MegaMenger Graphs

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Can you build a fractal out of paper? This article is inspired by the MegaMenger project to build models of the Menger sponge out of business cards. It uses graph theory and recurrence relations to analyze fractals called the Sierpinski carpet and Menger sponge.

Fractals

A fractal is a type of mathematical object. Many well-known fractals are self-similar. That is, part of the object is isomorphic to the whole. Often, a fractal can be formed by iterating some operation infinitely many times on a set.

For example, consider the Cantor set. Begin with the interval $[0, 1]$ and remove the middle third. This leaves the set $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, which contains two intervals of length $\frac{1}{3}$. Next, remove the middle thirds of both of these subintervals. At each step, remove the middle third of each subinterval remaining. Iterating this operation infinitely many times produces the Cantor set. Note that the portion of the Cantor set in $\left[0, \frac{1}{3}\right]$ is isomorphic to the entire set.

The Cantor set is a subset of a one dimensional real line. We can generalize it to higher dimensional fractals. One way of doing this is to start with a square, divide it into nine smaller equal-sized squares, and remove the middle square. Then repeat the operation on each smaller square ad infinitum. This produces a fractal called the Sierpinski carpet. This can be seen to contain many copies of the Cantor set.

One way of generalizing this idea is to start with a three dimensional cube, and divide it into 27 smaller equal-sized cubes. Then remove the middle cube from each face and the center cube,

Figure 1. Constructing the Cantor Set. Source: Wikipedia (By 127 "rect" - From en.wikipedia.org Image:Cantor_set_in_seven_iterations.svg, Public Domain, https://commons.wikimedia.org/w/index.php?curid=1576217).

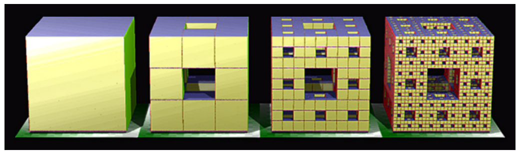

Figure 2. Constructing the Menger Sponge (levels 0-3). The faces are levels of the Sierpinski carpet. Source: Wikipedia (Public Domain, https://commons.wikimedia.org/w/index.php?curid=64192)

leaving 20 smaller cubes. Then repeat the operation on each smaller cube ad infinitum. This produces a fractal called the Menger sponge. This can be seen to contain many copies of the Sierpinski carpet. It was first described in 1926 by Karl Menger, who is famous in graph theory for Menger's Theorem on connectivity.

Discrete Models of Fractals

If we want to build real world models of these fractals, we cannot perform infinitely many iterations of an operation. Even performing finitely many operations eventually becomes impossible if it requires us to remove smaller and smaller portions of a real world object. An alternative way to construct real world models of fractals is to start with a given unit and build larger and larger models out of more and more copies of this unit. We will refer to these as levels of the fractal model. Thus level 0 for the Sierpinski carpet is a square, and level 1 is built by arranging eight of these squares into a ring. Then level 2 is built by arranging eight copies of level 1 into a ring, and so on. For the Menger sponge, level 0 is a cube, and level 1 is built out of twenty of these cubes. Level two is built out of twenty copies of level 1, and so on.

We would like a way to describe the locations of the level 0 units in these constructions. To describe the location of a subinterval in the Cantor set, label it 0, 1, or 2 depending which third of the interval it is contained in. Then append a 0, 1, or 2 for which third of this subinterval it is contained in. Continuing in this fashion, we obtain a string of 0's, 1's, and 2's that describes the location of the subinterval. But if the subinterval is actually in the Cantor set, 1 cannot appear. We can think of such a string of length i as a ternary number between 0 and $3^i - 1$ describing the location of a subinterval in level i of the model of the Cantor set. For example, 0220 represents the subinterval $\left[\frac{24}{81}, \frac{25}{81}\right]$, since $0 \cdot 27 + 2 \cdot 9 + 2 \cdot 3 + 0 \cdot 1 = 24$.

Since the Sierpinski carpet is a subset of the plane, we assign two coordinates to indicate the location of a square in it. Each coordinate is described by a ternary string constructed analogously to that for the Cantor set. A square is contained in the Sierpinski carpet if and only if its coordinates do not both contain a 1 in the same position.

For the Menger sponge, we assign three coordinates to indicate the location of a cube in it. Each coordinate is described by a ternary string constructed analogously to that for the Cantor set. A cube is contained in the Menger sponge if and only there is at most one 1 in any position of its three coordinates. For example, $(10, 01, 02)$ is in the Menger sponge, but $(10, 01, 12)$ is not.

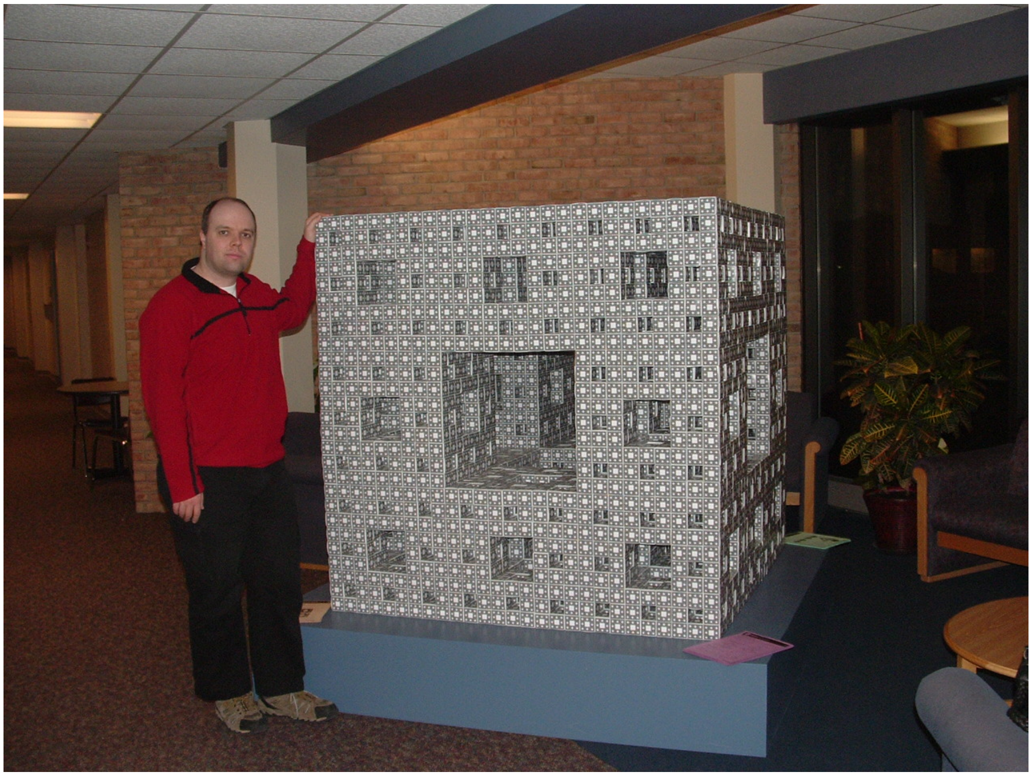

Figure 3. The author with Calvin College's Menger Sponge.

The goal of the MegaMenger project was to build twenty level 3 models of the Menger sponge out of business cards at sites around the world. Each level one cube was constructed out of six cards, has side length two inches with two flaps on each face that were used for linking it with other cubes. Once all the cubes were linked together, another card was attached to each face to hide the exposed flaps, add aesthetic appeal, and provide additional structural support. The level 3 model has side length 4.5 feet and weighs approximately 170 pounds.

The MegaMenger project was organized by Queen Mary University of London. I participated in Calvin College's project, which was organized by Randy Pruim and Gerard Venema and took place during October 2014. More information is available at [**4**] and [**2**]. This project inspired me to ask and answer a number of questions about models of the Sierpinski carpet and Menger sponge.

Graph Theory Models

We will use graph theory to model the real world models of the Sierpinski carpet and Menger sponge. A graph is composed of a set of vertices and a set of edges, which is a subset of the set of 2-element subsets of the vertex set [**6**]. In a Sierpinski carpet graph, each vertex represents a square of the carpet. Vertices are adjacent if their squares share a common edge (See Figures 4 and

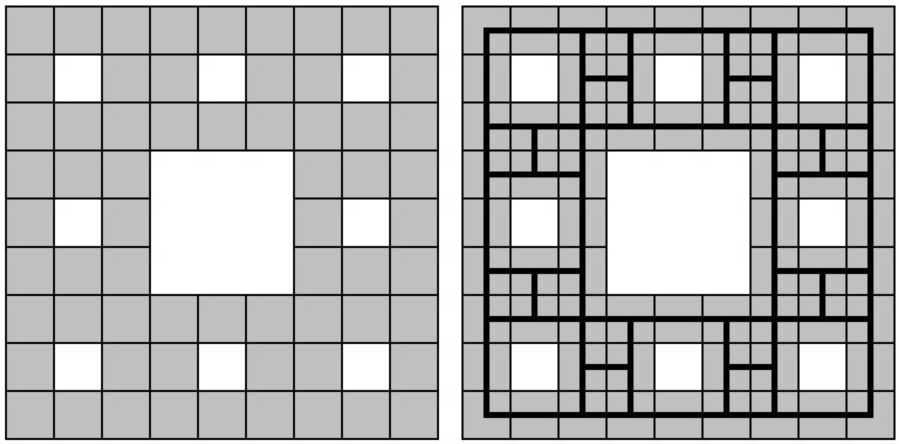

Figure 4. Constructing the Graph SC_2

5). In a Menger sponge graph, each vertex represents a cube of the sponge. Vertices are adjacent if their cubes share a common face. We will denote the level i Sierpinski carpet graph as SC_i, and the level i Menger sponge graph as MS_i. We will use graph theory to find information about the structure of the Sierpinski carpets and Menger sponges. We begin by calculating some parameters of these graphs.

The order of a graph, $n(G)$, is the number of vertices in graph G. Each iteration in the construction of the Cantor set can be formed from two copies of the previous one. If c_i is the number of subintervals after i iterations constructing the Cantor set, we find the recurrence relation $c_i = 2c_{i-1}$, with initial condition $c_0 = 1$. This immediately gives the exponential solution $c_i = 2^i$. The Sierpinski carpet is formed from eight copies of the previous level, so the order $n\left(SC_i\right) = 8n\left(SC_{i-1}\right)$ and $n\left(SC_0\right) = 1$. Thus $n\left(SC_i\right) = 8^i$. The Menger sponge is formed from twenty copies of the previous level, so $n\left(MS_i\right) = 20n\left(MS_{i-1}\right)$ and $n\left(MS_0\right) = 1$. Thus $n\left(MS_i\right) = 20^i$.

Sizes of the Graphs

The size $m(G)$ of a graph is the number of edges it contains. To determine the sizes of the Sierpinski carpet graphs, we must develop a recurrence relation (see [**5**] for more). Obviously, $m\left(SC_0\right) = 0$ and $m\left(SC_1\right) = 8$. Each level is formed from eight copies of the previous level with some added edges in between. There are 3^{i-1} edges between each copy of SC_{i-1}. Since there are eight edges in SC_1, there are eight sets of 3^{i-1} edges added. Thus we find the recurrence relation $m\left(SC_i\right) = 8m\left(SC_{i-1}\right) + 8 \cdot 3^{i-1}$. This is a nonhomogeneous linear recurrence relation. The solution to the related homogeneous relation $a_i - 8a_{i-1} = 0$ is exponential, and has the form $A8^i$. A theorem on recurrence relations tells us that a particular solution to the nonhomogenous recurrence relation has the form $B3^i$. Thus the general solution should have the form $m\left(SC_i\right) = A8^i + B3^i$.

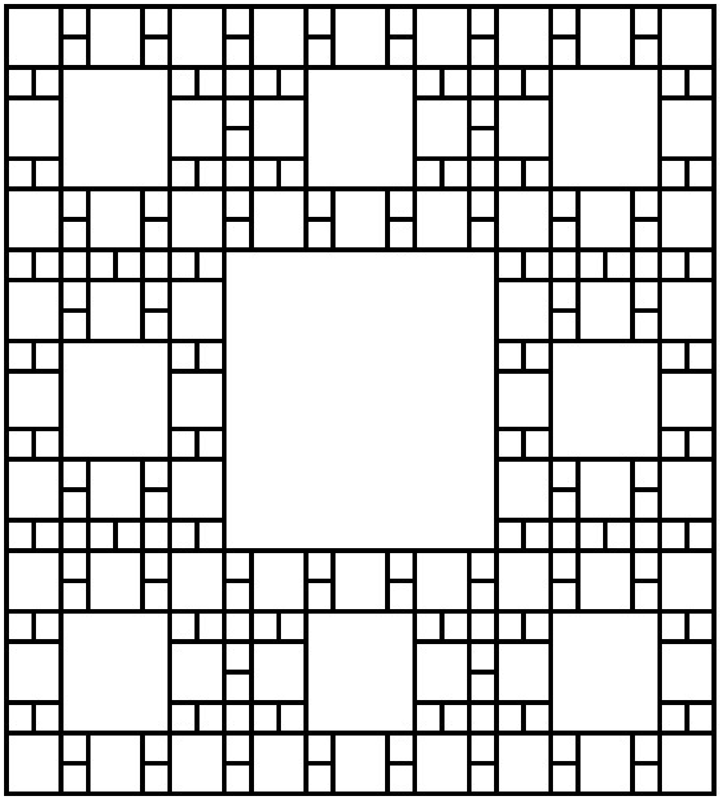

Figure 5. The Sierpinski Carpet graph SC_3

Plugging in the initial conditions, we find $0 = A + B$ and $8 = 8A + 3B$. Solving this system of linear equations, we find $A = \frac{8}{5}$ and $B = -\frac{8}{5}$. Thus $m(SC_i) = \frac{8}{5}8^i - \frac{8}{5}3^i$. The first few terms of this sequence are 0, 8, 88, 776, 6424, ... This also implies that the average degree of the Sierpinski carpet graphs approaches $\frac{16}{5} = 3.2$ as i grows large.

To determine the sizes of the Menger sponge graphs, we use a similar method. We see $m(MS_0) = 0$ and $m(MS_1) = 24$. Each level is formed from 20 copies of the previous level with some added edges in between. There are 8^{i-1} edges between each copy of MS_{i-1}, since each face of the Menger sponge is a level $i-1$ Sierpinski carpet. Since there are 24 edges in MS_1, there are 24 sets of 8^{i-1} edges added. Thus we find the recurrence relation $m(MS_i) = 20m(MS_{i-1}) + 24 \cdot 8^{i-1}$. The solution to the related homogeneous relation $a_i - 20a_{i-1} = 0$ is exponential, and has the form $A20^i$. A particular solution to the nonhomogenous recurrence relation has the form $B8^i$. Thus the general solution should have the form $m(MS_i) = A20^i + B8^i$. Plugging in the initial conditions, we find $0 = A + B$ and $24 = 20A + 8B$. Solving this system of linear equations, we find $A = 2$ and $B = -2$. Thus $m(MS_i) = 2 \cdot 20^i - 2 \cdot 8^i$. The first few terms of this sequence are 0, 24, 672, 14976, 311808, ... This also implies that the average degree of the Menger sponge graphs approaches 4 as i grows large.

Surface Area of Menger Sponges

In constructing a model of the Menger sponge, we added extra cards on the outside faces to improve its aesthetics and stability. To determine how many cards were needed, we want to find the surface area of the Menger sponge. When constructing a level from the twenty copies, we see that some outside faces will be covered up. Thus it is convenient to split the surface area into the inside surface area and outside surface area.

For the inside surface area, we note that $IS\left(MS_1\right) = 24$, since each of the six holes in the sides of the cube is surrounded by four interior faces. Now level i is composed of twenty copies of level $i-1$. In addition, there are 24 inside faces that are level $i-1$ Sierpinski carpets. Thus we find the recurrence relation $IS\left(MS_i\right) = 20 \cdot IS\left(MS_{i-1}\right) + 24 \cdot 8^{i-1}$. This is the same recurrence relation as for the size of the Menger sponge graph! It also has the same initial condition, so it has the same solution, $IS\left(MS_i\right) = 2 \cdot 20^i - 2 \cdot 8^i$. One way of illustrating the relationship between the size and inside surface area is to note that in the level one cube, each vertex of degree two is incident with two edges (and this counts all the edges exactly once) and the corresponding cube has two inside faces.

For the outside surface area, we note that there are six outside faces. Each face is a level i Sierpinski carpet, which has order 8^i. Thus the outside surface area is $6 \cdot 8^i$. Adding the inside and outside surface areas, we see that the total surface area of MS_i is $2 \cdot 20^i + 4 \cdot 8^i$. The first few terms of this sequence are 6, 72, 1056, 18048, 336384, ...

Partite Sets of the Menger Sponges

Calculating the chromatic number of the Sierpinski carpet and Menger sponge graphs is not an interesting problem since these graphs are easily seen to be bipartite. That is, the vertices can be separated into two sets so that there are no edges between vertices in the same set. A more interesting problem is determining the cardinalities of the two partite sets. First we note that all of the corners of any Menger sponge graph are in the same partite set since any two corners are an even distance apart. We denote this partite set as V_i', with cardinality n_i', and the other partite set as V_i'', with cardinality n_i''. We see that $n_0' = 1$, $n_0'' = 0$, $n_1' = 8$, and $n_1'' = 12$. When we construct level i, set V_i' consists of the sets V_{i-1}' for the corner level $i-1$ cubes of level i, and also contains the sets V_{i-1}'' of the non-corner level $i-1$ cubes of level i. Thus some of the sets 'switch places', so the recurrences for each partite set depend on both. Specifically, we find $n_i' = 8n_{i-1}' + 12n_{i-1}''$ and $n_i'' = 12n_{i-1}' + 8n_{i-1}''$. We can write this in matrix form.

$$\begin{bmatrix} n_i' \\ n_i'' \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 12 & 8 \end{bmatrix} \begin{bmatrix} n_{i-1}' \\ n_{i-1}'' \end{bmatrix}$$

To solve this discrete dynamical system, we find the eigenvalues of the coefficient matrix A (see [**3**] for more). We see $\det\left(A - \lambda I\right) = \left(8 - \lambda\right)^2 - 12^2 = \lambda^2 - 16\lambda - 80 = \left(\lambda - 20\right)\left(\lambda + 4\right)$, so the eigenvalues are $\lambda_1 = 20$ and $\lambda_2 = -4$. For λ_1, we see $A - \lambda_1 I = \begin{bmatrix} -12 & 12 \\ 12 & -12 \end{bmatrix}$, so the corresponding eigenvector is $\overrightarrow{v}_1 = (1, 1)$. For λ_2, we see $A - \lambda_2 I = \begin{bmatrix} 12 & 12 \\ 12 & 12 \end{bmatrix}$, so the corresponding eigenvector is $\overrightarrow{v}_2 = (1, -1)$. It is known in linear algebra that $(n'_i, n''_i) = A\,(\lambda_1)^i\,\overrightarrow{v}_1 + B\,(\lambda_2)^i\,\overrightarrow{v}_2$. We plug in the initial conditions, finding

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} + B \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

It is easy to see that $A = \frac{1}{2}$, $B = -\frac{1}{2}$ is the solution. Thus we find that $n'_i = \frac{1}{2}20^i + \frac{1}{2}(-4)^i$ and $n''_i = \frac{1}{2}20^i - \frac{1}{2}(-4)^i$. Notably, which set is larger alternates between successive levels. The first few terms of the sequence for n' are 1, 8, 208, 3968, 80128, ... and the first few terms of the sequence for n'' are 0, 12, 192, 4032, 79872, ...

Conclusion

There is much more to explore about Sierpinski carpet and Menger sponge graphs. Further investigation occurs in my article determining their vertex degrees and degeneracies [**1**].

Acknowledgments

Thanks to Gerard Venema and Allen Schwenk for their helpful comments.

Summary. In 2014, faculty and students at colleges around the world participated in the MegaMenger project to build a model of the Menger sponge, a type of fractal, out of business cards. This model can itself be modeled using graph theory, with each vertex representing a small cube, and an edge between two vertices whenever they share a face. We study graphs representing different steps of building the Menger sponge and Sierpinski carpet to determine their order, size, and chromatic number, along with the surface area of the Menger sponge. Calculating these quantities requires solving many recurrence relations, so we review techniques for this along the way.

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