i x - (-1)^j \varphi^{k-2j}). \tag{2.4}$$

Sometimes it will be more useful to write $P_{k,d}(x) = \prod_{i=0}^{d-1} P_{k,1}(\omega^i x)$. This can be done because $\omega^0 = \omega^d = 1$.

**Theorem 2.2.** *The $d$ symmetrization of $P_{k,1}$ is well-defined, that is, $P_{k,d}$ is independent of the primitive $d$ root of unity chosen.*

*Proof.* Let $\omega_1$ and $\omega_2$ be primitive $d$ roots of unity. It follows that $\omega_1^{m_1} = \omega_2$ and $\omega_2^{m_2} = \omega_1$, for some integers $0 < m_1, m_2 < d$, so that $\omega_2^{m_1 m_2} = \omega_2$, therefore $m_1 m_2 - 1 = jd$, for some integer $j$. In particular $m_1$ and $m_2$ are relatively primes with $d$ and

$$\prod_{i=1}^{d} P_{k,1}(\omega_1^i x) = \prod_{i=1}^{d} P_{k,1}(\omega_2^{m_2 i} x) = \prod_{i=1}^{d} P_{k,1}(\omega_2^{m_2 i - qd} x),$$

where $q$ is any integer. Given an index $i$, pick $q = q_i$ as the quotient of the division of $m_2 i$ by $d$, then the number $r_i = m_2 i - qd$ satisfies $0 \leq r_i < d$. Now observe that the number $r_i$ obtained by division depends on the initial number $i$, but different numbers $i$ produce different numbers $r_i$, because if $m_2 i - q_i d = m_2 j - q_j d$, where $1 \leq i, j \leq d$, then $m_2(i-j) = d(q_i - q_j)$. Since $m_2$ and $d$ are relatively prime, then $d$ must divide $i - j$. But since $1 \leq i, j \leq d$, then $i - j$ is an integer in the interval $-(d-1) \leq i - j \leq d - 1$. The only number divisible by $d$ in that interval is 0, so that $i - j = 0$, and $i = j$. Therefore the sequence of remainders of the division of $m_2 i$ by $d$ is a one-to-one function from $\{1, 2, \ldots, d\}$ to $\{0, 1, \ldots, d-1\}$ and therefore it must be onto. This means, we must have

$$\prod_{i=1}^{d} P_{k,1}(\omega_2^{m_2 i - qd} x) = \prod_{r=0}^{d-1} P_{k,1}(\omega_2^r x) = \prod_{r=1}^{d} P_{k,1}(\omega_2^r x),$$

therefore $\prod_{i=1}^{d} P_{k,1}(\omega_1^i x) = \prod_{i=1}^{d} P_{k,1}(\omega_2^i x)$, and $P_{k,d}(x)$ is independent of the $d$ primitive root of unity chosen. $\square$

## 3. Properties of $P_{k,d}$

In this section we will prove factorization and divisibility properties of the polynomial $P_{k,d}$ constructed in section 2. We prove first some basic properties of $P_{k,d}$.

**Theorem 3.1.** *Let $k$ and $d$ be positive integers, then $P_{k,d}$ is a polynomial of degree $(k+1)d$ that can be expanded only in powers of $x^d$.*

*Proof.* That $P_{k,d}$ has degree $(k+1)d$ follows directly from its definition.

In order to prove that $P_{k,d}$ can be expanded in powers of $x^d$, let $\lambda \neq 1$ be a complex number that satisfies $\lambda^d = 1$ and $\omega$ be a primitive $d$ root of unity, then there exists an exponent $e$ such that $\lambda = \omega^e$, and $0 < e < p$, therefore

$$\begin{aligned}
P_{k,d}(\lambda x) &= \prod_{i=1}^{d} P_{k,1}(\omega^i \lambda x) = \prod_{i=1}^{d} P_{k,1}(\omega^{i+e} x) \\
&= \prod_{i=1}^{d-e} P_{k,1}(\omega^{i+e} x) \cdot \prod_{i=d-e+1}^{d} P_{k,1}(\omega^{i+e} x) \\
&= \prod_{i=e+1}^{d} P_{k,1}(\omega^{i} x) \cdot \prod_{i=d+1}^{d+e} P_{k,1}(\omega^{i} x) \\
&= \prod_{i=e+1}^{d} P_{k,1}(\omega^{i} x) \cdot \prod_{i=d+1}^{d+e} P_{k,1}(\omega^{i-d} x) \\
&= \prod_{i=e+1}^{d} P_{k,1}(\omega^{i} x) \cdot \prod_{i=1}^{e} P_{k,1}(\omega^{i} x) \\
&= \prod_{i=1}^{e} P_{k,1}(\omega^{i} x) \cdot \prod_{i=e+1}^{d} P_{k,1}(\omega^{i} x) \\
&= \prod_{i=1}^{d} P_{k,1}(\omega^{i} x) = P_{k,d}(x).
\end{aligned}$$

$\square$

Now we establish some divisibility properties of the family of polynomials $P_{k,d}$.

**Theorem 3.2.** *If $d_1$ divides $d_2$, then $P_{k,d_1}$ divides $P_{k,d_2}$. Moreover, if $\omega$ is a primitive $d_2$ root of unity, then*

$$P_{k,d_2}(x) = P_{k,d_1}(x)P_{k,d_1}(\omega x)\cdot \ldots \cdot P_{k,d_1}(\omega^{d_2/d_1-1}x). \tag{3.1}$$

*Proof.* Let $m = d_2/d_1$. Since $\omega$ is a primitive $d_2$ root of unity, then $\omega^m$ is a primitive $d_1$ root of unity.

Write $P_{k,d_2}$ as a product of $d_2$ polynomials of the form $P_{k,1}(\omega^j x)$, where $0 \le j \le d_2 - 1$. In order to simplify this proof, write this multiplication in a rectangular array of $d_1$ rows, each with $m$ factors as shown below. Note that the last factor in this product is $P_{k,1}(\omega^{d_2-1}x)$, by the definition of $P_{k,d_2}(x)$, but it is written below as $P_{k,1}(\omega^{(d_1-1)m}\omega^{m-1}x)$, because $d_2 - 1 = d_1 m - 1 = (d_1-1)m + m - 1$. Therefore

$$P_{k,d_2}(x) = \begin{array}{llll} P_{k,1}(x) & P_{k,1}(\omega x) & \cdot\ldots\cdot & P_{k,1}(\omega^{m-1}x) \\ P_{k,1}(\omega^m x) & P_{1,1}(\omega^m \omega x) & \cdot\ldots\cdot & P_{k,1}(\omega^m\omega^{m-1}x) \\ \vdots & \vdots & \vdots & \vdots \\ P_{k,1}(\omega^{(d_1-1)m}x) & P_{k,1}(\omega^{(d_1-1)m}\omega x) & \cdot\ldots\cdot & P_{k,1}(\omega^{(d_1-1)m}\omega^{m-1}x). \end{array}$$

By equation (2.4) the multiplication of the polynomials in the first (leftmost) column is $P_{k,d_1}(x)$, while the multiplication of the polynomials in the second column is $P_{k,d_1}(\omega x)$, etc. Multiplying all columns we obtain equation (3.1). □

The following factorization of $x^d - \lambda^d$ will be useful in the remainder of this paper.

**Theorem 3.3.** *Let $d$ be a positive integer, $\lambda$ a complex number and $\omega$ be a primitive $d$ root of unity, then*

$$\prod_{p=1}^{d}(\omega^p x - \lambda) = (-1)^{d+1}(x^d - \lambda^d). \tag{3.2}$$

*Proof.* Let $P(x) = \prod_{p=1}^{d}(\omega^p x - \lambda)$, then $P$ is a polynomial of degree $d$ with roots $x_p = \lambda/\omega^p$, where $p = 1, \ldots, d$. These are the same roots of the polynomial $Q(x) = x^d - \lambda^d$, so that $P(x) = C(x^d - \lambda^d)$, where $C$ is the coefficient of the highest power of $x$, which is $\omega^{1+2+\ldots+d} = \omega^{d(d+1)/2} = (-1)^{d+1}$. □

The previous theorem implies that we can write equation (2.4) as

$$P_{k,d}(x) = (-1)^{(k+1)(d+1)}\prod_{j=0}^{k}(x^d - (-1)^{jd}\varphi^{(k-2j)d}). \tag{3.3}$$

The next theorem tells us that when simplifying the previous product, it is convenient to multiply factors whose indices add up to $k$.

**Theorem 3.4.** *Let $k$ be an integer, and assume that $j_1$ and $j_2$ are integers such that $j_1+j_2 = k$, then*

$$(x^d - (-1)^{j_1 d}\varphi^{(k-2j_1)d})(x^d - (-1)^{j_2 d}\varphi^{(k-2j_2)d}) = x^{2d} - (-1)^{j_1 d}L_{(k-2j_1)d}x^d + (-1)^{kd}. \tag{3.4}$$

*Proof.* Let $A_1 = (-1)^{j_1 d}\varphi^{(k-2j_1)d}$ and $A_2 = (-1)^{j_2 d}\varphi^{(k-2j_2)d}$, then $(x^d - A_1)(x^d - A_2) = x^{2d} - (A_1 + A_2)x^d + A_1A_2$. We compute and simplify $A_1 + A_2$ and $A_1A_2$.

We start by simplifying $A_1A_2$

$$\begin{aligned} A_1A_2 &= (-1)^{j_1 d}\varphi^{(k-2j_1)d}(-1)^{j_2 d}\varphi^{(k-2j_2)d} \\ &= (-1)^{(j_1+j_2)d}\varphi^{(2k-2(j_1+j_2))d} = (-1)^{kd}. \end{aligned}$$

Now we simplify $A_1 + A_2$, then

$$\begin{aligned} A_1 + A_2 &= (-1)^{j_1 d}\varphi^{(k-2j_1)d} + (-1)^{(k-j_1)d}\varphi^{(k-2(k-j_1))d} \\ &= (-1)^{j_1 d}\varphi^{(k-2j_1)d} + (-1)^{(k-j_1)d}\varphi^{-(k-2j_1)d} \end{aligned}$$

But, since $\varphi^p = F_p\varphi + F_{p-1}$ for any integer $p$, then we can simplify the above equality as

$$= (-1)^{j_1 d}(F_{(k-2j_1)d}\varphi + F_{(k-2j_1)d-1}) + (-1)^{(k-j_1)d}(F_{-(k-2j_1)d}\varphi + F_{-(k-2j_1)d-1})$$

Now, since $F_{-p} = (-1)^{p-1}F_p$, we can rewrite the above equality as

$$\begin{aligned} &= (-1)^{j_1 d}(F_{(k-2j_1)d}\varphi + F_{(k-2j_1)-1}) \\ &\quad +(-1)^{(k-j_1)d}((-1)^{(k-2j_1)d-1}F_{(k-2j_1)d}\varphi + (-1)^{(k-2j_1)d}F_{(k-2j_1)d+1}) \\ &= (-1)^{j_1 d}(F_{(k-2j_1)d}\varphi + F_{(k-2j_1)d-1}) + (-1)^{(2k-3j_1)d-1}F_{(k-2j_1)d}\varphi \\ &\quad +(-1)^{(2k-3j_1)d}F_{(k-2j_1)d+1} \\ &= (-1)^{j_1 d}(F_{(k-2j_1)d}\varphi + F_{(k-2j_1)d-1}) - (-1)^{j_1 d}F_{(k-2j_1)d}\varphi + (-1)^{j_1 d}F_{(k-2j_1)d+1} \\ &= (-1)^{j_1 d}F_{(k-2j_1)d-1} + (-1)^{j_1 d}F_{(k-2j_1)d+1} \\ &= (-1)^{j_1 d}L_{(k-2j_1)d} \end{aligned}$$

Hence

$$(-1)^{j_1 d}\varphi^{(k-2j_1)d} + (-1)^{j_2 d}\varphi^{(k-2j_2)d} = (-1)^{j_1 d}L_{(k-2j_1)d}. \tag{3.5}$$

This concludes the proof. □

An analogous conclusion to that of Theorem 3.2 is that if $k_1$ divides $k_2$, then $P_{k_1,d}$ divides $P_{k_2,d}$. This, however, is not true. For example, $P_{2,1}$ is not divisible by $P_{1,1}$. However, we prove the following

**Theorem 3.5.** *Let $k$ and $d$ be positive integers. Let $Q_{k,d}(x) = P_{k,d}(x^k)$, and $m$ be a positive integer, then $Q_{k,d}$ divides $Q_{mk,d}$.*

*Proof.* By equation (3.3)

$$Q_{k,d}(x) = (-1)^{(k+1)(d+1)}\prod_{j=0}^{k}(x^{kd} - (-1)^{jd}\varphi^{(k-2j)d}),$$

so that

$$Q_{km,d}(x) = (-1)^{(km+1)(d+1)}\prod_{j=0}^{km}(x^{kmd} - (-1)^{jd}\varphi^{(km-2j)d}).$$

Now for each $0 \le j \le k$, the polynomial $x^{kd} - (-1)^{jd}\varphi^{(k-2j)d}$ divides the polynomial $x^{kmd} - (-1)^{jmd}\varphi^{(km-2jm)d}$. If we pick $j' = jm$, then $x^{kd} - (-1)^{jd}\varphi^{(k-2j)d}$ divides the polynomial $x^{kmd} - (-1)^{j'd}\varphi^{(km-2j')d}$. Since $0 \le j \le k$, then $0 \le j' \le mk$. This means that $x^{kmd} - (-1)^{j'd}\varphi^{(km-2j')d}$ is a factor of $Q_{km,d}$. Therefore every different factor of $Q_{k,d}$ divides a different factor of $Q_{km,d}$, so that $Q_{k,d}$ divides $Q_{km,d}$. □

In particular, the family of polynomials $Q_{k,d}$ satisfies that $Q_{k_1,d_1}$ divides $Q_{k_2,d_2}$, whenever either $k_1$ divides $k_2$ or $d_1$ divides $d_2$.

The following theorem establishes the link needed to obtain relations between coefficients of the polynomials $P_{k,d}$ and $P_{k+1,d}$.

**Theorem 3.6.** *Let $P_{k,d}$ be defined by (2.4), then*

$$P_{k+1,d}(\varphi x) = (-1)^{d+1}P_{k,d}(x)(\varphi^{(k+2)d}x^d - (-1)^{(k+1)d}). \tag{3.6}$$

*Proof.* In fact, by equation (2.4) we must have

$$P_{k+1,d}(x) = \prod_{p=1}^{d}\prod_{j=0}^{k+1}(\omega^p x - (-1)^j \varphi^{k+1-2j}),$$

so that

$$\begin{array}{rcl} P_{k+1,d}(\varphi x) & = & \prod_{p=1}^{d}\prod_{j=0}^{k+1}(\omega^p \varphi x - (-1)^j \varphi^{k+1-2j}) \\ & = & \prod_{p=1}^{d}\prod_{j=0}^{k+1}\varphi(\omega^p x - (-1)^j \varphi^{k-2j}) \\ & = & \varphi^{(k+2)d}\prod_{p=1}^{d}\prod_{j=0}^{k+1}(\omega^p x - (-1)^j \varphi^{k-2j}) \\ & = & \varphi^{(k+2)d}\prod_{p=1}^{d}\prod_{j=0}^{k}(\omega^p x - (-1)^j \varphi^{k-2j})\prod_{p=1}^{d}(\omega^p x - (-1)^{k+1}\varphi^{k-2(k+1)}) \\ & = & \varphi^{(k+2)d}P_{k,d}(x)\prod_{p=1}^{d}(\omega^p x - (-1)^{k+1}\varphi^{-(k+2)}). \end{array}$$

But $\prod_{p=1}^{d}(\omega^p x - (-1)^{k+1}\varphi^{-(k+2)}) = (-1)^{d+1}(x^d - (-1)^{(k+1)d}/\varphi^{(k+2)d})$, by the factorization Theorem 3.3, so that

$$P_{k+1,d}(\varphi x) = \varphi^{(k+2)d}P_{k,d}(x)(-1)^{d+1}(x^d - (-1)^{(k+1)d}/\varphi^{(k+2)d}),$$

which implies equation (3.6). □

Now we are ready to expand $P_{k,d}$ in powers of $x^d$.

**Theorem 3.7.** *If $P_{k,d}$ is defined by equation (2.4), then*

$$P_{k,d}(x) = (-1)^{(k+1)(k+2)d/2}\sum_{j=0}^{k+1}(-1)^{j+jkd-j(j-1)d/2}\binom{k+1}{j}_d x^{jd}. \tag{3.7}$$

*Proof.* As a consequence of Theorem 3.4 and equality (3.3), if $k$ is odd, then

$$P_{k,d}(x) = \prod_{j=0}^{(k-1)/2}(x^{2d} - (-1)^{jd}L_{(k-2j)d}x^d + (-1)^d), \tag{3.8}$$

and if $k$ is even, then

$$P_{k,d}(x) = (-1)^{d+1}(x^d - (-1)^{kd/2})\prod_{j=0}^{k/2-1}(x^{2d} - (-1)^{jd}L_{(k-2j)d}x^d + 1). \tag{3.9}$$

Equations (3.8) and (3.9) prove that whether $k$ is even or odd, $P_{k,d}$ is a product of polynomials that have integer coefficients, and therefore $P_{k,d}$ has integer coefficients too.

Since $P_{k,d}$ has degree $(k+1)d$ and can be expanded only in powers of $x^d$, let us write

$$P_{k,d}(x) = \sum_{j=0}^{k+1}C_{k,d,j}x^{jd}, \tag{3.10}$$

where the coefficients $C_{k,d,j}$ are integers. Since $C_{k,d,0} = P_{k,d}(0)$, we conclude that

$$\begin{array}{rcl} C_{k,d,0} & = & P_{k,d}(0) = \prod_{p=1}^{d}\prod_{j=0}^{k}(-(-1)^j\varphi^{k-2j}) \\ & = & (-1)^{(k+1)d}\varphi^{k(k+1)d}\prod_{p=1}^{d}\prod_{j=0}^{k}(-1)^j\varphi^{-2j} \\ & = & (-1)^{(k+1)d}\varphi^{k(k+1)d}\prod_{p=1}^{d}(-1)^{k(k+1)/2}\varphi^{-k(k+1)} \\ & = & (-1)^{(k+1)d}\varphi^{k(k+1)d}(-1)^{k(k+1)d/2}\varphi^{-k(k+1)d} \\ & = & (-1)^{(k+1)d+k(k+1)d/2} \\ & = & (-1)^{(k+1)(k+2)d/2}, \end{array}$$

so that

$$C_{k,d,0} = (-1)^{(k+1)(k+2)d/2}. \tag{3.11}$$

On the other hand, equation (3.6) can be written as

$$\sum_{j=0}^{k+2} C_{k+1,d,j}(\varphi x)^{jd} = (-1)^{d+1}(\varphi^{(k+2)d}x^d - (-1)^{(k+1)d}) \sum_{j=0}^{k+1} C_{k,d,j}x^{jd}.$$

Simplifying the multiplication in the right hand side, we get

$$\begin{aligned} &= (-1)^{d+1}\left(\sum_{j=0}^{k+1} C_{k,d,j}\varphi^{(k+2)d}x^{(j+1)d} - \sum_{j=0}^{k+1}(-1)^{(k+1)d}C_{k,d,j}x^{jd}\right) \\ &= (-1)^{d+1}\left(\sum_{j=1}^{k+2} C_{k,d,j-1}\varphi^{(k+2)d}x^{jd} - \sum_{j=0}^{k+1}(-1)^{(k+1)d}C_{k,d,j}x^{jd}\right) \end{aligned}$$

Comparing coefficients of $x^{jd}$, when $1 \leq j \leq k+1$ leads to the equations

$$C_{k+1,d,j}\varphi^{jd} = (-1)^{d+1}\left(C_{k,d,j-1}\varphi^{(k+2)d} - (-1)^{(k+1)d}C_{k,d,j}\right), \tag{3.12}$$

for all $1 \leq j \leq k+1$.

Dividing equation (3.12) by $\varphi^{jd}$ we obtain

$$C_{k+1,d,j} = (-1)^{d+1}\varphi^{(k+2-j)d}C_{k,d,j-1} + (-1)^{kd}\varphi^{-jd}C_{k,d,j}. \tag{3.13}$$

Given that $\varphi^p = F_p\varphi + F_{p-1}$, for any integer $p$, the right hand side of equation (3.13) is of the form $a + b\varphi$, where $a$ and $b$ are integers. Since $\varphi$ is irrational, then an equation of the form $c = a + b\varphi$, where $a$, $b$ and $c$ are integers implies $c = a$ and $b = 0$. This, in turn, gives us the following two equations

$$0 = (-1)^{d+1}F_{(k+2-j)d}C_{k,d,j-1} - (-1)^{(k-j)d}F_{jd}C_{k,d,j} \tag{3.14}$$

$$C_{k+1,d,j} = (-1)^{d+1}F_{(k+2-j)d-1}C_{k,d,j-1} + (-1)^{(k-j)d}F_{jd+1}C_{k,d,j}, \tag{3.15}$$

for any $1 \leq j \leq k+1$. From equation (3.14) it follows that

$$C_{k,d,j} = -(-1)^{(k-j+1)d}\frac{F_{(k+2-j)d}}{F_{jd}}C_{k,d,j-1}. \tag{3.16}$$

for any $1 \leq j \leq k+1$. Solving this equation leads to the equality

$$C_{k,d,j} = (-1)^{j+jkd-j(j-1)d/2+(k+1)(k+2)d/2}\binom{k+1}{j}_d, \tag{3.17}$$

for all $1 \leq j \leq k+1$. Observe that formula (3.17) is also valid when $j = 0$ because $\binom{k+1}{0}_d = 1$, therefore formula (3.17) is valid for $0 \leq j \leq k+1$. □

We summarize our analysis in the following

**Theorem 3.8.** *Let $G_n$ be a generalized Fibonacci sequence, then for every integer $n$ and for all $d, k \geq 1$,*

$$\sum_{j=0}^{k+1}(-1)^{j+jkd-j(j-1)d/2}\binom{k+1}{j}_d G_{n+jd}^k = 0. \tag{3.18}$$

*Proof.* By the construction of $P_{k,d}$, we have $P_{k,d}((-1)^j\varphi^{k-2j}) = 0$, for any $0 \leq j \leq k$. This implies $P_{k,d}(S)G^k = 0$, for any generalized Fibonacci sequence $G$.

Since $P_{k,d}$ can be expanded in powers of $x^d$ by a formula given by equation (3.7), then the equation $P_{k,d}(S)G^k = 0$ is

$$\sum_{j=0}^{k+1}(-1)^{j+jkd-j(j-1)d/2}\binom{k+1}{j}_d G^k_{n+jd} = 0,$$

is true for any integer $n$, and for all $d, k \geq 1$, as we wanted to prove. □

More divisibility properties among members of the $P_{k,d}$ family are implied by the next theorem, which implies that $P_{k,d}$ divides $P_{k+2,d}$ when $d$ is even, and that $P_{k,d}$ divides $P_{k+4,d}$, for any $d$.

**Theorem 3.9.** *Let $k$ and $d$ be positive integers, then*

$$P_{k+2,d}(-x) = (-1)^{(k+1)d}P_{k,d}(x)(x^{2d} - (-1)^d L_{(k+2)d}x^d + (-1)^{kd}). \tag{3.19}$$

*Proof.* Observe that by equation (3.3)

$$P_{k+2,d}(x) = (-1)^{(k+3)(d+1)}\prod_{j=0}^{k+2}(x^d - (-1)^{jd}\varphi^{(k+2-2j)d}),$$

so that simplifying we can write

$$\begin{aligned} P_{k+2,d}(x) &= (-1)^{(k+1)(d+1)}\textstyle\prod_{j=0}^{k+2}(x^d - (-1)^{jd}\varphi^{(k-2(j-1))d}) \\ &= (-1)^{(k+1)(d+1)}\textstyle\prod_{j=-1}^{k+1}(x^d - (-1)^{(j+1)d}\varphi^{(k-2j)d}). \end{aligned}$$

Replacing $x$ with $-x$, we obtain

$$\begin{aligned} P_{k+2,d}(-x) &= (-1)^{(k+1)(d+1)}\textstyle\prod_{j=-1}^{k+1}((-1)^d x^d - (-1)^{(j+1)d}\varphi^{(k-2j)d}) \\ &= (-1)^{(k+1)(d+1)}(-1)^{(k+3)d}\textstyle\prod_{j=-1}^{k+1}(x^d - (-1)^{jd}\varphi^{(k-2j)d}) \\ &= (-1)^{(k+1)d}(-1)^{(k+1)(d+1)}\textstyle\prod_{j=-1}^{k+1}(x^d - (-1)^{jd}\varphi^{(k-2j)d}). \end{aligned}$$

If we separate, from the product above, the terms with indices $j = -1$ and $j = k+1$, those terms have indices that add up to $k$, so that by Theorem 3.4 their product is $x^{2d} - (-1)^d L_{(k+2)d} + (-1)^{kd}$. The multiplication of the remaining terms from $j = 0$ to $j = k$ is $P_{k,d}(x)$, so we obtain

$$P_{k+2,d}(-x) = (-1)^{(k+1)d}P_{k,d}(x)(x^{2d} - (-1)^d L_{(k+2)d} + (-1)^{kd}).$$ □

The final divisibility property we will establish is given in the following

**Theorem 3.10.** *The polynomial $P_{1,kd}$, divides $Q_{k,d}$.*

*Proof.* By equation (3.8) with $k = 1$, we obtain

$$P_{1,d}(x) = x^{2d} - L_d x^d + (-1)^d = R_d(x^d),$$

where

$$R_d(x) = x^2 - L_d x + (-1)^d = (x - \varphi^d)(x - (-1/\varphi)^d). \tag{3.20}$$

Observe that by the previous factorization of $R_d$, $P_{1,d}(\varphi) = 0$. This implies that equations (3.8) and (3.9) can be written as

$$P_{k,d}(x) = \prod_{j=0}^{(k-1)/2} R_{(k-2j)d}((-1)^j x^d), \tag{3.21}$$

when $k$ is odd, and

$$P_{k,d}(x) = (-1)^{d+1}(x^d - (-1)^{kd/2}) \prod_{j=0}^{k/2-1} R_{(k-2j)d}((-1)^j x^d), \tag{3.22}$$

when $k$ is even. In both products in equations (3.21) and (3.22), the term with $j = 0$ is $R_{kd}(x^d)$, so the term with $j = 0$ in the product defining $Q_{k,d}(x) = P_{k,d}(x^k)$ is $R_{kd}(x^{kd}) = P_{1,kd}(x)$. It follows that $P_{1,kd}$ divides $Q_{k,d}$. □

## 4. Applications

If we go back to equation (3.15), and we use equation (3.17) we obtain an equation of the form

$$(-1)^A \binom{k+2}{j}_d = (-1)^B F_{(k+2-j)d-1} \binom{k+1}{j-1}_d + (-1)^C F_{jd+1} \binom{k+1}{j}_d, \tag{4.1}$$

where

$$\begin{aligned} A &= j + j(k+1)d - j(j-1)d/2 + (k+2)(k+3)d/2 \\ B &= d + j + (j-1)kd - (j-1)(j-2)d/2 + (k+1)(k+2)d/2 \\ C &= (k-j)d + j + jkd - j(j-1)d/2 + (k+1)(k+2)d/2. \end{aligned}$$

Dividing both sides of equation (4.1) by $(-1)^A$, and noticing that $B - A = -2kd + 2jd + 2d$ and $C - A = -2jd - 2d$, it follows that $(-1)^{B-A} = (-1)^{C-A} = 1$, and

$$\binom{k+2}{j}_d = F_{(k+2-j)d-1} \binom{k+1}{j-1}_d + F_{jd+1} \binom{k+1}{j}_d. \tag{4.2}$$

The previous argument gives another proof of a classical identity between consecutive rows of fibonomial coefficients for a fixed $d$. The next theorem gives an identity between three consecutive fibonomial coefficients and one that is two rows apart.

**Theorem 4.1.** *Let $k$ and $d$ be positive numbers such that $k \geq 2$, then for any $j$ such that $0 \leq j \leq k$*

$$\binom{k+2}{j+2}_d = (-1)^{(k+j)d} \binom{k}{j}_d + L_{(k+1)d} \binom{k}{j+1}_d + (-1)^{jd} \binom{k}{j+2}_d. \tag{4.3}$$

*Proof.* In order to simplify the writing of this proof, let us write $P_{k,d}$ using equation (3.10) where $C_{k,d,j}$ are given by equation (3.17).

Therefore, by theorem 3.9

$$\sum_{j=0}^{k+3} C_{k+2,d,j} (-1)^{jd} x^{jd} = (-1)^{(k+1)d} \sum_{j=0}^{k+1} C_{k,d,j} x^{jd} (x^{2d} - (-1)^d L_{(k+2)d} x^d + (-1)^{kd}).$$

Distributing the terms inside the parenthesis into the sum, the previous equation is equal to

$$= \sum_{j=0}^{k+1} (-1)^{(k+1)d} C_{k,d,j} x^{(j+2)d} - (-1)^{kd} L_{(k+2)d} C_{k,d,j} x^{(j+1)d} + (-1)^d C_{k,d,j} x^{jd}.$$

Now we split this sum into three sums, and reindex each sum, we obtain

$$\begin{aligned} \textstyle\sum_{j=0}^{k+3} C_{k+2,d,j} (-1)^{jd} x^{jd} &= \textstyle\sum_{j=2}^{k+3} (-1)^{(k+1)d} C_{k,d,j-2} x^{jd} - \sum_{j=1}^{k+2} (-1)^{kd} L_{(k+2)d} C_{k,d,j-1} x^{jd} \\ &\quad + \textstyle\sum_{j=0}^{k+1} (-1)^d C_{k,d,j} x^{jd}. \end{aligned} \tag{4.4}$$

In this equation, let us compare the terms with power $x^{jd}$ in both sides of the equation where $2 \le j \le k+1$, then

$$(-1)^{jd} C_{k+2,d,j} = (-1)^{(k+1)d} C_{k,d,j-2} - (-1)^{kd} L_{(k+2)d} C_{k,d,j-1} + (-1)^{d} C_{k,d,j}. \tag{4.5}$$

Now recall that by equation (3.17) the coefficients $C_{k,d,j}$ are a multiplication of a power of $-1$ and a fibonomial, so that equation (4.5) can be written in the form

$$(-1)^{A} \binom{k+3}{j}_d = (-1)^{B} \binom{k+1}{j-2}_d + (-1)^{C} L_{(k+2)d} \binom{k+1}{j-1}_d + (-1)^{D} \binom{k+1}{j}_d, \tag{4.6}$$

where $A$, $B$, $C$ and $D$ are given by

$$\begin{aligned} A &= jd + j + j(k+2)d - j(j-1)d/2 + (k+3)(k+4)d/2, \\ B &= (k+1)d + j + jkd - (j-2)(j-3)d/2 + (k+1)(k+2)d/2, \\ C &= kd + j + (j-1)kd - (j-1)(j-2)d/2 + (k+1)(k+2)d/2, \\ D &= d + j + jkd - j(j-1)d/2 + (k+1)(k+2)d/2. \end{aligned}$$

Dividing equation (4.6) by $(-1)^A$, yields an equation of the form

$$\binom{k+3}{j}_d = (-1)^{B-A} \binom{k+1}{j-2}_d + (-1)^{C-A} L_{(k+2)d} \binom{k+1}{j-1}_d + (-1)^{D-A} \binom{k+1}{j}_d. \tag{4.7}$$

Since

$$\begin{aligned} B - A &= (-3k - 7 - j)d - 2, \\ C - A &= (-2k - 2j - 6)d, \\ D - A &= (-2k - 3j - 4)d, \end{aligned}$$

then replacing these equations into equation (4.7), and reindexing by switching $k$ to $k-1$, and $j$ to $j+2$ we obtain equation (4.3) for $k \ge 2$ and $0 \le j \le k$. ☐

**Theorem 4.2.** *Let $G_n$ be a generalized Fibonacci sequence, and let $k$ and $d$ be positive integers, then*

$$\sum_{j=0}^{k+1} (-1)^{j+jkd-j(j-1)d/2} \binom{k+1}{j}_d G_{n+jkd} = 0. \tag{4.8}$$

*Proof.* By the definition of $Q_{k,d}$, it follows that $Q_{k,d}(\varphi) = 0$, so that

$$\sum_{j=0}^{k+1} (-1)^{j+jkd-j(j-1)d/2} \binom{k+1}{j}_d \varphi^{jkd} = 0. \tag{4.9}$$

Multiplying by $\varphi^n$, and using that $\varphi^p = F_p \varphi + F_{p-1}$ for any integer $p$, we obtain the identity

$$\sum_{j=0}^{k+1} (-1)^{j+jkd-j(j-1)d/2} \binom{k+1}{j}_d (F_{n+jkd}\varphi + F_{n+jkd-1}) = 0. \tag{4.10}$$

Since $\varphi$ is irrational, it follows that

$$\sum_{j=0}^{k+1} (-1)^{j+jkd-j(j-1)d/2} \binom{k+1}{j}_d F_{n+jkd} = 0, \tag{4.11}$$

and

$$\sum_{j=0}^{k+1} (-1)^{j+jkd-j(j-1)d/2} \binom{k+1}{j}_d F_{n+jkd-1} = 0. \tag{4.12}$$

Given that any generalized Fibonacci sequence is a combination of $F_n$ and $F_{n-1}$, then equations (4.11) and (4.12) imply equation (4.8) for any generalized Fibonacci sequence. □

Finally, we show an application of Theorem 3.2. In order to state this theorem in a simple form, we follow the convention that $\binom{k+1}{i}_d = 0$ if $i > k+1$.

**Theorem 4.3.** *Let $k$ and $d$ be positive integers, then*

$$\binom{k+1}{j}_{2d} = \sum_{i=0}^{2j} (-1)^{(i-j)(d+1)} \binom{k+1}{i}_d \binom{k+1}{2j-i}_d, \tag{4.13}$$

*for any $0 \le j \le k+1$.*

*Proof.* By Theorem 3.2, $P_{k,2d}(x) = P_{k,d}(x)P_{k,d}(\omega x)$, where $\omega$ is a $2d$ primitive root of unity. In order to simply the writing of the proof, let us write $P_{k,d}$ using equation (3.10) where $C_{k,d,j}$ are given by equation (3.17). Then the equation $P_{k,2d}(x) = P_{k,d}(x)P_{k,d}(\omega x)$ becomes

$$\sum_{j=0}^{k+1} C_{k,2d,j} x^{2jd} = \sum_{j=0}^{k+1} C_{k,d,j} x^{jd} \sum_{j=0}^{k+1} C_{k,d,j} (\omega x)^{jd}.$$

But since $\omega$ is a $2d$ primitive root of unity, then $\omega^d = -1$, so that the previous equation can be written as

$$\sum_{j=0}^{k+1} C_{k,2d,j} x^{2jd} = \sum_{j=0}^{k+1} C_{k,d,j} x^{jd} \sum_{i=0}^{k+1} C_{k,d,i} (-1)^i x^{id}.$$

Multiplying both sums, we obtain

$$\sum_{j=0}^{k+1} C_{k,2d,j} x^{2jd} = \sum_{j=0}^{k+1} \sum_{i=0}^{k+1} C_{k,d,i} C_{k,d,j} (-1)^i x^{(i+j)d}. \tag{4.14}$$

Comparing both sides of the previous equation, we notice that the left hand side does not have any odd powers, so collecting terms with odd powers in the right hand side will produce 0. Therefore, we will collect only terms that contain even powers. Doing this, we can rewrite equation (4.14) as

$$\begin{aligned} \textstyle\sum_{j=0}^{k+1} C_{k,2d,j} x^{2jd} &= \textstyle\sum_{p=0}^{k+1} \sum_{\substack{0 \le i,j \le k+1 \\ i+j=2p}} C_{k,d,i} C_{k,d,j} (-1)^i x^{2pd} \\ &= \textstyle\sum_{p=0}^{k+1} \sum_{i=q_{p,k}}^{r_{p,k}} C_{k,d,i} C_{k,d,2p-i} (-1)^i x^{2pd}, \end{aligned}$$

where $q_{p,k} = \max(0, 2p-k-1)$ and $r_{p,k} = \min(k+1, 2p)$. Equating coefficients we obtain

$$C_{k,2d,p} = \sum_{i=q_{p,k}}^{r_{p,k}} C_{k,d,i} C_{k,d,2p-i} (-1)^i, \tag{4.15}$$

for any $0 \le p \le k+1$.

Let us substitute equation (3.17) into equation (4.15), then we obtain

$$(-1)^j \binom{k+1}{j}_{2d} = \sum_{i=q_{p,k}}^{r_{p,k}} (-1)^{A(i,j,k,d)} \binom{k+1}{i}_d \binom{k+1}{2j-i}_d$$

where $A(i,j,k,d)$ is given by

$$\begin{aligned} A(i,j,k,d) &= i + ikd - i(i-1)d/2 + (k+1)(k+2)d/2 + 2j - i + (2j-i)kd \\ &\quad -(2j-i)(2j-i-1)d/2 + (k+1)(k+2)d/2 + i \\ &= i + 2jkd - i(i-1)d/2 + (k+1)(k+2)d + 2j - (2j-i)(2j-i-1)d/2 \\ &= i + 2jkd - (i^2 + 2j^2 - 2ij - j)d + 2j + (k+1)(k+2)d. \end{aligned}$$

Therefore

$$(-1)^{A(i,j,k,d)-j} = (-1)^{(i-j)(d+1)}.$$

It follows that

$$\binom{k+1}{j}_{2d} = \sum_{i=q_{p,k}}^{r_{p,k}} (-1)^{(i-j)(d+1)} \binom{k+1}{i}_d \binom{k+1}{2j-i}_d.$$

This identity implies identity (4.13) when we interpret $\binom{k+1}{j}_d = 0$, whenever $j > k+1$. □

## 5. Generalization to Fibonacci Polynomials

A Generalized Fibonacci Function Sequence $G_n(x)$ is a sequence that satisfies the recurrence relation $G_n(x) = xG_{n-1}(x) + G_{n-2}(x)$, where $G_1(x)$ and $G_2(x)$ are functions (with real or complex domain, and with real or complex codomain.) In particular, the sequence of Fibonacci polynomials $F_n(x)$ starts with $F_1(x) = 1$, and $F_2(x) = x$, while the sequence of Lucas Polynomials $L_n(x)$ starts with $L_1(x) = x$ and $L_2(x) = x^2 + 2$.

Binet's formula generalizes for generalized Fibonacci Function sequences as

$$G_n(x) = A(x)\alpha^n(x) + B(x)(-1/\alpha(x))^n,$$

where $A(x)$ and $B(x)$ are functions, and $\alpha(x)$ plays the role of the Golden ratio $\varphi$, and is given by

$$\alpha(x) = \frac{x + \sqrt{x^2+4}}{2}.$$

The polynomial $P_{k,d}$ can be defined as the symmetrization of $P_{k,1}$ as

$$P_{k,d}(x,t) = \prod_{i=1}^{d} \prod_{j=0}^{k} (\omega^i t - (-1)^j \alpha^{k-2j}(x)).$$

Observe that in this case, $P_{k,d}$ depends on two variables. What we know at this moment about $P_{k,d}$ is that it is defined as a polynomial in $t$ for fixed $x$. We will not talk about the nature of the coefficients of that polynomial as functions yet, so we will hold this discussion until later in this paper. At this moment, all that matters is that the coefficients of $P_{k,d}(x,t)$ are functions of $x$.

When we discuss below analogous results to the theorems in Section 4 of this paper, we mean to say that we keep $x$ fixed and consider $P_{k,d}(x,t)$ as a polynomial in its variable $t$.

The analogous theorem to Theorem 3.1 holds without change. The divisibility result in Theorem 3.2 is a consequence of the definition of $P_{k,d}$ as a symmetrization of $P_{k,1}$, so it also holds. The analogous to equation (3.3) is

$$P_{k,d}(x,t) = (-1)^{(k+1)(d+1)} \prod_{j=0}^{k} (t^d - (-1)^{jd} \alpha^{(k-2j)d}(x)), \tag{5.1}$$

and also holds true, because its proof depends on the factorization of $t^n - \lambda^n$ that is deduced in Theorem 3.3.

Equation (5.1) can also be simplified when we multiply terms whose indices add up to $k$. In this case Theorem 3.4 holds because its proof only uses that $F_{-n}(x) = (-1)^{n-1}F_n(x)$, $\alpha^p(x) = F_p(x)\alpha(x) + F_{p-1}(x)$, $L_p(x) = F_{p+1}(x) + F_{p-1}(x)$, which are properties that are also true for Fibonacci and Lucas polynomials.

The analogous to Theorem 3.5 also holds because its proof only depends on equation (5.1) and the fact that $t^k - \lambda^k$ is a factor of $t^{mk} - \lambda^{mk}$ for every positive integer $m$.

The link between consecutive rows of Fibonomial coefficients established in Theorem 3.6 also holds, as this depends on the definition of $P_{k,d}$ as a symmetrization of $P_{k,1}$. The analogous factorization is

$$P_{k+1,d}(x, \alpha(x)t) = (-1)^{d+1}P_{k,d}(x,t)(\alpha^{(k+2)d}(x)t^d - (-1)^{(k+1)d}). \tag{5.2}$$

The main formula for $P_{k,d}(x,t)$ follows the same lines of the proof of Theorem 3.7, and is

$$P_{k,d}(x,t) = (-1)^{(k+1)(k+2)d/2}\sum_{j=0}^{k+1}(-1)^{j+jkd-j(j-1)d/2}\binom{k+1}{j}_{d,x}t^{jd}. \tag{5.3}$$

In this case the coefficients $\binom{k}{j}_{d,x}$ are given by

$$\binom{k}{j}_{d,x} = \frac{F_{kd}(x)}{F_d(x)}\cdot\frac{F_{(k-1)d}(x)}{F_{2d}(x)}\cdot\frac{F_{(k-2)d}(x)}{F_{3d}(x)}\cdot\ldots\cdot\frac{F_{(k-j+1)d}(x)}{F_{jd}(x)},$$

when $j > 0$ and $\binom{k}{0}_{d,x} = 1$. The proof of this formula is done along the same lines of the proof of Theorem 3.7. The needed changes are the following. First, due to equation (5.1) the analogous of equations (3.8) and (3.9) are

$$P_{k,d}(x,t) = \prod_{j=0}^{(k-1)/2}(t^{2d} - (-1)^{jd}L_{(k-2j)d}(x)t^d + (-1)^d), \tag{5.4}$$

when $k$ is odd, and

$$P_{k,d}(x,t) = (-1)^{d+1}(t^d - (-1)^{kd/2})\prod_{j=0}^{k/2-1}(t^{2d} - (-1)^{jd}L_{(k-2j)d}(x)t^d + 1). \tag{5.5}$$

when $k$ is even. It follows that the coefficients of the polynomial $P_{k,d}(x,t)$, as a polynomial in $t$, are polynomials in $x$, because they are combinations of products of Lucas polynomials (in the $x$ variable.) Observe that the analogous equations to equations (3.14) and (3.15) also holds in this case because $\alpha(x)$ is not a rational function. This can be established very easily using the fact that $\lim_{x\to\infty}\alpha(x) = \infty$, and $\lim_{x\to-\infty}\alpha(x) = 0$, so $\alpha(x)$ has only one horizontal asymptote at $-\infty$, while any rational function has the same horizontal asymptote at $-\infty$ and $\infty$.

In particular, we deduce that for any generalized Fibonacci Function sequence $G_n(x)$ we must have an analogous equation to (3.18), namely

$$\sum_{j=0}^{k+1}(-1)^{j+jkd-j(j-1)d/2}\binom{k+1}{j}_{d,x}G^k_{n+jd}(x) = 0. \tag{5.6}$$

The analogous of Theorem 3.9 also holds, and gives us the equation

$$P_{k+2,d}(x,-t) = (-1)^{(k+1)d}P_{k,d}(x,t)(t^{2d} - (-1)^dL_{(k+2)d}(x)t^d + (-1)^{kd}). \tag{5.7}$$

Finally, the proof that $P_{1,kd}(x,t)$ divides $Q_{k,d}(x,t)$ generalizes immediately also.

As a consequence of the previous discussion, all generalizations of the theorems in section 5 hold. For example the analogous identity to identity (4.1) must be

$$\binom{k+2}{j+2}_{d,x} = (-1)^{(k+j)d}\binom{k}{j}_{d,x} + L_{(k+1)d}(x)\binom{k}{j+1}_{d,x} + (-1)^{jd}\binom{k}{j+2}_{d,x}, \tag{5.8}$$

for any integers $k, d$ such that $k, d \geq 1$. We also have an analogous to identity (4.2), namely that for every generalized Fibonacci Function Sequence $G_n(x)$, we have

$$\sum_{j=0}^{k+1}(-1)^{j+jkd-j(j-1)d/2}\binom{k+1}{j}_{d,x} G_{n+jkd}(x) = 0, \tag{5.9}$$

for any integers $k, d$ such that $k, d \geq 1$ and any integer $n$. Finally, we also have an analogous of identity (4.13), namely

$$\binom{k+1}{j}_{2d,x} = \sum_{i=0}^{2j}(-1)^{(i-j)(d+1)}\binom{k+1}{i}_{d,x}\binom{k+1}{2j-i}_{d,x}, \tag{5.10}$$

for any $0 \leq j \leq k+1$ and any $d \geq 1$.

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