Degrees of Menger and Sierpinski Graphs

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This article uses graph theory and recurrence relations to analyze fractals called the Sierpinski carpet and Menger sponge. It is inspired by the MegaMenger project to build models of the Menger sponge out of business cards.

1 Discrete Models of Fractals

Many well-known fractals are self-similar. That is, part of the object is isomorphic to the whole. Often, a fractal can be formed by iterating some operation infinitely many times on a set.

For example, consider the Cantor set. Begin with the interval $[0,1]$ and remove the middle third. This leaves the set $\left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right]$, which contains two intervals of length $\frac{1}{3}$. At each step, remove the middle third of each subinterval remaining. Iterating this operation infinitely many times produces the Cantor set.

The Cantor set is a subset of a one dimensional real line. We can generalize it to higher dimensional fractals. One way of doing this is to start with a square, divide it into nine smaller equal-sized squares, and remove the middle square. Then repeat the operation on each smaller square ad infinitum. This produces a fractal called the Sierpinski carpet. This can be seen to contain many copies of the Cantor set.

One way of generalizing this idea is to start with a three dimensional cube, and divide it into 27 smaller equal-sized cubes. Then remove the middle cube from each face and the center cube, leaving 20 smaller cubes. Then repeat the operation on each smaller cube ad infinitum. This produces a fractal called the Menger sponge. This can be seen to contain many copies of the Sierpinski carpet. It was first described in 1926 by Karl Menger, who is famous in graph theory for Menger's Theorem on connectivity.

If we want to build real world models of these fractals, we cannot perform infinitely many iterations of an operation. Even performing finitely many operations eventually becomes impossible if it requires us to remove smaller and smaller portions of a real world object. An alternative way to construct real world models of fractals is to start with a given unit and build larger and larger models out of more and more copies of this unit. We will refer to these as levels of the fractal model. Thus level 0 for the Sierpinski carpet is a square, and level 1 is built by arranging eight of these squares into a ring. Then level 2 is built by arranging eight copies of level 1 into a ring, and so on. For the Menger sponge, level 0 is a cube, and level 1 is built out of twenty of these cubes. Level two is built out of twenty copies of level 1, and so on.

The goal of the MegaMenger project was to build twenty level 3 models of the Menger sponge out of business cards at sites around the world. I participated in Calvin College's project, which took place during October 2014 (see [2] and [4]). This project inspired me to ask and answer a number of questions about models of the Sierpinski carpet and Menger sponge.

We will use graph theory to model the real world models of the Sierpinski carpet and Menger sponge. In a Sierpinski carpet graph, each vertex represents a square of the carpet. Vertices are adjacent if their squares share a common edge. In a Menger sponge graph, each vertex represents a cube of the sponge. Vertices are adjacent if their cubes share a common face. We will denote the level i Sierpinski carpet graph as SC_i, and the level i Menger sponge graph as MS_i.

The Sierpinski carpet is formed from eight copies of the previous level, so the order is $n\left(SC_i\right) = 8^i$. The Menger sponge is formed from twenty copies of the previous level, so $n\left(MS_i\right) = 20^i$.

In another paper, I determined the sizes of the Sierpinski carpet graphs are $m\left(SC_i\right) = \frac{8}{5}8^i - \frac{8}{5}3^i$ and the sizes of the

Menger sponge graphs are $m(MS_i) = 2 \cdot 20^i - 2 \cdot 8^i$. I also determined the surface area of the Menger Sponge models and cardinalities of the partite sets of the Menger sponge graphs.

2 Degrees of Vertices in the Sierpinski Carpets

Examining the vertices of the Sierpinski carpet graphs, we find vertices of degree two, three, and four. How many are there of each? We denote by $d_k^2(i)$ the number of vertices of degree k in SC_i. We note that SC_1 has eight vertices of degree two and none of degree three or four. Each level i Sierpinski carpet is formed from eight copies of level $i-1$ with eight sets of 3^{i-1} edges added in between. We refer to the set of vertices for which one of the dimensions is as large (or small) as possible as a boundary of the carpet. Each set of edges added between level $i-1$ carpets will increase the degrees of the vertices in two boundary sets. Thus we must add in or subtract out the vertices in these 16 boundary sets to count the vertices of a particular degree correctly. In addition, there are four vertices contained in two boundary sets, so we must also make sure that these vertices are counted correctly.

We need to know the degrees of the vertices in the boundary sets of a level i carpet. For level 1, we have three vertices of degree two. To construct the next level, we take three copies of this side, which makes four vertices of degree three. Let $d_2^1(i)$ be the number of vertices of degree two in the boundary of the level i Sierpinski carpet and $d_3^1(i)$ be the number of vertices of degree three. Then we see that $d_3^1(i) = 3d_3^1(i-1) + 4$ and $d_2^1(i) = 3d_2^1(i-1) - 4$. Thus $d_2^1(i) = A3^i + B$, $d_2^1(1) = 3$, and $d_2^1(2) = 5$. Thus $3 = 3A + B$ and $5 = 9A + B$. Solving, we find $A = \frac{1}{3}$ and $B = 2$, so $d_2^1(i) = \frac{1}{3}3^i + 2 = 3^{i-1} + 2$. Also, $d_3^1(i) = A3^i + B$, $d_3^1(1) = 0$, and $d_3^1(2) = 4$. Thus $0 = 3A + B$ and $4 = 9A + B$. Solving, we find $A = \frac{2}{3}$ and $B = -2$, so $d_3^1(i) = \frac{2}{3}3^i - 2 = 2 \cdot 3^{i-1} - 2$.

To find the number of vertices of degree two in level i of the Sierpinski carpet, we see that 16 sets of $d_2^1(i-1)$ vertices have their degrees increased from two to three and must be removed. Note however that this leads to four vertices whose degrees increase from two to four. They are subtracted out twice and must be added back in once. Thus the recurrence relation is $d_2^2(i) = 8d_2^2(i-1) - 16\left(3^{i-2}+2\right) + 4 = 8d_2^2(i-1) - 16 \cdot 3^{i-2} - 28$.

We would suspect that this recurrence relation has a solution of the form $A8^i + B3^i + C$. We can solve for the coefficients using linear algebra. For the number of vertices of degree two, this leads to the solution $d_2^2(i) = \frac{1}{10}8^i + \frac{16}{15}3^i + 4$. To verify that this is correct, we check that $d_2^2(1) = 8$ and plug the formula into the recurrence relation. We see $d_2^2(i+1) = 8\left(\frac{1}{10}8^i + \frac{16}{15}3^i + 4\right) - 16 \cdot 3^{i-1} - 28 = \frac{1}{10}8^{i+1} + \frac{16}{15}3^{i+1} + 4$, verifying the formula. The first few terms of this sequence are 8, 20, 84, 500, 3540, ...

For the number of vertices of degree three, we see that 16 sets of $d_2^1(i-1)$ vertices have their degrees increased from two to three, and so must be added in. Meanwhile, 16 sets of $d_3^1(i-1)$ vertices have their degrees increased from three to four, and so must be removed. There are also four vertices of degree two which have their degrees increased to four. They were added in twice, and so must be subtracted out twice. Thus the recurrence relation is $d_3^2(i) = 8d_3^2(i-1) + 16\left(3^{i-2}+2\right) - 16\left(2 \cdot 3^{i-2} - 2\right) - 8 = 8d_3^2(i-1) - 16 \cdot 3^{i-2} + 56$. Using a similar method to that for the previous problem, we find the solution $d_3^2(i) = \frac{3}{5}8^i + \frac{16}{15}3^i - 8$. We check that $d_3^2(1) = 0$ and $d_3^2(i+1) = 8\left(\frac{3}{5}8^i + \frac{16}{15}3^i - 8\right) - 16 \cdot 3^{i-1} + 56 = \frac{3}{5}8^{i+1} + \frac{16}{15}3^{i+1} - 8$, verifying the formula. The first few terms of this sequence are 0, 40, 328, 2536, 19912, ...

For the number of vertices of degree four, we see that 16 sets of $d_3^1(i-1)$ vertices have their degrees increased from three to four, and so must be added in. There are also four vertices of degree two which have their degrees increased to four, and so must be added in. Thus the recurrence relation is $d_4^2(i) = 8d_4^2(i-1) + 16\left(2 \cdot 3^{i-2} - 2\right) + 4 = 8d_4^2(i-1) + 32 \cdot 3^{i-2} - 28$. We find the solution $d_4^2(i) = \frac{3}{10}8^i - \frac{32}{15}3^i + 4$. We check that $d_3^2(1) = 0$ and $d_4^2(i+1) = 8\left(\frac{3}{10}8^i - \frac{32}{15}3^i + 4\right) + 32 \cdot 3^{i-1} - 28 = \frac{3}{10}8^{i+1} - \frac{32}{15}3^{i+1} + 4$, verifying the formula. The first few terms of this sequence are 0, 4, 100, 1060, 9316, ...

Summarizing, we have found that for $i \geq 1$,

$$d_2^2(i) = \frac{1}{10}8^i + \frac{16}{15}3^i + 4$$

$$d_3^2(i) = \frac{3}{5}8^i + \frac{16}{15}3^i - 8$$

$$d_4^2(i) = \frac{3}{10}8^i - \frac{32}{15}3^i + 4$$

How can we be certain that we have not made a mistake in deriving these formulas? These formulas should count each vertex exactly once between them. Thus we see that $d_2^2(i) + d_3^2(i) + d_4^2(i) = 8^i = n(SC_i)$. But perhaps we have moved some vertices from one group to another, resulting in offsetting errors. As a further check, we use the First Theorem of Graph Theory. Thus we check that $2d_2^2(i) + 3d_3^2(i) + 4d_4^2(i) = \frac{16}{5}8^i - \frac{16}{5}3^i = 2m(MC_i)$. These formulas also imply that as i grows large, approximately $\frac{1}{10}$ of the vertices will have degree 2, $\frac{3}{5}$ of the vertices will have degree 3, and $\frac{3}{10}$ of the vertices will have degree 4.

3 Degrees of Vertices in the Faces of Menger Sponges

We would like to determine similar formulas for the Menger sponge graphs. Before we can do this, we need to know how many vertices of each degree are in the faces of Menger sponge graphs. This is related to the formulas for the Sierpinski carpet graphs, but not identical since some of the vertices in the face of a Menger sponge will be adjacent to a vertex in its interior. We denote by $f_k^2(i)$ the number of vertices of degree k in a face of MS_i. We find that all of the vertices of degree four in the Sierpinski carpet become vertices of degree five in the face of a Menger sponge, so $f_5^2(i) = d_4^2(i) = \frac{3}{10}8^i - \frac{32}{15}3^i + 4$. All of the vertices of degree two in the Sierpinski carpet remain such in the face of a Menger sponge except for the four corners. Thus $f_2^2(i) = d_2^2(i) - 4 = \frac{1}{10}8^i + \frac{16}{15}3^i$.

For vertices of degree three and four, we will need to use recurrence relations. We note that all the degree three vertices in the face of the Menger sponge still have degree three when the faces are combined, except perhaps for the four corners. In addition, the 16 sets of $d_2^1\left(i-1\right)$ vertices have their degrees increased from two to three, and so must be added in. There are also four corners that must be added in. Thus the recurrence relation is $f_3^2\left(i\right)=8\left(f_3^2\left(i-1\right)-4\right)+16\cdot3^{i-2}+4=8f_3^2\left(i-1\right)+16\cdot3^{i-2}-28$. We find the solution $f_3^2\left(i\right)=\frac{2}{5}8^i-\frac{16}{15}3^i+4$. We check that $f_3^2\left(1\right)=4$ and $f_3^2\left(i+1\right)=8\left(\frac{2}{5}8^i-\frac{16}{15}3^i+4-4\right)+16\cdot3^{i-1}+4=\frac{2}{5}8^{i+1}-\frac{16}{15}3^{i+1}+4$, verifying the formula. The first few terms of this sequence are 4, 20, 180, 1556, 12852, ...

For the number of vertices of degree four, we see that 16 sets of $d_3^1\left(i-1\right)$ vertices have their degrees increased from four to five, and so must be subtracted out. There are also $4\cdot4+2\cdot4$ corners which have their degrees increased from three to four, and so must be added in. Thus the recurrence relation is $f_4^2\left(i\right)=8f_4^2\left(i-1\right)-16\left(2\cdot3^{i-2}-2\right)+24=8f_3^2\left(i-1\right)-32\cdot3^{i-2}+56$. We find the solution $f_4^2\left(i\right)=\frac{1}{5}8^i+\frac{32}{15}3^i-8$. We check that $f_3^2\left(1\right)=0$ and $f_4^2\left(i+1\right)=8\left(\frac{1}{5}8^i+\frac{32}{15}3^i-8\right)-32\cdot3^{i-1}+56=\frac{1}{5}8^{i+1}+\frac{32}{15}3^{i+1}-8$, verifying the formula. The first few terms of this sequence are 0, 24, 152, 984, 7064, ...

Summarizing, we have found that for $i\geq1$,

$$f_2^2\left(i\right)=\frac{1}{10}8^i+\frac{16}{15}3^i$$

$$f_3^2\left(i\right)=\frac{2}{5}8^i-\frac{16}{15}3^i+4$$

$$f_4^2\left(i\right)=\frac{1}{5}8^i+\frac{32}{15}3^i-8$$

$$f_5^2\left(i\right)=\frac{3}{10}8^i-\frac{32}{15}3^i+4$$

As a check, we note that $f_2^2\left(i\right)+f_3^2\left(i\right)+f_4^2\left(i\right)+f_5^2\left(i\right)=8^i=n\left(SC_i\right)$.

4 Degrees of Vertices in the Menger Sponges

Now we are ready to determine how many vertices of degree two through six are in the Menger sponge graphs. We denote by $d_k^3\left(i\right)$ the number of vertices of degree k in MS_i. The level i Menger sponge graph is formed from twenty copies of level $i-1$. There are 24 edges in MS_1, and so there are 48 copies of faces of level $i-1$ that are joined in building level i. There are also $8 \cdot 3 = 24$ boundaries of level $i-1$ Sierpinski carpets that are joined to two level $i-1$ cubes. There are $8 \cdot 3 = 24$ corners that are joined to two level $i-1$ cubes, and eight vertices that are joined to three level $i-1$ cubes.

To determine the number of vertices of degree two, we must subtract out $f_2^2\left(i-1\right)$ vertices of degree two on the faces of the Menger sponge. This will subtract out 3^{i-2} vertices on the boundaries of these faces twice, so we must add them back in. Thus we see that the recurrence relation is $d_2^3\left(i\right) = 20d_2^3\left(i-1\right) - 48\left(\frac{1}{10}8^{i-1} + \frac{16}{15}3^{i-1}\right) + 24 \cdot 3^{i-2} = 20d_2^3\left(i-1\right) - \frac{3}{5}8^i - \frac{72}{5}3^i$. We find the solution $d_2^3\left(i\right) = \frac{1}{17}20^i + \frac{2}{5}8^i + \frac{216}{85}3^i$. We check that $d_2^3\left(1\right) = 12$ and $d_2^3\left(i+1\right) = 20\left(\frac{1}{17}20^i + \frac{2}{5}8^i + \frac{216}{85}3^i\right) - \frac{3}{5}8^{i+1} - \frac{72}{5}3^{i+1} = \frac{1}{17}20^{i+1} + \frac{2}{5}8^{i+1} + \frac{216}{85}3^{i+1}$, verifying the formula. The first few terms of this sequence are 12, 72, 744, 11256, 201960, ...

For the number of vertices of degree three, we must subtract out $f_3^2\left(i-1\right)$ vertices of degree three on the faces of the Menger sponge. We must also add in $f_2^2\left(i-1\right)$ vertices of degree two on the faces of the Menger sponge. The vertices of degree two on the boundaries of the faces will be increased to four, so we must subtract out 3^{i-2} vertices on the boundaries of these faces twice. There are also eight corner vertices subtracted out three times that must be added back in twice, and 24 corner vertices that were subtracted out twice, and must be added back in once. Thus the recurrence relation is $d_3^3\left(i\right) = 20d_3^3\left(i-1\right) - 48\left(\frac{2}{5}8^{i-1} - \frac{16}{15}3^{i-1} + 4\right) + 48\left(\frac{1}{10}8^{i-1} + \frac{16}{15}3^{i-1}\right) - 48 \cdot 3^{i-2} + 40 = 20d_3^3\left(i-1\right) - \frac{9}{5}8^i + \frac{144}{5}3^i - 152$. We find the solution $d_3^3\left(i\right) = \frac{24}{85}20^i + \frac{6}{5}8^i - \frac{432}{85}3^i + 8$. We check that $d_3^3\left(1\right) = 8$ and $d_3^3\left(i+1\right) = 20\left(\frac{24}{85}20^i + \frac{6}{5}8^i - \frac{432}{85}3^i + 8\right) - \frac{9}{5}8^i + \frac{144}{5}3^i - 152 =$

$\frac{24}{85}20^{i+1} + \frac{6}{5}8^{i+1} - \frac{432}{85}3^{i+1} + 8$, verifying the formula. The first few terms of this sequence are 8, 152, 2744, 49688, 941624, ...

For the number of vertices of degree four, we must subtract out $f_4^2(i-1)$ vertices of degree four on the faces of the Menger sponge. We must also add in $f_3^2(i-1)$ vertices of degree three on the faces of the Menger sponge. The vertices of degree two on the boundaries of the faces will be increased to four, so we must add in 3^{i-2} vertices on the boundaries of these faces. The $2 \cdot 3^{i-2} - 2$ vertices of degree four on the boundaries of the faces were subtracted out twice, and so must be added back in once. There are also eight corner vertices added in three times that must be subtracted out three times, and 24 corner vertices that were added in twice, and must be subtracted out twice. Thus the recurrence relation is $d_4^3(i) = 20d_4^3(i-1) - 48\left(\frac{1}{5}8^i + \frac{32}{15}3^i - 8\right) + 48\left(\frac{2}{5}8^{i-1} - \frac{16}{15}3^{i-1} + 4\right) + 24 \cdot (3^{i-1} - 2) - 72 = 20d_4^3(i-1) + \frac{6}{5}8^i - \frac{216}{5}3^i + 456$. We find the solution $d_4^3(i) = \frac{32}{85}20^i - \frac{4}{5}8^i + \frac{648}{85}3^i - 24$. We check that $d_4^3(1) = 0$ and $d_4^3(i+1) = 20\left(\frac{32}{85}20^i - \frac{4}{5}8^i + \frac{648}{85}3^i - 24\right) + \frac{6}{5}8^{i+1} - \frac{216}{5}3^{i+1} + 456 = \frac{32}{85}20^{i+1} - \frac{4}{5}8^{i+1} + \frac{648}{85}3^{i+1} - 24$, verifying the formula. The first few terms of this sequence are 0, 144, 2784, 57552, 1180320, ...

For the number of vertices of degree five, we must subtract out $f_5^2(i-1)$ vertices of degree five on the faces of the Menger sponge. We must also add in $f_4^2(i-1)$ vertices of degree four on the faces of the Menger sponge. The $2 \cdot 3^{i-2} - 2$ vertices of degree four on the boundaries of the faces were added in twice, and so must be subtracted out once. There are also 24 corner vertices whose degrees increase from three to five that must be added in. Thus the recurrence relation is $d_5^3(i) = 20d_5^3(i-1) - 48\left(\frac{3}{10}8^i - \frac{32}{15}3^i + 4\right) + 48\left(\frac{1}{5}8^i + \frac{32}{15}3^i - 8\right) - 48 \cdot (2 \cdot 3^{i-2} - 2) + 24 = 20d_5^3(i-1) - \frac{3}{5}8^i + \frac{288}{5}3^i - 456$. We find the solution $d_5^3(i) = \frac{14}{85}20^i + \frac{2}{5}8^i - \frac{864}{85}3^i + 24$. We check that $d_5^3(1) = 0$ and $d_5^3(i+1) = 20\left(\frac{14}{85}20^i + \frac{2}{5}8^i - \frac{864}{85}3^i + 24\right) - \frac{3}{5}8^{i+1} + \frac{288}{5}3^{i+1} - 456 = \frac{14}{85}20^{i+1} + \frac{2}{5}8^{i+1} - \frac{864}{85}3^{i+1} + 24$, verifying the formula. The first few terms of this sequence are 0, 24, 1272, 27192, 537720, ...

For the number of vertices of degree six, we must add in $f_5^2(i-1)$ vertices of degree five on the faces of the Menger sponge. We must also add in the $2 \cdot 3^{i-2} - 2$ vertices of degree four on the boundaries of the faces. There are also 8 corner vertices whose degrees increase from three to six that must be added in. Thus the recurrence relation is $d_6^3(i) = 20d_6^3(i-1) + 48\left(\frac{3}{10}8^i - \frac{32}{15}3^i + 4\right) + 24 \cdot (2 \cdot 3^{i-2} - 2) + 8 = 20d_6^3(i-1) + \frac{9}{5}8^i - \frac{144}{5}3^i + 152$. We find the solution $d_6^3(i) = \frac{2}{17}20^i - \frac{6}{5}8^i + \frac{432}{85}3^i - 8$. Now $d_6^3(1) = 0$ and $d_6^3(i+1) = 20\left(\frac{2}{17}20^i - \frac{6}{5}8^i + \frac{432}{85}3^i - 8\right) + \frac{9}{5}8^{i+1} - \frac{144}{5}3^{i+1} + 152 = \frac{2}{17}20^{i+1} - \frac{6}{5}8^{i+1} + \frac{432}{85}3^{i+1} - 8$, verifying the formula. The first few terms of this sequence are 0, 8, 456, 14312, 338376, ...

Summarizing, we have found that for $i \geq 1$,

$$d_2^3(i) = \frac{1}{17}20^i + \frac{2}{5}8^i + \frac{216}{85}3^i$$

$$d_3^3(i) = \frac{24}{85}20^i + \frac{6}{5}8^i - \frac{432}{85}3^i + 8$$

$$d_4^3(i) = \frac{32}{85}20^i - \frac{4}{5}8^i + \frac{648}{85}3^i - 24$$

$$d_5^3(i) = \frac{14}{85}20^i + \frac{2}{5}8^i - \frac{864}{85}3^i + 24$$

$$d_6^3(i) = \frac{2}{17}20^i - \frac{6}{5}8^i + \frac{432}{85}3^i - 8$$

As a check, we note that $d_2^3(i)+d_3^3(i)+d_4^3(i)+d_5^3(i)+d_6^3(i) = 20^i = n(MS_i)$, so the order is correct. Further, we see that $2d_2^3(i) + 3d_3^3(i) + 4d_4^3(i) + 5d_5^3(i) + 6d_6^3(i) = 4 \cdot 20^i - 4 \cdot 8^i = 2m(MS_i)$, so the First Theorem of Graph Theory is satisfied. These formulas also imply that as i grows large, approximately $\frac{1}{17}$ of the vertices will have degree 2, $\frac{24}{85}$ of the vertices will have degree 3, $\frac{32}{85}$ of the vertices will have degree 4, $\frac{14}{85}$ of the vertices will have degree 5, and $\frac{2}{17}$ of the vertices will have degree 6.

5 3-Cores of Menger Sponge Graphs

Suppose we successively delete a vertex of minimum degree of a graph until none remain. The maximum degree of any of these vertices when deleted is called the degeneracy $D(G)$ of a graph G [3]. A graph is k-degenerate if its degeneracy is at most k. By reversing the sequence of vertices formed by deleting a vertex of minimum degree, we can construct the graph so that each vertex, when added, is adjacent to at most $D(G)$ vertices. This is relevant to constructing a physical model of the Menger sponge, since if it constructed by adding one level 0 cube at a time, it is easier to add a cube adjacent to as few other previously added cubes as possible.

It is not hard to see that any Menger sponge graph is 3-degenerate. Note that MS_i is a subgraph of $P_{3^i}\square P_{3^i}\square P_{3^i}$. Any subgraph of $P_a\square P_b\square P_c$ has a vertex of degree at most three, which can be found by maximizing (or minimizing) the first coordinate, then maximizing (or minimizing) the second coordinate, then maximizing (or minimizing) the third coordinate. (Aside from from some degenerate cases, any such graph will have at least eight such vertices.) Thus any Menger sponge graph is 3-degenerate.

We see MS_1 is 2-degenerate, but what about larger Menger sponge graphs? The k-core of a graph is the maximal subgraph of minimum degree at least k of the graph (if it exists) [1]. Building a level 2 cube, we see that any face of a level 1 cube has a ring of eight vertices. Some of these faces will be face-to-face with other faces of level 1 cubes. In the Menger sponge graphs, this creates the subgraph $C_8\square K_2$, which is 3-regular. Thus for $i \geq 2$, the Menger sponge graph MS_i has a 3-core, so $D(MS_i) = 3$.

How many vertices are in the 3-core of a Menger sponge graph? As seen above, any vertex in a face of a level 1 cube adjacent to a vertex in another level 1 cube is in the 3-core. Vertices for which this is not the case must have degree 2 or 3 in the entire graph. Vertices with degree 2 are certainly not in the 3-core. A vertex of degree 3 is is the corner of a level 1 cube. Since it is not adjacent to vertices in any of the three faces it is contained it, its level 1 cube must be in the corner of its level 2 cube. Proceeding similarly, we see that this vertex must be in the corner of the entire cube. Then its three neighbors all have degree two, so it is not in the 3-core. Thus the only degree three vertices not in the 3-core are the eight corners of the cube. Thus the order of the 3-core $n\left(C_3\left(MS_i\right)\right) = n\left(MS_i\right) - d_2^3\left(i\right) - 8 = 20^i - \left(\frac{1}{17}20^i + \frac{2}{5}8^i + \frac{216}{85}3^i\right) - 8 = \frac{16}{17}20^i - \frac{2}{5}8^i - \frac{216}{85}3^i - 8$. The first few terms of this sequence are 0, 320, 7248, 148736, 2998032, ...

What can we say about the structure of the 3-core of a Menger sponge graph? We note that any Menger sponge has three planes of symmetry that are parallel to its faces. In any nontrivial sponge, the vertices on these planes all have degree 2. This is easily verified by induction. This implies that the 3-core is disconnected, and its number of components is divisible by 8. For MS_2, there are exactly eight components, each of which is composed of three partially overlapping copies of $C_8 \Box K_2$.

We note that each of these components contains vertices in three of the faces of the level 2 cube. When level 2 cubes are combined to form level three cubes, vertices that are in the three faces of a corner cube that are face-to-face with other faces, or vertices in those faces, will be contained in a single component of the 3-core. Seven of the eight components of the 3-core of the corner cube will be part of this larger component, along with four components of the 3-cores of each of the three cubes adjacent to it. This larger component also contains vertices in three of the faces of the level 3 cube. Proceeding similarly, we find that for $i \geq 1$, the 3-core of MS_i has $8\left(i-1\right)$ components, which can be divided into $i-1$ classes of eight isomorphic components.

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