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CONTINUOUS NON-DIFFERENTIABLE FUNCTIONS

by Brent Hailpern
University of Denver

Historically, there have been two types of mathematical discovery, intuitive reasoning and rigorous proof. These two methods of thinking complement each other perfectly. Intuitive thought can help one make great leaps of understanding. However, one simple error can lead intuitive thought on a wild goose chase. Rogorous proof, on the other hand, can harbor few, if any, mistakes because of the contradictions that inevitably arise. Unfortunately, rigorous proof implies something to prove. The object of the proof must, in some sense, be intuitively reasonable. The most profitable approach to understanding mathematics in general, and continuous non-differentiable functions in particular, is an alternating progression of intuition and rigor.

Before the nineteenth century, the terms continuity and differentiability were only intuitive ideas. Euler and Leibniz used "continuous" to describe "a function specified by an analytic formula" ([1], p. 405). In 1817, Bernhard Bolzano gave a workable definition of continuity, in the modern sense ([1], p. 951). Cauchy, in 1821, also defined continuity, in a usable form, in *Cours d'analyse algébrique*. Finally, K. W. T. Weierstrass gave what we call the modern ε, δ definition of continuity: A function $f(x)$ is continuous at $X = x_0$ if given any positive number ε, there exists a δ such that for every x in the interval $|x - x_0| < 6$, $|f(x) - f(x_0)| < \varepsilon$ ([1], p. 952). Intuitively this is often described as a function whose graph one can "draw" without lifting one's pencil off the paper.

Even after continuity had been rigorously defined, the connection between continuity and differentiability was not well understood. Most mathematicians of Cauchy's time believed that continuity implied differentiability except at isolated points ([1], p. 955). On July 18, 1872, Weierstrass presented his classic example of a continuous non-differentiable function:

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x) ,$$

where a is an odd integer and b is a positive constant less than 1 such that $ab > 1 + (3\pi/2)$ ([1], p. 956, where the following comment occurs:

> The historical significance of the discovery that continuity does not imply differentiability...was great. It made mathematicians all the more fearful of trusting intuition....)

Since that time, many mathematicians have devised rigorous proofs showing that various functions are continuous but not differentiable. The proofs are elegant in themselves. However, they are not easily understood without a knowledge of advanced calculus. The beginning calculus student is usually limited to understanding the $f(x) = |x|$ is not differentiable at $x = 0$ because of the sudden change in slope. A rigorous understanding may be beyond a beginner's grasp, but intuitive insight need not be. For example, in 1927, Fred W. Perkins of Harvard University published a proof for an elementary example of a continuous non-differentiable function [2]. His proof consisted of four parts. He first defined his function by an interpolation method over a dense domain. Secondly, the definition was extended to the entire real domain. Perkins then proved that the function was continuous. Finally, he demonstrated that his function was not differentiable. Though the proof involves the limit-point concept and related theorems, an intuitive understanding can be gained from the original definition with the aid of a few graphs.

The function is defined as follows. Given two points (x_a, y_a), (x_b, y_b) with $x_a \neq x_b$, define two new interpolated points (x_1, y_1) and (x_2, y_2) by:

$$x_1 = x_a + \frac{1}{3}(x_b - x_a), \qquad y_1 = y_a + \frac{5}{6}(y_b - y_a),$$

$$x_2 = x_a + \frac{2}{3}(x_b - x_a), \qquad y_2 = y_a + \frac{1}{6}(y_b - y_a).$$

Let $M(a,b) = \dfrac{y_b - y_a}{x_b - x_a}$ be the slope of the line from (x_a, y_a) to (x_b, y_b).

Algebraic calculation gives:

$$M(a,1) = \frac{y_1 - y_a}{x_1 - x_a} = \frac{\frac{5}{6}(y_b - y_a)}{\frac{1}{3}(x_b - x_a)} = \frac{5}{2}M(a,b),$$

and

$$M(1,2) = -2M(a,b), \quad M(2,b) = \frac{5}{2}M(a,b).$$

Notice that each interpolation causes the slopes to at least double in absolute value. Also, by the defining equations:

$$|y_1 - y_a| \le \frac{5}{6}|y_b - y_a|, \qquad |y_2 - y_1| \le \frac{5}{6}|y_b - y_a|,$$

$$|y_b - y_2| \le \frac{5}{6}|y_b - y_a|. \tag{1}$$

To define the function on $0 \le x \le 1$ choose $f(0) = 0$ and $f(1) = 1$. The interpolation process defines the function on the domain:

$$\{x : x = \frac{p}{3^n};\ p \text{ is a non-negative integer and } p \le 3^n\}.$$

The variable n is the number of interpolations required to evaluate the function at that point. The integer n is called the order of the interpolation. For example, the 0th order is (0,0) and (1,1) because no interpolations are required to obtain the arbitrarily chosen points. Interpolating once returns $\left(\frac{1}{3}, \frac{5}{6}\right)$ and $\left(\frac{2}{3}, \frac{1}{6}\right)$ as the first order points. The second order points are: $\left(\frac{1}{9}, \frac{25}{36}\right)$, $\left(\frac{2}{9}, \frac{5}{36}\right)$, etc. From (1) it can be seen that

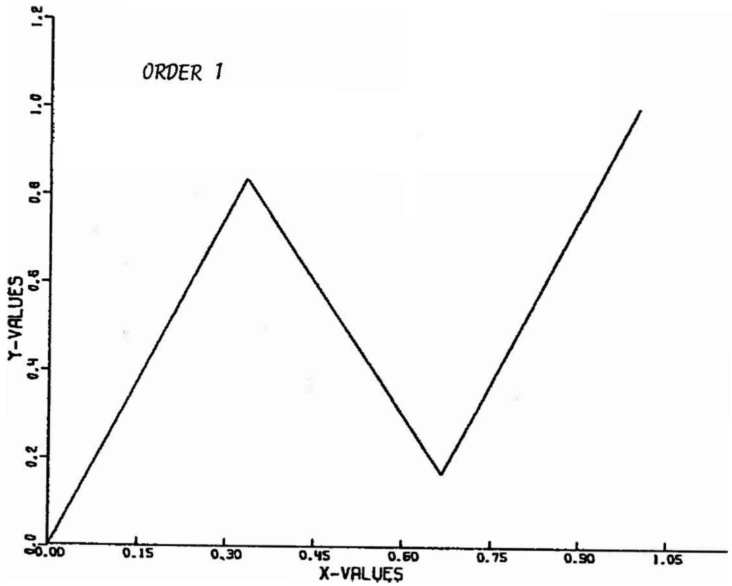
$$\left|f\left(\frac{p+1}{3^n}\right) - f\left(\frac{p}{3^n}\right)\right| \le \left(\frac{5}{6}\right)^n. \tag{2}$$

As a result of (2) the definition of the function can be extended to the real domain by a limiting process [3].

One can reason that the function is continuous from the fact that in (2) as two points are closer together (as $n \to \infty$) the function values grow closer and closer together. The fact that in each order of interpolation the slopes at least double shows that as $n \to \infty$ the slopes become infinitely large.

However, by following the four defining interpolation equations that Perkins gives, one can plot the first several orders of interpolation of the function, as shown on the following pages. It has often been said that a picture is worth a thousand words.

The successive orders of interpolation show the constantly increasing slopes and the tendency toward an infinite number of "absolute-value-like" bends in the graph. One can therefore sense intuitively that the infinite orders of interpolation would form a non-differentiable function.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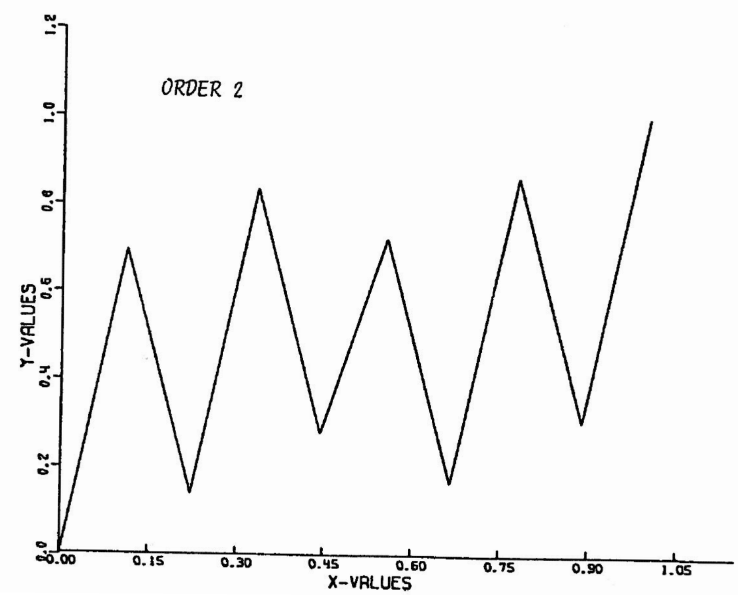
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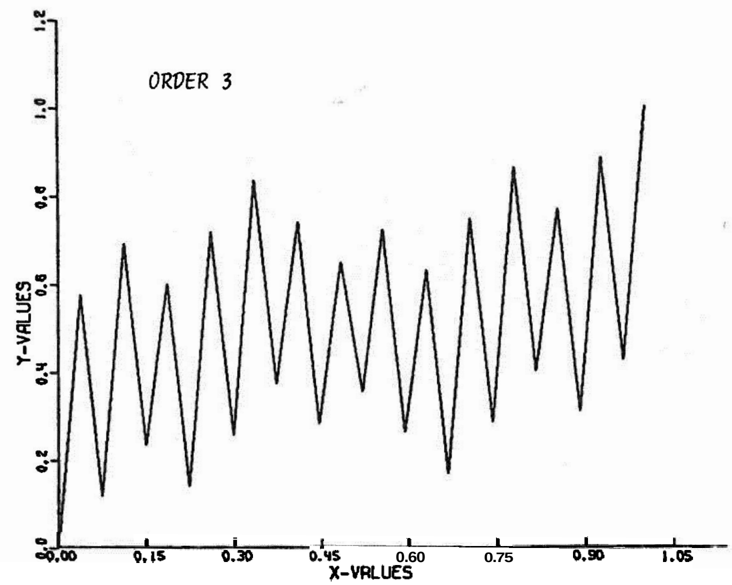
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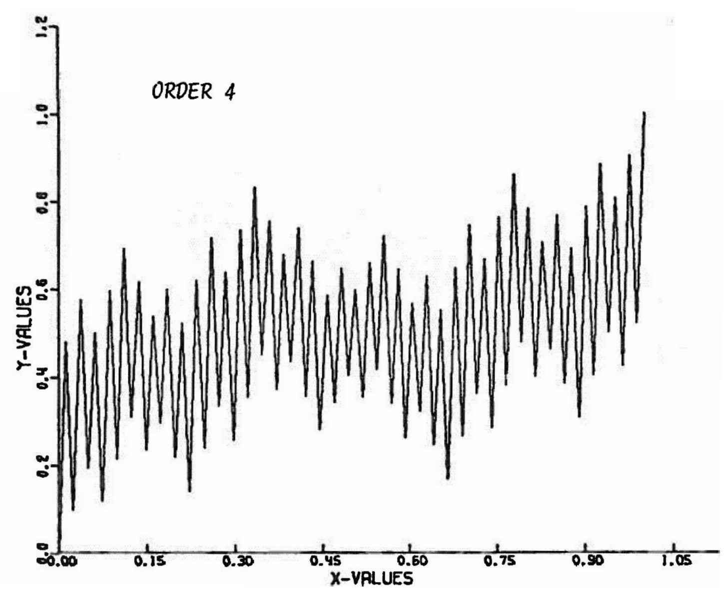
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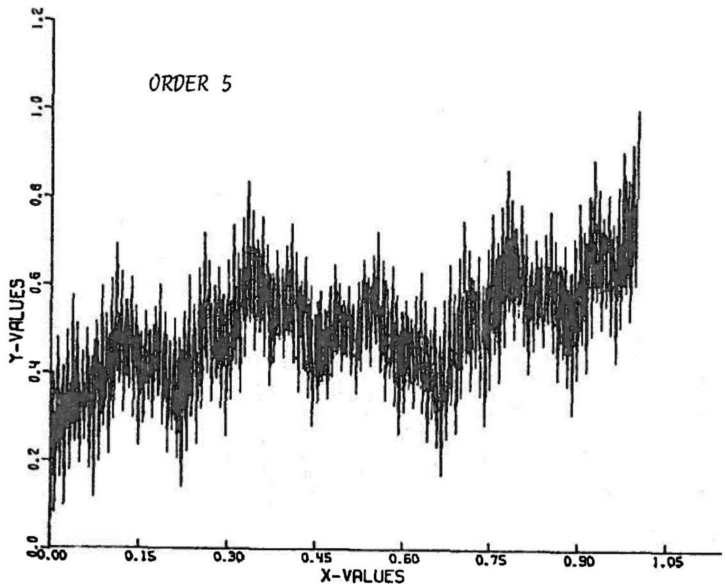
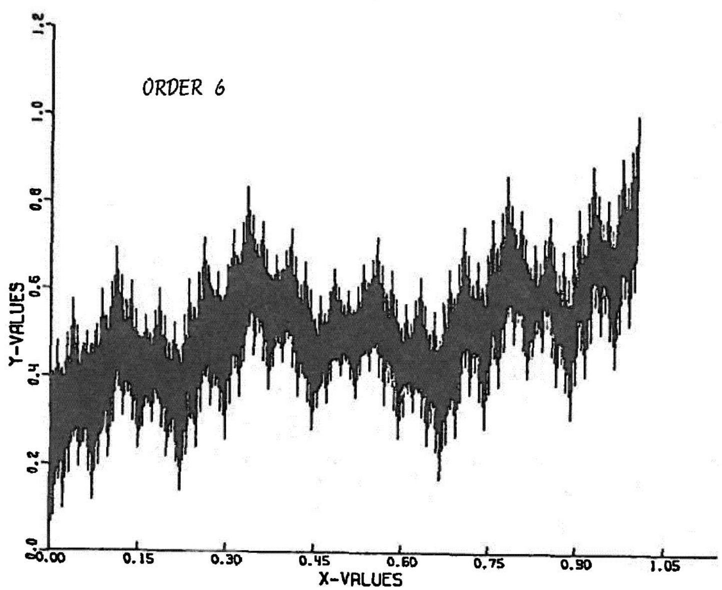




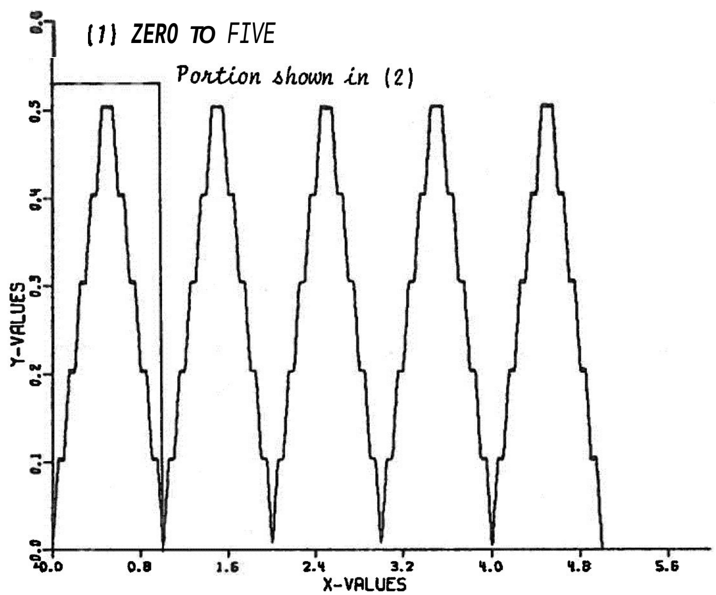
Also as the order increases, neighboring points become closer together. hinting that one could "draw" the graph of the function without lifting one's pencil from the paper. In other words, it can be intuitively seen as a continuous function that is not differentiable.

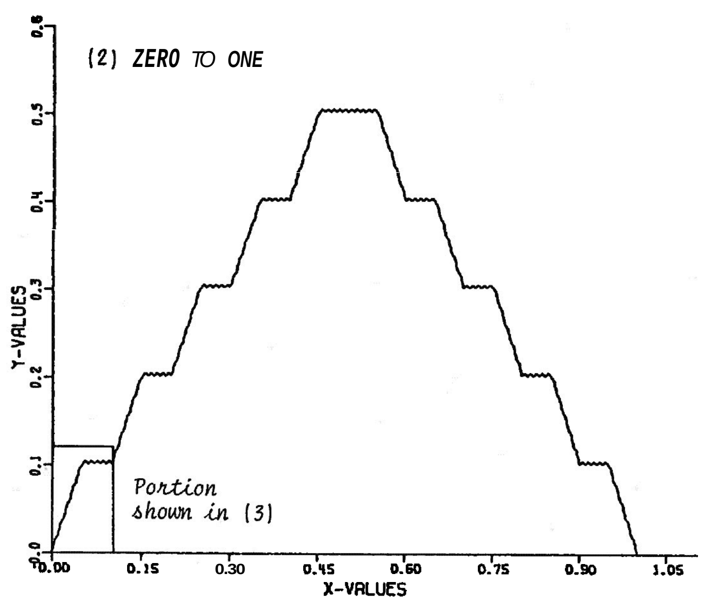
A second and different example was devised by B. A. Van der Waerden [4] in 1930. Rather than using an interpolating process? this function is devined by summation. Let

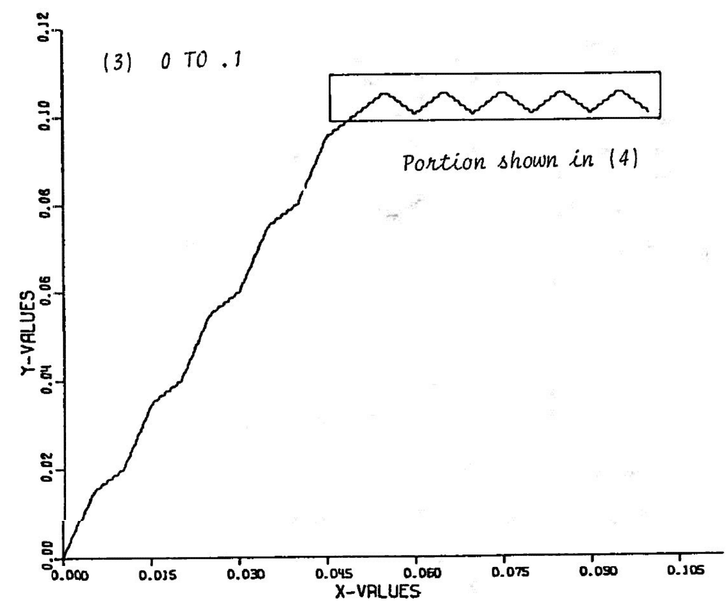
$$f(x) = \sum_{n=0}^{\infty} \frac{g(10^n x)}{10^n}$$

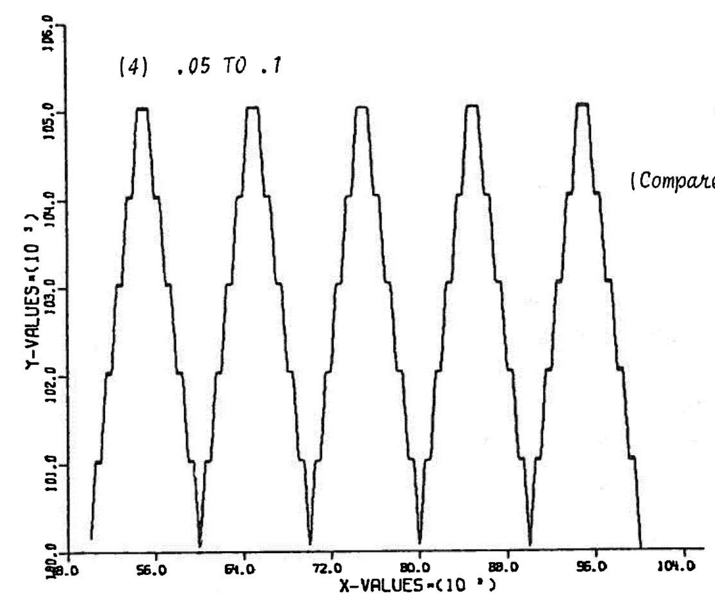
where $g(x)$ = distance from x to the nearest integer. The proof of continuity follows from the fact that the infinite sum of a sequence of continuous functions which converges uniformly is itself continuous. Van der Waerden proved that the series was not differentiable by finding a sequence of values x $+ h_n$ (which approach x as n goes to infinity) such that $\dfrac{f(x+h_n) - f(x)}{h_n}$ does not approach a limit as $n \to \infty$ (see Appendix). The proof is not easy to understand. An intuitive grasp of the qualities of the function, however, may be obtained from the graph of the function. Upon plotting, a series of pyramid-like shapes appears. The interesting fact is that upon magnification, the same pattern reappears. The small "bumps" on the horizontal steps of the pyramid turn out to be a repetition of the original pyramid. These smaller pyramids have "bumps" which are again a repetition of the original pattern. This indicates intuitively that the function is not differentiable, because taking a derivative involves taking smaller and smaller intervals on the domain in an attempt to have the slope of the function approach a constant value. This function approaches no constant value as smaller intervals are taken. It merely repeats itself ad *infinitum.* (For graphs, see following pages.)

The use of such plots yields a different type of beauty that can be lost in a proof. Beginning students can get a glimpse of what is involved in further studies. The experienced mathematician is sometimes surprised in realizing that this is what the graph of a continuous non-differentiable function looks like. Thus, the cycle is complete. The initial concepts of continuity and differentiability were impotent without rigorous definitions. Once the rigor is established, one can concentrate on the intuitive beauty that these ideas represent.









(Compare with (1))

APPENDIX

Proof That Van der Waerden's Function Is Non-differentiable

Let $f(x) = \sum_{n=0}^{\infty} \frac{g(10^n x)}{10^n}$ where $g(x)$ = distance from x to the nearest integer. We shall consider only the case $0 \le x \le 1$. We write x in the form $x = 0.a_1 a_2 \cdots a_n \cdots$ with the agreement that x will be written as a finite decimal completed with zeros, should the option arise. Then we can write

$$g(10^n x) = \begin{cases} 0.a_{n+1}a_{n+2}\cdots & \text{if } 0.a_{n+1}a_{n+2}\cdots \le \frac{1}{2} \\ 1 - 0.a_{n+1}a_{n+2}\cdots & \text{if } 0.a_{n+1}a_{n+2}\cdots > \frac{1}{2} \end{cases}$$

To show that $f(x)$ is not differentiable at a point x we need only exhibit a sequence $h_m \to 0$ such that the limit of

$$\frac{f(x + h_m) - f(x)}{h_m}$$

does not exist. Consider the sequence $\{h_m\}$ where

$$h_m = \begin{cases} -10^{-m}, & \text{if } m = 4 \text{ or } 9, \\ 10^{-m}, & \text{otherwise} \end{cases}$$

Note that as $m \to \infty$, $h_m \to 0$ and $x + h_m \to x$. Let

$$r_m(x) = \frac{f(x + h_m) - f(x)}{h_m} \; ;$$

then

$$r_m(x) = 10^m \sum_{n=0}^{\infty} \pm \frac{g[10^n(x \pm 10^{-m})] - g(10^n x)}{10^n}$$

where $\pm$ depends on whether $a_m = 4$ or 9.

For $n \ge m$ the terms of the sum $r_m(x)$ are equal to zero. This follows since the factor $\frac{10^n}{10^m}$ with $n - m \ge 0$ simply translates the domain of g by an integral amount and therefore $g\left(10^n x + \frac{10^n}{10^m}\right) = g(10^n x)$. Hence we need only consider $n < m$.

If $0 \le a_m < 4$ or $5 \le a_m < 9$ then,

$$r_m(x) = 10^m \sum_{n=0}^{m-1} \frac{g\left(10^n x + \frac{10^n}{10^m}\right) - g(10^n x)}{10^n}$$

But adding $\frac{10^n}{10^m}$ to $(10^n x)$ adds a one to the digit a_m. Therefore, $1 \le a_m + 1 < 5$ or $6 \le a_m + 1 < 10$. But in these two intervals

$$g\left(10^n x + \frac{10^n}{10^m}\right) = g(10^n x) \pm \frac{10^n}{10^m}, \qquad \begin{cases} + \text{ if } 0 \le a_m < 4 \\ - \text{ if } 5 \le a_m < 9 \end{cases}$$

which implies that

$$r_m = 10^m \sum_{n=0}^{m-1} \frac{\pm 10^n / 10^m}{10^n} = \sum_{n=0}^{m-1} \pm 1 \; .$$

The case where $a_m = 4$ or 9 is more involved. If we had tried

$$r_m(x) = 10^m \sum_{n=0}^{m-1} \frac{g\left(10^n x + \frac{10^n}{10^m}\right) - g(10^n x)}{10^n}$$

then we could not have guaranteed that each term of $r_m = \pm 1$ (as the reader can verify by trying $x = 0.444\cdots$). By using

$$r_m(x) = 10^m \sum_{n=0}^{m-1} \frac{-\left| g\left(10^n x - \frac{10^n}{10^m}\right) - g(10^n x) \right|}{10^n}$$

and an argument similar to the above, it can be shown that

$$r_m(x) = \sum_{n=0}^{m-1} \pm 1 \; .$$

Hence, for both cases, $r_m(x) = \sum_{n=0}^{m-1} \pm 1$. If $m - 1$ is even, then the sum consists of an odd number of terms and $r_m(x)$ is an odd integer. Similarly, if $m - 1$ is odd, then $r_m(x)$ is an even integer. Therefore, the sequence $\{r_m(x)\}$ is a sequence of integers, alternating even and odd, and does not converge. This implies that the derivative does not exist.

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REFEREES FOR THIS ISSUE

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TWO APPLICATIONS OF PSEUDOINVERSES

by Philip D. Olivier
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The concept of matrices dates back to the 1850's. Ever since, mathematicians have been concerned with the question "Since I can multiply two (conformable) matrices to get a third, how do I undo this multiplication?" It was soon discovered that every square non-singular matrix has associated with it another matrix, called its inverse. This inverse is the most natural extension of the idea of an inverse from ordinary multiplication. It was also found (E. H. Moore, 1920) that a rectangular or singular matrix also has associated with it another matrix, called its pseudoinverse. This pseudoinverse is the most natural extension of the matrix inverse.

This paper has two purposes: The first is to introduce to undergraduates who have taken Linear Algebra the pseudoinverse and second, to outline two simple, though non-trivial, applications of them. The first application should be accessible to anyone in an advanced calculus course (see Buck [3]), the second to anyone who has taken a course on ordinary differential equations (see Kreider et. al. [4]).

Introduction to Pseudoinverses

First we give the definitions of the inverse and pseudoinverse and some properties of each.

Definition 1. The *inverse* of a square, non-singular matrix A is that matrix X that satisfies the following two equations:

I1) $XA = I$ I2) $AX = I$

where I is the identity matrix of proper dimensions, usually written A^{-1}.

These two equations imply that A is square and non-singular.

Definition 2. The ***pseudoinverse*** of an arbitrary real matrix A is that matrix X that satisfies the following four equations:

PI1) $AXA = A$ PI3) $(AX)^T = AX$

PI2) $XAX = X$ PI4) $(XA)^T = XA$

where $(AX)^T$ signifies the transpose of the matrix AX. The usual notation for the matrix X is A .

Before going on it might be advisable for the reader to take a column and row matrix (vectors) and verify that if A is that chosen vector then

$$A^+ = \frac{A^T}{|A|^2}$$

where $|A|^2$ is the squared Euclidean norm of A.

In the theorems that follow we state some facts about pseudoinverses; the proofs are omitted because they are easy exercises and can be found in the references [1, pp. 2-3].

Theorem 1. If A is a square non-singular matrix then $A^+ = A^{-1}$.

Theorem 2. If the matrix equation AX = B represents any set of consistent linear equations then

$$AA^+B = B.$$

Theorem 3. If AX = B is as in Theorem 2 then

$$X = A^+B + (I - A^+A)Z$$

where I is the proper identity matrix and Z is any conformable matrix. (Z must be conformable with respect to both multiplication and addition.)

Theorem 1 says that if you know how to calculate the pseudoinverse you know how to calculate the inverse. Theorem 2 lets you check for consistency, which is very important in overdetermined systems, and Theorem 3 shows how to use the pseudoinverse in problem solving.

Theorems 2 and 3 are very useful. For example, when analyzing a complicated electric circuit using Kirchoff's Laws one always end up with more equations than unknowns, that is, an overdetermined system. To solve for the circuit parameters one merely checks for consistency (they should be consistent, otherwise a goof was made in setting up the equations). Then one uses Theorem 3. There is no need to eliminate the superfluous equations.

Applications

Example 1. In the theory of functions of several real variables the total (as opposed to partial) derivative is defined as follows.

Definition 3. If $f : R^n \to R^m$ then f' is that linear transformation, if it exists, that satisfies the following equation

$$\lim_{h\to 0} \frac{|f(x + h) - f(x) - f'h|}{|h|} = 0$$

where x, h $\in R^n$. Since h $\in R^n$ (considered as a column vector) and $f(x + h) - f(x) \in R^m$, f' must be an n by m matrix. The limit exists if it is the same for all vectors h as they go to zero.

Using pseudoinverses it is possible to state an equivalent definition that has the advantage of having the same appearance as the definition of the derivative for real functions of a real variable.

Definition 4. If $f : R^n \to R^m$, then

$$f' = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

where division by h means multiplication by the pseudoinverse of h. The order of multiplication is taken so as to obtain the object with the largest matrix dimensions.

For clarity take $f(x)$, x and h to be column vectors; then f' is formed by post multiplying $[f(x + h) - f(x)]$ by h^+ and taking the limit.

So far it has only been claimed that the two definitions agree. Rudin [5, pg. 215] shows that the ijth component of f' (according to Definition 3) is

$$(f')_{ij} = \frac{\partial f_i}{\partial x_j}$$

where f_i is the ith component of the m-vector f and x_j is the jth coordinate variable. In the following theorem we prove the equivalence of the two definitions by showing that $(f')_{ij}$ is the partial derivative of f_i with respect to x_j according to Definition 4 also.

Theorem 4. Definitions 3 and 4 are equivalent.

Proof. Let $f(x) \in R^m$, x $\in R^n$ be column vectors. Then

$$f' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \begin{pmatrix} f_1(x+h) - f_1(x) \\ f_2(x+h) - f_2(x) \\ \vdots \\ f_m(x+h) - f_m(x) \end{pmatrix} \frac{(h_1, h_2, \ldots, h_n)}{\sum_{k=1}^n |h_k|^2}$$

We focus our attention on the ijth component:

$$(f')_{ij} = \lim_{h\to 0} \frac{[f_i(x+h) - f_i(x)]}{\Sigma_{k=1}^n |h_k|^2} h_j. \qquad (1)$$

For the limit to exist it must be the same regardless of the manner in which it is approached. So let $h = h_j E_j$ where E_j is the unit vector whose components are all zeroes except for the jth component, which is a one. Then (1) becomes

$$(f')_{ij} = \lim_{h_j\to 0} [f_i(x + h_j E_j) - f_i(x)] \frac{h_j}{h_j^2} = \frac{\partial f_i}{\partial x_j}.$$

The idea of the total derivative of a function of several variables is now seen to be the obvious extension of the derivative. Whenever a concept makes rigor intuitive it is well worth the time to teach it.

Example 2. Schields [6, pp. 180-181] states that any linear differential operator can be expressed as a matrix with respect to a given set of vectors (functions). He then shows what the first and second derivative matrices look like with respect to the basis vectors $1, x, x^2, \cdots$. But he gives up at last because the matrices he obtained are singular. We show how pseudoinverses can be used to fill in that gap. To illustrate, we solve the ordinary second order linear differential equation

$$y' - y'' = 2x$$

or

$$Ly \equiv (D - D^2)y = 2x$$

In terms of the basis polynomials the matrix for L is

$$L = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using Greville's method in [2] the pseudoinverse of L is given by

$$L^+ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}.$$

Using Theorem 3 we have

$$y = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix}$$

or, in polynomial form,

$$y = a + 2x + x^2.$$

There are numerous other applications of pseudoinverses. They are useful virtually wherever matrices are.

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THE GEOMETRIC INVARIANCE OF TANGENTS TO CURVES $y = ax^r$

by Louis I. Alpert
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We present a simple geometric definition and construction of the tangent, at a given point P, to a curve C in the xy plane given by the equation $y = ax^r$, a real and r a positive rational. Our definition will not require calculus or analytic geometry (although we use the latter to simplify our notation), and it is entirely based upon a very interesting property of *geometric invariance* which the tangent λ to C at P possesses.

We begin with a parabola C (see Figure 1), with a vertical axis chosen for convenience. Starting at point P, we first move d units towards the axis of C, and then drop a perpendicular. Q is its intersection with C. Similarly, we locate R on C, this time moving d units away from the axis. If d is "small", we see that the acute angle α

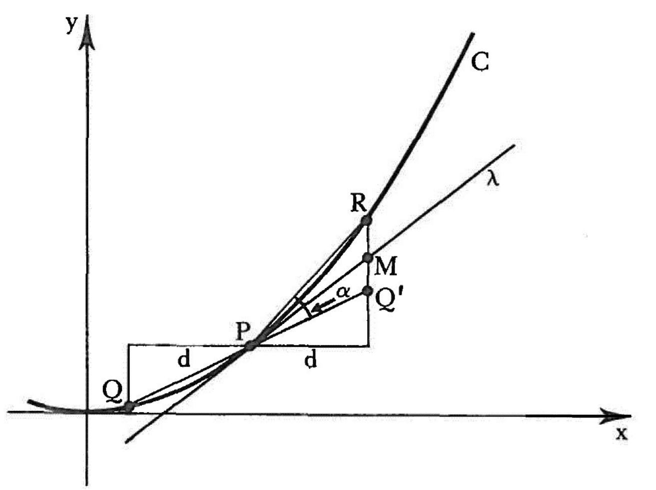

FIGURE 1

between the chords PQ and PR is also "small". Historical approaches to solving the *Problem of Tangents* were often concerned with the meaning of the "limiting position" of the chords PQ and PR as either d or α

approached 0. (It took more than two thousand years in the development of mathematics and the invention of analytic geometry and calculus in the seventeenth century before a precise interpretation of "limiting position" was given).

We do not concern ourselves here with any concept of "limiting position". Instead, the basic concept presented in this article is purely geometric, and although it will apply to the family of curves $y = ax^r$, a real and r a positive rational, we initially restrict $a = 1$ and r to positive integral values, n. Therefore, our initial concept is a property of *geometric invariance* for a well-defined class of, straight lines which pass through points of $y = x^n$, $n \geq 1$.

Returning to Figure 1, we first consider the case $n = 2$. We construct λ by simply joining P to M, the midpoint of the vertical line-segment which joins R to Q', where Q' is the symmetric image of Q in P. It is now possible to establish that λ is invariant with respect to the choice of d (see the proof of theorem 1 given below).

Remark. While Theorem 1 could have been proven by the ancient Greeks using synthetic geometry, we more conveniently obtain a proof using analytic geometry.

Theorem 1. Let M be any point of the vertical line-segment RQ', as shown in Figure 1. Line PM is invariant with respect to d if and only if M is the midpoint of RQ'.

Proof. For brevity, we take the parabola C of Figure 1 as the curve $y = x^2$. Then

$$P = (p,\ p^2),\quad Q = (p-d,\ p^2-2pd+d^2),\quad Q' = (p+d,\ p^2+2pd-d^2),$$
$$R = (p+d,\ p^2+2pd+d^2),\quad M' = (p+d,\ p^2+2pd).$$

We may represent $M = (p + d,\ p^2 + 2pd + wd^2)$, where $-1 \leq w \leq 1$. Our computation directly shows that PM is invariant with respect to d if and only if $w = 0$, that is, if and only if $M = M'$. This is seen by computing the slope of PM as $(2pd + wd^2)/d = 2p + wd$, noting that it will be invariant with respect to d, if and only if $w = 0$.

If we now bring calculus into our presentation, we may note that the value of the derivative at p is 2p, so that the usual definition of the tangent to C at P is precisely $\lambda = PM$. This establishes the following:

Corollary 1. Line PM is invariant with respect to d if and only if PM is the tangent to C at P.

In view of Theorem 1, we now define the tangent to the parabola C at the point P as the (unique) straight line through P which possesses our property of geometric invariance. Corollary 1 merely states that our "new" definition of tangent is equivalent to the "old" definition based upon analytic geometry and the calculus.

We now proceed to show that our "new" definition generalizes in a very natural way to the entire family of curves, $C : y = x^n$, $n \geq 1$. To this end, we first introduce

$$\bar{n} = \sqrt[n-1]{n} \, ,$$

which may be interpreted as the $(n-1)st$ geometric mean of n and $n-2$ units. Now, our proposed construction of the tangent to C at P may be detailed as follows:

1. Move d units from P towards the left (say) and construct a perpendicular from this new position until it intersects C at some point Q.
2. Move $\bar{n}d$ units from Q to the right and construct a perpendicular from this new position until it intersects C at some point R.
3. Define the $(n-1)st$ geometric image of Q in P, called Q', and located to the right of P on the extension of line PQ, at a distance from Q equal to $\bar{n}$ times the distance from P to Q.
4. Locate the point M on the vertical line RQ', whose distance from Q' is $1/\bar{n}$ times the distance from R to Q'. Join the points P and M and let $\lambda = PM$.

Note that our proposed construction above implies that both the horizontal position of P and the vertical position of M will be $1/\bar{n}$ times the distance from Q to R and R to Q', respectively. We note that the role of M as the midpoint of RQ' when $n = 2$ (the special case of the parabola) may now be viewed more generally as that point on RQ' whose distance from Q' is $1/\bar{n}$ times the distance from R to Q'. Thus, the reciprocal of λ represents a "geometric average position" along the vertical line RQ' to which the point P (located at the same "geometric average position" between Q and R) is joined to produce the tangent to C at P.

With the above motivation, we now introduce the following notation. Let

$$P = (p, p^n), \quad Q = (p-d, (p-d)^n), \quad Q' = (p_1, \bar{n}p^n-(\bar{n}-1)(p-d)^n),$$
$$R = (p_1, p_1^n), \quad M = (p_1, p^n+n(\bar{n}-1)dp^{n-1})$$

where $p_1 = p + (\bar{n}-1)d$.

Theorem 2. Let N be any point in the vertical line-segment RQ', with notation as defined above. Then PN is invariant with respect to d if and only if $N = M$ in RQ'.

With analytic geometry, this can be proved in a manner similiar to the proof of Theorem 1.

We now extend our "new" definition of tangent to the family of curves $C : y = x^n$, $n \geq 1$, by stating that λ is the tangent to C at P if it is the (unique) straight line through P which possesses the property of geometric invariance described in Theorem 2.

It is a consequence of the exercise in analytic geometry used to prove Theorem 2 that λ may be recognized as the "old" tangent to C at P, so that we obtain a generalization of Corollary 1 which establishes the equivalence of the "old" and "new" definitions of tangents to curves $C : y = x^n$, $n \geq 1$.

It may be of interest to note that all of the foregoing may be established using only synthetic geometry, resulting in a purely geometric characterization of the concept of tangent to a curve $y = x^n$, $n \geq 1$. (Because of this, the ancient Greek geometers quite probably could have understood and appreciated our concept of tangent.) An important advantage here is that, using this definition, a ruler and compass construction of the tangent exists for all curves of the form $y = x^n$, $n = 2^k + 1$, $k \geq 0$.

As an example, this definition may be used for a ruler and compass construction of the tangent to the cubic at any point (a curve whose existence was known to the ancient Greeks, but who were unable to construct its tangent-lines). Let $P = (p, p^3)$, $p > 0$, be a given point on the cubic C as shown in Figure 2. Following the details of our construction in the special case $n = 3$, we first move d units from P to the left

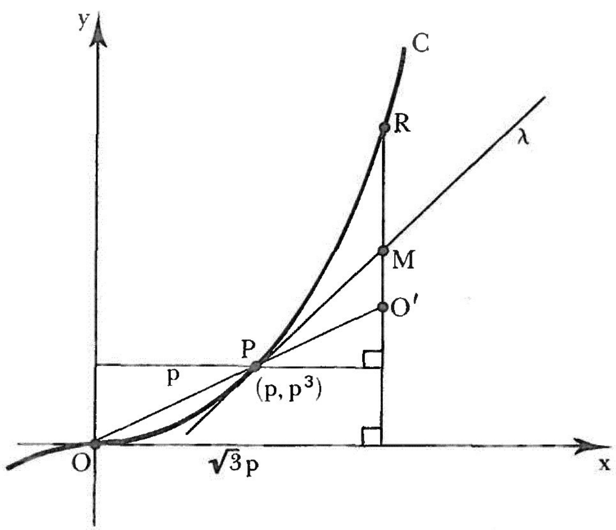

FIGURE 2

(on account of our invariance with respect to d, we conveniently choose $d = p$) and construct a perpendicular from this new position until it intersects C at $O = (0, 0)$. We then move $\sqrt[2]{3} \equiv \sqrt{3}$ times p units from O to the right and construct a perpendicular from this new position until it intersects C at R as shown. Next, we locate O' to the right of P on the extension of line OP, at a distance from O equal to $\sqrt{3}\,\overline{OP}$, where $\overline{OP}$ is the distance from O to P. Finally, we locate M on the vertical line-segment RO' by moving down from R a distance equal to $\sqrt{3}/3 \cdot \overline{RO'}$, where $\overline{RO'}$ is the distance from R to O'. Our tangent λ is now obtained by joining P to M as shown.

A definition and construction of the tangent, at a given point P, to a curve $y = x^r$, r a positive rational, may be suitably generalized from that given in the case $r = n$, a positive integer. The definitions of the notation $\bar{r}$, P, Q, Q', R, and M given prior to Theorem 2 apply directly when $r \geq 1$, and Theorem 2 may be verified for such r. For positive values of $r < 1$, we may identify the graphs of $y = x^p$ and $x = y^{1/r}$ (in the first quadrant) so that our definition and construction shall

apply in the case of all positive rational values of r.

We emphasize that while our development does not require calculus or analytic geometry, it does require the construction of the reciprocal of the $(r-1)$st root of r; a procedure which is not so simple (except, possibly, when $r = 2^k + 1$, k a positive integer or 0). Since $|a|^{1/r}$ may be considered a "scale factor", we note that our definition and construction applies to all curves of the form $y = ax^r$, where a is any real number, and r, any positive rational number. We remark, finally, that by restricting r to (positive) rationals, we avoid all limiting processes implied, say, in the definition of an irrational root of a real number.

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A NONSTANDARD MODEL OF THE REAL NUMBERS
WITH APPLICATIONS TO LIMITS AND CONTINUITY

by Paul Raymond Patten
University of Oklahoma

I. *Filters, Ultrafilters, and Sequences*

The purpose of this paper is to give a description of a nonstandard model of the real numbers using a set theoretic construction, and to give applications of this model to limits and continuous functions.

The main purpose in developing this model is to resurrect the idea of an infinitesimal element in an ordered field which satisfies the same statements in a lower predicate calculus language as does the real numbers. The lower predicate calculus or first order language includes those sentences where the quantifiers there exists and for all are applied to variables ranging over the set of reals, R, but not to variables ranging over proper subsets of R, such as the natural numbers, N. For example, consider the sentence

> for all x **and** for all y exactly one of the following is true: x is less than y, x = y, or y is less than x,

and the sentence

> for all x there exists a natural number n such that x *is* less than n.

In the first sentence the variables x and y are allowed to range over all possible values in R (the trichotomy property for R), while in the second sentence one of the variables (n) is restricted to a proper subset (N) of R (the Archimedean property). Thus while any model of the real numbers will be required to satisfy properties like the trichotomy property it is not necessary for the model to satisfy properties like the Archimedean property. For a more careful treatment of the model theoretic aspects of nonstandarc analysis, see Robinson [3].

This paper will avoid the explicit use of model theory by giving a *particular* model to be constructed using set theory (Zermelo-Fraenkel) with the axiom of choice in the form of Zorn's lemma included. This construction is based on the methods found in Luxemburg [2].

The model to be constructed will consist of the set

$$^*R = \{f \mid f \text{ is a real valued sequence}\}$$

under certain equivalence and order relations to be defined in the following discussion. As a first step toward the definition of an appropriate equivalence relation consider equivalence with respect to the family of subsets

$$F = \{N - F \mid F \text{ is a finite subset of } N\}$$

defined by

$$f =_F g \text{ if and only if } \{i \mid f(i) = g(i)\} \in F.$$

To show that $=_F$ is an equivalence relation first notice that F satisfies the following three properties of a *filter* on N:

($F1$) $\emptyset$ is not in F and N belongs to F.

($F2$) If A is in F and $A \subseteq B \subseteq N$ then B belongs to F.

($F3$) If A belongs to F and B belongs to F then $A \cap B$ belongs to F.

These properties can be used to show that $=_F$ is reflexive, symmetric, and transitive. For example, we prove the

<u>Transitive Property</u>. If $f =_F g$ and $g =_F h$ then $f =_F h$.

Proof. Since $\{i \mid f(i) = g(i)\}$ belongs to F and $\{i \mid g(i) = h(i)\}$ belongs to F, by property F3 $A = \{i \mid f(i) = g(i)\} \cap \{i \mid g(i) = h(i)\}$ is in F. Since $f(i) = h(i)$ for all i in A, $A \subseteq \{i \mid f(i) = h(i)\}$ so that by property F2 $f =_F h$.

In a similar manner one may define an order $<_F$ with respect to F by stating $f <_F g$ if and only if $\{i \mid f(i) < g(i)\}$ belongs to F. It is clear by a proof similar to the one given above that $<_F$ is transitive and is therefore an order relation.

Some examples of these relations will now be considered.

<u>*Example 1*</u>. Let $f(1) = 100$, $f(2) = 2$, and for n > 2, $f(n) = (1/2)^{n-2}$. Also let $g(1) = 500$, $g(2) = 2$, $g(3) = 5$, and for n > 3, $g(n) = (1/2)^{n-2}$. Then $f =_F g$ since f and g agree except on a finite set. Define the sequence r by $r(n) = r$ for $0 < r$ in R. Then, with the above f, and for any positive real number r, $f <_F r$.

Example 2. Let $f(n) = 1/2$ if n is odd and $f(n) = 2$ if n is even. Let $g(n) = 2$ for all n. Since $\{i \mid f(i) = g(i)\} = \{2n \mid n \text{ is in } N\}$ has an infinite complement $f \neq_F g$. Also, since $\{i \mid f(i) > g(i)\} = \emptyset$, $g \not<_F f$, and since $\{i \mid f(i) < g(i)\} = \{2n - 1 \mid n \in N\}$ has an infinite complement, $f \not<_F g$.

Example 2 shows one shortcoming of using the filter F: $^*R$ under the order $<_F$ does not satisfy the trichotomy property. Hence $^*R$ under $<_F$ is not a model of R satisfying the first order language.

In order to construct an ordering of $^*R$ which satisfies the trichotomy property a filter $U \supseteq F$ which contains as an element either the odd natural numbers or the even natural numbers is needed. In fact since examples using infinite sets other than the even or odd natural numbers can be constructed it is necessary to have a filter $U \supseteq F$ such that given a subset A of N then either A or N - A is in U. This property (UF) is guaranteed if U is taken to be a maximal filter (under set inclusion) containing F as a subset (U is called an *ultrafilter*). Such an ultrafilter is guaranteed by Zorn's lemma. It should be noted at this point that the ultrafilter U is not unique and that it may be possible, if $A \subseteq N$, for A to be in U or N - A to be in U without knowing explicitly which is the case. That is, there may be an ultrafilter which contains the even numbers and another ultrafilter which contains the odd numbers; thus given an arbitrary ultrafilter it may not be possible in Example 2 to decide whether $f <_U g$ or $f =_U g$. All that is known is that exactly one of these possibilities must hold.

Let an ultrafilter U containing F be fixed. Then $=_U$ and $<_U$ are defined in the same way as before; however, $f =_U g$ will be denoted as $f = g$ and $f <_U g$ as $f < g$. We note that if $r = s$ in R then $r = s$ as constant sequences in R^* (we say that $=_U$ extends =). Similarly, if $r < s$ in R then $r < s$ as constant sequences ($<_U$ extends <). The following properties are thus observed to hold for = and < on $^*R$:

($U1$) = is an equivalence relation on $^*R$ extending = on R.

($U2$) < is an order relation on $^*R$ extending < on R.

($U3$) If f and g are in $^*R$ then exactly one of the following is true: $g < f$, $g = f$, or $f < g$.

($U4$) If $g = f$, $h = k$, and $f < k$ then $g < h$ (< is compatible with =).

The only properties that have not been handled previously are $U3$ and U4. U3 will be proved for the case $g \not\approx f$ since the other cases can be

handled similarly.

Proof of $U3$ if $g \not\approx f$. Thus, $A = \{i \mid f(i) \geq g(i)\}$ does not belong to U. By property UF the complement of A, $\{i \mid f(i) < g(i)\}$, belongs to U. Thus $f < g$. On the other hand if $f < g$ then by property F2 it can be seen that $g \not< f$ and $g \neq f$.

(For the proof of U4 use the filter properties of U to show that $\{i \mid g(i) < k(i)\}$ is in U.)

2. Extension of Functions to $^*R$

The next property verifies that real valued functions of a finite number of real variables can be extended to $^*R$. First it is necessary to define the extension of subsets of R to subsets of $^*R$.

Definition. Let $A \subseteq R$. Then define $^*A = \{g \in {}^*R \mid \{i \mid g(i) \text{ is in } A\} \in U\}$.

Property EXT. Let $A_k \subseteq R$ for $k = 1, 2, \ldots, n$. Let $f: A_1 \times A \times \ldots \times A_n \to R$ be a function. Then f can be extended to a function $^*f$ where $^*f: {}^*A_1 \times {}^*A_2 \times \ldots \times {}^*A_n \to {}^*R$ is defined by $^*f(g_1, \ldots, g_n)(i) = f(g_1(i), \ldots, g_n(i))$ if $g_k(i)$ is in A_k for $k = 1, \ldots, n$ and $^*f(g_1, \ldots, g_n)(i) = 1$ for any other i. Notice that if $(a_1, \ldots, a_n)$ is a constant sequence in $A_1 \times \ldots \times A_n$ then $^*f(a_1, \ldots, a_n)(i) = f(a_1, \ldots, a_n)$ for all i in N.

Proof. It is clear that $^*f(g_1, \ldots, g_n)$ is in $^*R$. To show that $^*f$ is well defined suppose $g_k = h_k$ for $k = 1, \ldots, n$ (i.e. $(g_1, \ldots, g_n) = (h_1, \ldots, h_n)$). Then for each k, $\{i \mid g_k(i) = h_k(i)\}$ is in U. Hence, $B = \{i \mid (g_1(i), \ldots, g_n(i)) = (h_1(i), \ldots, h_n(i))\}$ contains $\bigcap_{k=1}^{n} \{i \mid g_k(i) = h_k(i)\}$ which can be shown to be in U inductively. Since $\{i \mid f(g_1(i), \ldots, g_n(i)) = f(h_1(i), \ldots, h_n(i))\} \supseteq \bigcap_{k=1}^{n} \{i \mid g_k(i) \in A_k\} \cap B$, $^*f(g_1, \ldots, g_n) = {}^*f(h_1, \ldots, h_n)$ by property $F2$.

The property EXT applied to the cases where $n = 1$ and $n = 2$ shows that functions $f : A \to R$ can be extended to $^*R$, and operations such as $+, \cdot, -, \div$ can also be extended to $^*R$ (here as in R division by 0 is still undefined). Using examples 1 and 2 on pages 2 and 3 $(100, 2, 1/2, 1/4, \ldots) + (2, 2, 2, 2, \ldots) = (102, 4, 2+1/2, 2+1/4, \ldots)$. Notice that $(500, 2, 5, 1/4, \ldots, 2^{-(n-2)}, \ldots) + (2, 2, 2, 2, \ldots) = (502, 4, 7,$

$2-1/4, \ldots) = (102,\ 4,\ 2+1/2,\ 2+1/4, \ldots)$ so that in this example addition is well defined. In fact it can be shown that $^*R$ is an ordered ($<$) field in which R is embedded by the order preserving field monomorphism sending r in R to the constant sequence r in $^*R$.

At this point one should notice that there are certain properties of R which do not hold in $^*R$. It is well known that R satisfies the completeness property: If $\emptyset \neq A \subseteq R$ and there is an ℓ in R such that for all r in A $\ell \leq r$, then there is a greatest such ℓ in R called the greatest lower bound of A. The following example shows that $^*R$ is not complete:

Define a sequence of elements, f_n, of $^*R$ as follows. $f_1(i) = i$,

$$f_2(i) = \begin{cases} i-1 & \text{if } i \geq 2 \\ 1 & \text{otherwise} \end{cases}, \ldots, f_n(i) = \begin{cases} i-(n-1) & \text{if } i \geq n \\ 1 & \text{otherwise} \end{cases}, \ldots .$$

The first few terms appear as $f_1 : 1, 2, 3, 4, \ldots;$ $f_2 : 1, 1, 2, 3, \ldots;$ $f_3 : 1, 1, 1, 2, \ldots;$ etc. Notice that the constant sequence $1 < f_n$ for all n. Now if ℓ is a lower bound for $A = \{f_n \mid n \text{ is in } N\}$ then $\ell + 1$ is a larger lower bound for A. This fact may be verified by observing that for $n \geq 1, \ell + 1 \leq f_{n+1} + 1 = f_n$. Thus, there can be no greatest lower bound of A in $^*R$. Considering f_1 it can be seen that if n is in N then $n < f_1$; hence, $^*R$ is not Archimedean. These are examples of sentences which do not belong to the lower predicate calculus language for R.

3. *Infinitesimals* and *Standard Value*

At the beginning of this paper the goal of finding an extension of the real numbers such that non-zero infinitesimals would exist in this extension was set. To see that this goal has been achieved consider the sequence $f(n) = 1/n$. It is clear that given any real number $r > 0$ then $f(n) < r$ except on a finite set; hence, $f < r$ for all $0 < r$ belonging to R. Since $f \neq 0$, f is a non-zero infinitesimal. The set of infinitesimals, $\mathcal{I}$, is defined by $\mathcal{I} = \{f \in {}^*R \mid |f| < r \text{ for all real numbers } r > 0\}$. Another useful set in this connection is $\mathcal{B} = \{f \in {}^*R \mid \text{for some real number } r > 0,\ |f| < r\}$. In both of these sets absolute value is the extension of absolute value of real numbers to $^*R$ guaranteed by *EXT*. It can be shown that

$$|f| = \begin{cases} f & \text{if } f \geq 0 \\ -f & \text{if } f \leq 0 \end{cases}$$

just as in R. Thus, $|f| < r$ is equivalent to $-r < f < r$. For this reason

the elements of $\mathcal{B}$ are called the bounded elements of $^*R$. Notice that $\mathcal{I} \subseteq \mathcal{B}$.

The following is a list of some properties of $\mathcal{I}$ and $\mathcal{B}$:

($\mathcal{I}$1) If f, g belong to $\mathcal{I}$ then $f + g$ belongs to $\mathcal{I}$.

($\mathcal{I}$2) If f belongs to $\mathcal{I}$ and g belongs to $\mathcal{B}$ then fg belongs to $\mathcal{I}$.

($\mathcal{I}$3) f does not belong to $\mathcal{I}$ if and only if $1/f$ belongs to $\mathcal{B}$.

Proof. If f is not in $\mathcal{I}$ then for some real number $r > 0$ $|f| \geq 2/r > 1/r > 0$. Hence, $f \neq 0$ and $r > |1/f|$. Thus, $1/f$ belongs to $\mathcal{B}$. On the other hand if $1/f$ is in $\mathcal{B}$ then $f \neq 0$ (since 0 does not have a reciprocal in $^*R$). Now there is a real number $r > 0$ such that $0 < |1/f| < 1/r$. Hence, $r < |f|$, which implies that f is not in $\mathcal{I}$.

($\mathcal{I}$4) If f is in $\mathcal{I}$ then $-f$ is in $\mathcal{I}$.

($\mathcal{I}$5) If fg is in $\mathcal{I}$ then f is in $\mathcal{I}$ or g is in $\mathcal{I}$.

Proof. Let fg belong to $\mathcal{I}$ with f not in $\mathcal{I}$. Then $f \neq 0$ and there is a real number $r > 0$ such that $|f| \geq r$. Let s be a real number > 0. Then since fg is in $\mathcal{I}$ $|fg| < rs \leq |f|\ s$. Since $|f| > 0$, $|g| < s$; hence, g belongs to $\mathcal{I}$.

($\mathcal{I}$6) $\mathcal{B}$ is an integral domain.

Properties $\mathcal{I}$1, $\mathcal{I}$2, and $\mathcal{I}$4 guarantee that $\mathcal{I}$ is an ideal of $\mathcal{B}$. Property $\mathcal{I}$5 means that $\mathcal{I}$ is a prime ideal. Thus the ring $\mathcal{B}/\mathcal{I}$ is also an integral domain. Property $\mathcal{I}$3 implies that $\mathcal{B}/\mathcal{I}$ is actually a field. In fact it turns out that $\mathcal{B}/\mathcal{I}$ is isomorphic to R as an ordered field. To prove this fact, a new function, the standard value function st, will be defined from $\mathcal{B}$ onto R which will be a homomorphism of rings preserving $\leq$ with $\mathcal{I}$ as the kernel (that is $f \in \mathcal{I}$ if and only if $st(f) = 0$).

The definition of st is as follows: If f belongs to $\mathcal{B}$ then $st(f) = r$ in R where $f - r$ is in $\mathcal{I}$ if such a real number exists. (It is clear from this definition that $st(r) = r$ for all real numbers r so that st is onto R.) There are two properties which must be established:

(ST1) If there is such an r then r is unique, and

(ST2) the domain of st is $\mathcal{B}$ (i.e. given any f in $\mathcal{B}$ there is an r in R such that $f - r$ is in $\mathcal{I}$).

Proof of $ST1$. Notice that if $r \neq s$ and r, s belong to R with $f - r \in \mathcal{I}$ and $f - s \in \mathcal{I}$ then by $\mathcal{I}$1 and $\mathcal{I}$4 $f - s - (f - r) \in \mathcal{I}$. Hence, $s - r \in \mathcal{I}$, which is impossible since $|s - r| > |s - r|/2$, which belongs to R with $|s - r|/2 > 0$.

Proof of ST2. Assume first that $f \geq 0$. Then there is a real number $r > 0$ such that $f \leq r$. Let $A = \{r \in R \mid 0 \leq f \leq r\}$. Then $A \subseteq R$ and certainly A is bounded below by 0. Thus, by completeness in R, A has a greatest lower bound r_0 in R. The claim is that $f - r_0 \in I$. Let s be a real number > 0. Then either $s \leq |f - r_0|$ or $|f - r_0| < s$ by property U3. The case $s \leq |f - r_0|$ is now shown to be impossible. There are two subcases to consider: (a) $|f - r_0| = f - r_0$ (if $f \geq r_0$) or (b) $|f - r_0| = r_0 - f$ (if $f \leq r_0$).

(a) We have $s \leq f - r_0$ or $s + r_0 \leq f$ for all r in A. Since $s > 0$, $s + r_0$ is a lower bound contradicting the fact that r_0 is the greatest lower bound.

(b) We have $s \leq r_0 - f$ or $0 \leq f \leq r_0 - s$, which belongs to A contradicting the fact that r_0 is a lower bound for A. Thus, $f - r_0 \in I$. If $f < 0$ then the above argument guarantees an r_0 such that $-f - r_0 \in I$. By I4 $f - (-r_0)$ belongs to I. Hence $st(f) = -r_0 = -(st(-f))$. A corollary of this part of the argument is $st(-f) = -st(f)$.

Other properties which establish that st is the needed homomorphism are:

(ST3a) $st(f + g) = st(f) + st(g)$.

(ST3b) $st(f) = 0$ if and only if f belongs to I.

(ST4) If f, g belong to B then $st(fg) = st(f)st(g)$.

Proof of ST4. $f - st(f)$ is in I and $g - st(g)$ is in I. Since g is in B and $st(f)$ is in B, $(f - st(f))g$ belongs to I and $st(f)(g - st(g))$ belongs to I by I2. Now by I1 $fg - st(f)st(g) = (f - st(f))g + st(f)(g - st(g))$ is in I.

(ST5) If $f \notin I$, then $st(1/f) = 1/st(f)$.

(ST6) If $f \leq g$ then $st(f) \leq st(g)$.[1]

(ST7) A function $f : A \to R$ $(A \subseteq R)$ is bounded if and only if $*f(x)$ is in B for all x in $*A$.

Proof. First suppose f is bounded. Then there is an $r > 0$ in R such that $|f(x)| < r$ for all $x \in A$. If $x \in *A$ then $\{i \mid x(i)$ is in $A\} \in U$ and for such i, $|f(x(i))| < r$ which implies by the definition of $<$

[1] Notice that in this context $f < g$ does not necessarily imply $st(f) < st(g)$. For example let $f(n) = 1 + 1/(n + 1)$ and $g(n) = 1 + 1/n$. Then $st(f) = 1 = st(g)$, but $f < g$.

that $|*f(x)| < r$. Hence $*f(x)$ belongs to B. On the other hand suppose f is not bounded on A. Then for every n in N there is an x_n in A such that $|f(x_n)| > n$. Let $x(n) = x_n$ define a member of $*A$ (for some choice of the x_n). Then $*f(x)$ is not in B since given any real number $r > 0$, there is an n_0 such that $r < n_0$ and clearly $|*f(x)| > n_0 > r$.

4. The Nonstandard Definition of Limit and Its Equivalence to the Standard Definition

At this point a nonstandard definition of limit can be given.

Definition. The limit of $f(x)$ as x approaches a exists with

$$\lim_{x \to a} f(x) = st(*f(a + h))$$

for $0 \neq h \in I$ if and only if $st(*f(a + h))$ exists and is constant for all $0 \neq h \in I$.

The following theorem shows that this definition is equivalent to the standard ε, δ definition of limit.

Theorem L. The value $st(*f(a + h))$ exists and is a constant $L \in R$ for all nonzero $h \in I$ (where a is in R) if and only if for every real number $\varepsilon > 0$ there is a real number $\delta > 0$ such that for any real number t satisfying $0 < |t| < \delta$, $|f(a + t) - L| < \varepsilon$.

Proof. Suppose $st(*f(a + h)) = L$ for all $0 \neq h \in I$. Let $\varepsilon > 0$ be a given real number. Suppose contrary to the conclusion of this theorem that for each $0 < \delta$ in R there is a t in R such that $0 < |t_\delta| < \delta$ and $|f(a + t_\delta) - L| \geq \varepsilon$. For each $n \in N$ choose an $h(n)$ such that $0 < |h(n)| < 1/n$ $(=\delta)$ and $|f(a + h(n)) - L| \geq \varepsilon$. Then clearly h is in I and $|*f(a + h) - L| \geq \varepsilon$ so that $st(*f(a + h)) \neq L$ or $st(*f(a + h))$ does not exist. In either case the hypothesis is contradicted; hence, there is some $\delta_\varepsilon > 0$ such that if $0 < |t| < \delta_\varepsilon$ then $|f(a + t) - L| < \varepsilon$. On the other hand suppose that for every real $\varepsilon > 0$ there is a real $\delta > 0$ such that if $0 < |t| < \delta$ then $|f(a + t) - L| < \varepsilon$. Let h be a nonzero infinitesimal. Then $0 < |h| < \delta_\varepsilon$ for all $\varepsilon > 0$. Hence $|*f(a + h) - L| < \varepsilon$ for all real $\varepsilon > 0$. Thus, $*f(a + h) - L \in I$. By the definition of st, $st(*f(a + h)) = L$.

5. Applications

With this equivalence of the definitions of limit there are many elementary limit theorems which can be proved without epsilons and deltas. For example, by using ST3 and ST4 one can prove that if $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$ then $\lim_{x\to a} (f+g)(x) = L_1 + L_2$ and $\lim_{x\to a} (fg)(x) = L_1L_2$. Continuity can also be defined using nonstandard techniques.

Definition. The function $f : A \to R$ is continuous at a in A if and only if $st(*f(x)) = f(a)$ for all x in $*A$ with $st(x) = a$.

Other definitions associated with continuity are:

Open set: A is open if and only if for all x 6 8, $x \in *A$ if $st(x)$ 6 A.
Closed set: A is closed if and only if for all x 6 $*A \cap \mathcal{B}$, $st(x)$ 6 A.

Suppose $A \subseteq R$ is closed. Then $R - A$ is open. For a proof, suppose x is in $\mathcal{B}$ and $st(x)$ is in $R - A$. Suppose x is in $*A$. Then since A is closed $st(x)$ is also in A which is a contradiction. The following theorem is an application of these results.

Theorem. If $f : A \to R$ is continuous and $\emptyset \neq A \subseteq R$ is closed and bounded then $f(A)$ is also closed and bounded.

Proof. First $f(A)$ is bounded: Let x be in $*A$. Since there is a real number u such that $|a| \le u$ for all a in A $\{i \mid |x(i)| \le u\} \supseteq \{i \mid x(i)$ 6 $A\}$ belongs to $\mathcal{U}$; hence, $|x| \le u$. Thus x is in $\mathcal{B}$, and $st(x)$ exists. Since A is closed $st(x)$ is in A. Since f is continuous at $st(x)$, $st(*f(x)) = f(st(x))$ which is in R. Thus $*f(x)$ is in $\mathcal{B}$. By ST7 f is bounded on A.

To show that $f(A)$ is closed let y 6 $\mathcal{B} \cap *f(A)$. Thus, for some x 6 $*A$, $y = *f(x)$. On the set $\{i \mid y(i)$ 6 $f(A)\}$ let $x(i)$ be a preimage in A of $y(i)$; otherwise let $x(i) = 0$. Since A is bounded, $x \in \mathcal{B}$. Thus, since A is closed $st(x)$ 6 A. Since f is continuous $st(y) = st(*f(x)) = f(st(x))$ which is in $f(A)$. Thus $f(A)$ is closed.

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COMMENT BY EDITOR

The two articles immediately following were submitted at approximately the same time, both of them dealing with the problem of finding a closed formula for the series

$$\sum_{i=1}^{n} i^k = 1^k + 2^k + 3^k + \ldots + n^k,$$

but using completely different lines of attack. The result in the second paper actually provides a simple induction proof of the fact that there is a unique polynomial $P_k(x)$ for each integer $k \ge 0$ of degree $k + 1$ such that when x is a positive integer

$$P_k(x) = 1^k + 2^k + 3^k + \ldots + x^k.$$

This fact was used by L. S. Levy, "Summation of the Series $1^n + 2^n + \ldots + x^n$ Using Elementary Calculus" [*American Mathematical* Monthly, Vol. 77, No. 8 (1970), 840-847] to prove the interesting integral formula

$$P_n(x) = n\int_0^x P_{n-1}(t)\,dt + C_n x. \quad (C_n \text{ constant})$$

USING L'HOSPITAL'S RULE TO SUM A SERIES

by Norman Schaumberger
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The formulas

$$1 + 2 + 3 + \ldots + n = \cancel{\frac{n(n+1)}{2}} \tag{1}$$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{2}$$

are used in algebra and probability. They are particularly important in the history of the calculus where they stimulated interest in limits of sums. Archimedes used (2) as the basis of one of his quadratures of the parabola [1, pp. 50-51]. These along with the formula for the sum of cubes and higher powers are usually proved by mathematical induction or by somewhat obscure algebraic tricks. In this note we offer a technique for deriving these formulas which should be of interest to calculus students because it rests on differentiation of the exponential function and L'Hospital's Rule.

Let n be a positive integer. Since the left side of the following is a geometric series we see that

$$e^x + e^{2x} + \ldots + e^{nx} = \frac{e^x - e^{(n+1)x}}{1 - e^x}$$

Differentiating, with respect to x, we obtain, after some simplification,

$$(3)\quad e^x + 2e^{2x} + 3e^{3x} + \ldots + ne^{nx} = \frac{e^x - (n+1)e^{(n+1)x} + ne^{(n+2)x}}{(1 - e^x)^2}$$

Taking the limit of (3) as $x \to 0$ and using L'Hospital's Rule twice in order to evaluate the right side, we get (1). Now differentiating (3), we have

$$e^x + 2^2e^{2x} + 3^2e^{3x} + \cdots + n^2e^{nx} =$$

$$\frac{e^x + e^{2x} - (n+1)^2e^{(n+1)x} + (2n^2+2n-1)e^{(n+2)x} - n^2e^{(n+3)x}}{(1 - e^x)^3}$$

Letting $x \to 0$ and using L'Hospital's Rule three times on the right side gives (2).

Theoretically we could continue to differentiate and use L'Hospital's Rule to obtain the formulas for the sum of cubes and higher powers; unfortunately, however, the algebra becomes quite messy.

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A SIMPLE WAY OF EVALUATING $\sum_{i=1}^{k} i^n$

by R. S. Luthar
University of Wisconsin Center, Janesville

The following provides a very simple method for evaluating the series $\sum_{i=1}^{n} i^k$.

Let us write

$$S_k = 1 + 2^k + 3^k + \cdots + n^k$$

and

$$S_k = 1 + 2^k + \cdots + (n-1)^k + n^k$$

in the manner suggested above. Subtraction yields

$$0 = 1 + (2^k-1) + (3^k-2^k) + \cdots + (n^k-(n-1)^k) - n^k,$$

from which we obtain

$$n^k = 1 + (2^k-1) + (3^k-2^k) + \ldots + (n^k-(n-1)^k).$$

Thus, we can write

$$S_k = 1 + (1+(2^k-1)) + (1+(2^k-1)+(3^k-2^k)) + \cdots$$
$$+ (1+(2^k-1)+(3^k-2^k) + \cdots + (n^k-(n-1)^k))$$
$$= n + (n-1)(2^k-1) + (n-2)(3^k-2^k) + \cdots + (1)(n^k-(n-1)^k)$$
$$= \sum_{i=1}^{n} (n-(i-1))(i^k-(i-1)^k)$$
$$= \sum_{i=1}^{n} n(i^k-(i-1)^k) - \sum_{i=1}^{n} (i-1)(i^k-(i-1)^k)$$
$$= n\sum_{i=1}^{n} (i^k-(i-1)^k) - \sum_{i=0}^{n-1} i((i+1)^k-i^k)$$
$$= n(n^k) - \sum_{i=0}^{n-1} i[i^k + \tbinom{k}{1}i^{k-1} + \tbinom{k}{2}i^{k-2} + \cdots + 1 - i^k]$$
$$= n^{k+1} - \sum_{i=0}^{n-1} [\tbinom{k}{1}i^k + \tbinom{k}{2}i^{k-1} + \cdots + i]$$

$$= n^{k+1} - \sum_{i=1}^{n} [\binom{k}{1}i^k + \binom{k}{2}i^{k-1} + \cdots + i] + \binom{k}{1}n^k + \binom{k}{2}n^{k-1} + \cdots + n$$

$$= n[n^k + \binom{k}{1}n^{k-1} + \binom{k}{2}n^{k-2} + \cdots + 1] - \binom{k}{1}\sum_{i=1}^{n} i^k - \binom{k}{2}\sum_{i=1}^{n} i^{k-1} -$$

$$\cdots - \sum_{i=1}^{n} i;$$

$$S_k = n(n+1)^k - \binom{k}{1}S_k - \binom{k}{2}S_{k-1} - \binom{k}{3}S_{k-2} - \cdots - S_1$$

$$(1+k)S_k = n(n+1)^k - \binom{k}{2}S_{k-1} - \binom{k}{3}S_{k-2} - \cdots - S_1$$

Hence,

$$\boxed{S_k = \frac{1}{k+1}[n(n+1)^k - \binom{k}{2}S_{k-1} - \binom{k}{3}S_{k-2} - \cdots - S_1],}$$

which is valid for $k \geq 1$ (if $k = 1$ the terms $\binom{k}{i}S_{k-i+1}$ are omitted). Thus we have a formula connecting S_k with $S_{k-1}, S_{k-2}, \ldots, S_1$. For example,

$$S_1 = \frac{1}{1+1}[n(n+1)^1 - 0]$$

$$= \frac{n(n+1)}{2},$$

$$S_2 = \frac{1}{2+1}[n(n+1)^2 - \binom{2}{2}S_1]$$

$$= \frac{1}{3}[n(n+1)^2 - (1)\frac{n(n+1)}{2}]$$

$$= \frac{2n(n+1)^2 - n(n+1)}{6}$$

$$= \frac{n(n+1)[2(n+1) - 1]}{6}$$

$$= \frac{n(n+1)(2n+1)}{6},$$

and so on, ad infinitum.

NUMERICAL SOLUTION OF A NON-LINEAR ELECTRON CONDUCTION EQUATION WITH BOUNDARY VALUES

by James DeLucia
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Introduction

The equations which arise when discussing electron injection currents through a thin insulating film are in general non-linear. The equations are analytically intractable so that they were solved by numerical techniques on a digital computer. This paper is a presentation of the method used to solve the equations. These more exact solutions give results that have been experimentally verified but have not been explained by previous analyses[1]. Some of the previous solutions obtained under certain simplifying assumptions are also presented.

Conduction Equations

The structure that is analyzed is a thin insulating film sandwiched between two metal electrodes. The geometry is planar so that the analysis is one dimensional. The metal contacts are assumed to be made of the same material so that the structure is symmetric, and the boundary conditions at both metal-insulator faces are similar. Only steady state conditions are considered.

In the interior of the insulator, the current flow and charge distribution are governed by the current equation and Poisson's equation, as follows:

$$J = e\mu(nE + V_0\frac{dn}{dx}) \tag{1}$$

$$\frac{dE}{dx} = -\frac{e}{\varepsilon}(n - n_0) \tag{2}$$

where

$$E = -\frac{dV}{dx} \tag{3}$$

and

$$V_0 = kT/e \tag{4}$$

In the above relations J is the current density, E is the electric field intensity, V is the voltage, and n is the total free carrier density. (The quantities E, V and n vary over the displacement x.) Also, e is the electron charge, μ is the electron mobility, n_0 is the initial electron density, and ε is the insulator permittivity.

The current levels are assumed to be small enough so that the carrier concentrations at the metal-insulator surfaces are constant. Therefore, the boundary conditions are

$$n(0) = n(L) = \alpha n_0 \tag{5}$$

where L is the thickness of the insulator and a is a constant. Also, we can arbitrarily set

$$V(0) = 0. \tag{6}$$

Exact Results of Simplified Models

This structure was first analysed by Mott and Gurney [2] who assumed that the insulator was thick enough so that surface effects can be ignored. That is, the contribution to the current due to diffusion $(\frac{dn}{dx})$ was ignored and the electric field at the cathode vanishes. Also they considered an insulator whose insulating properties are good enough so that n always dominates n_0. The resulting equations are

$$J = e\mu nE \tag{7}$$

$$\frac{dE}{dx} = -\frac{en}{\varepsilon} \tag{8}$$

$$E = -\frac{dV}{dx} \tag{9}$$

with the boundary conditions

$$E(0) = 0 \tag{10}$$

$$V(0) = 0 \tag{6}$$

These equations have an exact solution of the form

$$J = \frac{9}{8}\mu\varepsilon V_a^2/L^3 \tag{11}$$

where $V_a = V(L)$ is the applied voltage.

Lampert [3] extends the above analysis to the case where n_0 cannot be ignored. Exact solutions can be obtained only in the limits of low applied voltage where n is dominated by n_0 and high applied voltage where n dominates n_0. The results are as follows.

For low voltages ($n << n_0$) we obtain ohmic conduction, or

$$J = e\mu n_0 V_a/L. \tag{12}$$

For high voltages ($n >> n_0$) this model reduces to the Mott-Gurney model, or

$$J = \frac{9}{8}\mu\varepsilon V_a^2/L^3. \tag{13}$$

Since the ohmic term is always present the current-voltage relationship for the Lampert model is essentially a quadratic, therefore

$$J = e\mu n_0 V_a/L + \frac{9}{8}\mu\varepsilon V_a^2/L^3. \tag{14}$$

However, for an insulator which is thin enough so that the surface effects cannot be ignored, equations (1), (2), (3) along with boundary conditions (5) and (6) must be solved in those forms. The remainder of this paper is a discussion of the methods used to solve these equations.

Normalization of the Equations

Equations (1), (2), (3), (5) and (6) can be made dimensionless and simpler by the proper choice of measurement units. Carrier concentration will be measured in units of the initial concentration of the insulator, and length, voltage, electric field and current will be measured in the following units:

$$L_D = (\varepsilon kT/e^2 n_0)^{\frac{1}{2}} \tag{15}$$

$$V_0 = kT/e \tag{4}$$

$$E_0 = V_0/L_D \tag{16}$$

$$J_0 = e\mu n_0 E_0 \tag{17}$$

The resulting normalized quantities are as follows:

$$g = n/n_0 \tag{18}$$

$$y = x/L_D \tag{19}$$

$$w = L/L_D \tag{20}$$

$$U = V/V_0 \tag{21}$$

$$F = E/E_0 \tag{22}$$

$$Z = J/J_0 \tag{23}$$

With these substitutions equations (1), (2), (3), (5), and (6) become

$$Z = gF + dg/dy \tag{24}$$

$$\frac{dF}{dy} = -(g - 1) \tag{25}$$

$$F = -\frac{dU}{dy} \tag{26}$$

$$g(0) = g(w) = \alpha \tag{27}$$

$$U(0) = 0 \tag{28}$$

The resulting equations are non-linear and are solved by numerical techniques on a digital computer.

Numerical Solution

a) Difference Equations

The film, of width W, is divided into 101 points, or 100 cells. The points are labled i where i goes from 1 to 101. The width of each cell is H where

$$H = w/100 \tag{29}$$

We want to find values of g, F and U at each point i. Relable $g(y)$ $F(y)$ and $U(y)$ as g_i, F_i and U_i.

Equations (24), (25) and (26) are approximated to the first order by difference equations. They become

$$g_{i+1} = g_i + H(Z - g_iF_i) \tag{30}$$

$$F_{i+1} = F_i + H(1 - g_i) \tag{31}$$

$$U_{i+1} = U_i - HF_i \tag{32}$$

where the boundary conditions become

$$g_1 = g_{101} = \alpha \tag{33}$$

$$U_1 = 0 \tag{34}$$

b) Starting Values

In order to obtain starting values for F and g, so that equations (30) and (31) can be iterated, we recognize the fact that the charge gradient($\frac{dg}{dy}$) must go to zero somewhere inside the film. This is a consequence of the boundary condition (33). Consider a point j and set dg/dy at j equal to zero. From (24) we obtain

$$F_j = z/g_j. \tag{35}$$

We can guess a value for Z and guess the starting value g_j. Equation (35) then gives us the starting value F_j.

c) General Procedure

The general procedure for obtaining a solution goes like this:

1) Consider a point j.
2) Guess Z and g_j.
3) Obtain F_3 from (35).
4) Obtain F_i and g_i for $i = 1$ to 101 by iterating (30) and (31) and using the appropriate starting values. This gives us computed values for g_1 and g_{101}.
5) Compare the computed values of g_1 and g_{101} with the required boundary conditions(33).
6) If both computed values of $g1$ and g_{101} match the required boundary conditions then the solution has been found. If not then go back to step 2 and guess again.

Once the solution has been found equations (32) and (34) are used to obtain the U_i, the voltage along the film. U_{102} is the applied voltage for the resulting current Z. The above procedure is repeated for points j = 10 through 51, thus giving us 42 values of Z versus U_{102}. The resulting Z versus U_{102} curve is the current-voltage characteristic of the device.

d) Block Diagram

The above section was just to give an idea of how equations (30) through (34) are used to obtain values for F_i, g. and U_i. A larger problem is trying to obtain the correct values of Z and g_3 that give a solution. What was done was to guess two values for Z (Z1 and Z2) and two values for g_j (g_{j1} and g_{j2}). These values were then systematically augmented in an attemptto obtain a straddle around the true values. Once a straddle is obtained it is a simple matter to interpolate to the real value.

A block diagram of the computer program used to solve the equation is presented in Figures 1 and 2. The program consists of a main program and a subroutine named *ZINTR*. Following is an explanation of the vari-

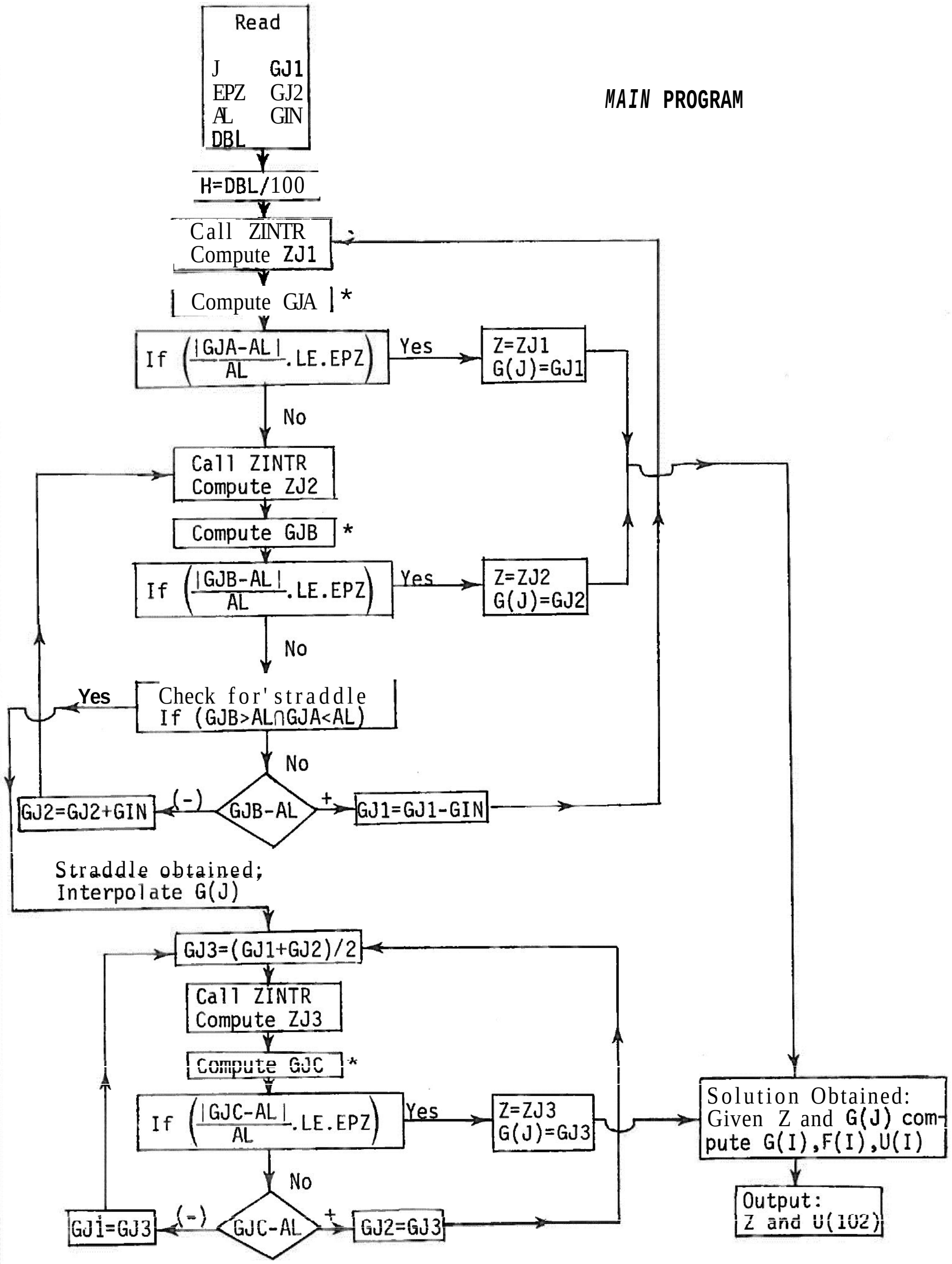

FIGURE I

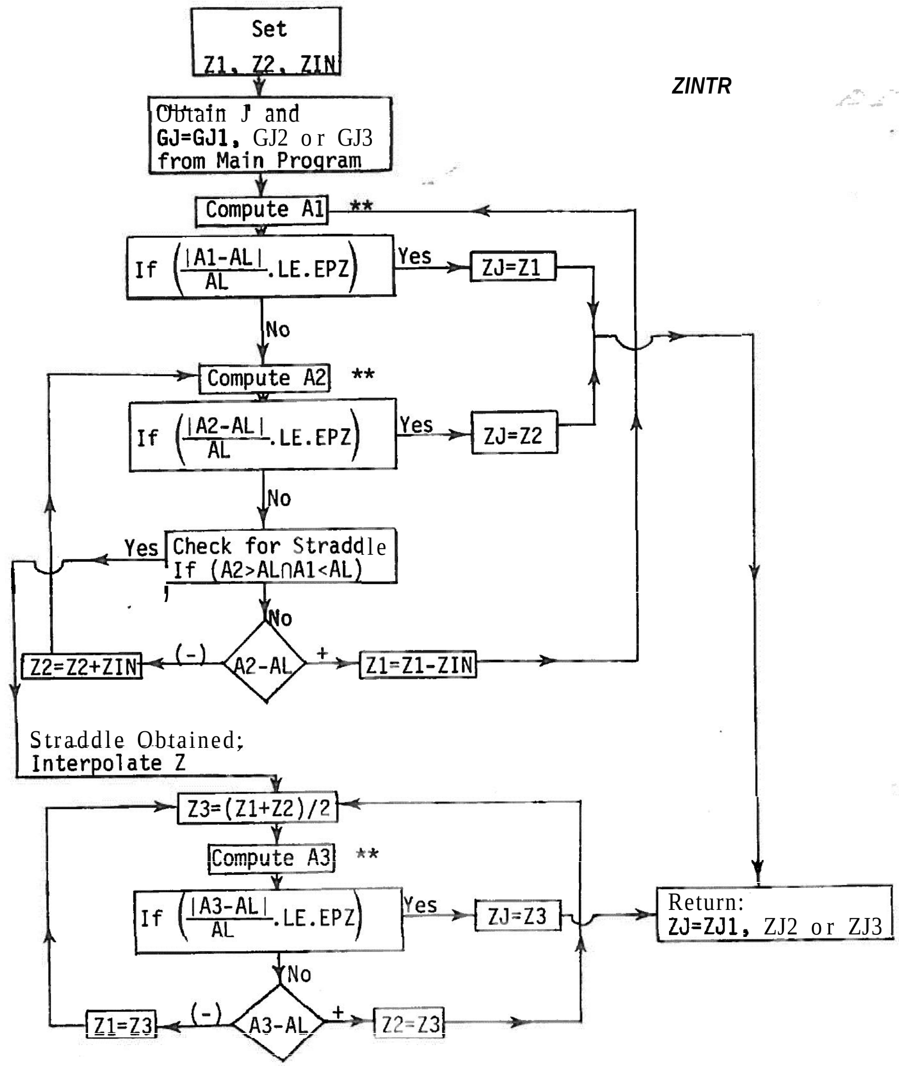

FIGURE 2

ables used.

Main Program (Finds the true value of g_d given the true value of Z from the subroutine)

J = point in consideration

EPZ = tolerance at the boundaries

DBL = w(width of the film)

AL = (boundary value)

$G(I)$ = charge density along the film   I = 1 to 101

$F(I)$ = electric field along the film   I = 1 to 101

$U(I)$ = voltage along the film   I = 1 to 102

$GJ1$ and GJ2 = two guesses for $G(J)$ ($GJ1 < GJ2$)

GJ3 = interpolated value of $G(J)$

GIN = augmenting value for $GJ1$ and $GJ2$

$ZJ1$ = value of Z that, along with $G(J) = GJ1$, fits the right boundary

ZJ2 = value of Z that, along with $G(J) = GJ2$, fits the right boundary

ZJ3 = value of Z that, along with $G(J) = GJ3$, fits the right boundary

GJA = computed value of $G(1)$ for $Z = ZJ1$ and $G(J) = GJ1$

GJB = computed value of $G(1)$ for Z = ZJ2 and $G(J)$ = GJ2

GJC = computed value of $G(1)$ for Z = ZJ3 and $G(J)$ = GJ3

The main program assumes that $GJ1 < GJ2 \Rightarrow GJA < GJB$.

*(Note from main program): This value is obtained by iterating equations (30) and (31) to the left hand boundary.

Subroutine ZINTR (Finds a value of Z (called ZJ) which along with some value of $G(J)$ (called GJ) fits the right hand boundary condition)

ZJ = the computed value of Z that is returned to the main program. It will be either $ZJ1$, ZJ2 or $ZJ3$ corresponding to $GJ1$, GJ2 or GJ3 respectively.

Z1 and Z2 = two guesses for ZJ ($Z1 < Z2$)

Z3 = interpolated value of ZJ

A1 = computed value of $G(101)$ for Z = Z1 and $G(J)$ = GJ

A2 = computed value of $G(101)$ for Z = Z2 and $G(J)$ = GJ

A3 = computed value of $G(101)$ for Z = Z3 and $G(J)$ = GJ

ZIN = augmenting value for Z1 and Z2

ZINTR assumes that $Z1 < Z2 \Rightarrow A1 < A2$

**(Note from ZINTR): This value is obtained by iterating (30) and (31) to the right hand boundary.

Results

Current-voltage curves (Z vs. U_{102}) were found for a = 10^8 and a = 10^6 for various values of w. The curves were each fit to a quadratic and to a cubic using the least squares criteria. Each curve followed a quadratic of the form

$$Z = a(\alpha, w)U + b(\alpha, w)U^2 \quad (36)$$

The values of a and b are presented in Tables 1 and 2. Figure 3 is the current-voltage curve for w = 0.2.

w	a	b
0.02	5.60×10^6	1.71×10^5
0.05	3.92×10^5	1.16×10^4
0.08	9.94×10^4	2.90×10^3
0.10	5.07×10^4	1.50×10^3
0.20	6.79×10^3	1.92×10^2
0.50	4.59×10^2	1.26×10^1
0.80	1.16×10^2	3.10
1.00	5.78×10^1	1.60

TABLE 1

w	a	b
0.20	5.90×10^3	163.5
0.50	3.90×10^2	11.7
0.80	9.85×10^1	2.9
1.00	5.13×10^1	1.5

TABLE 2

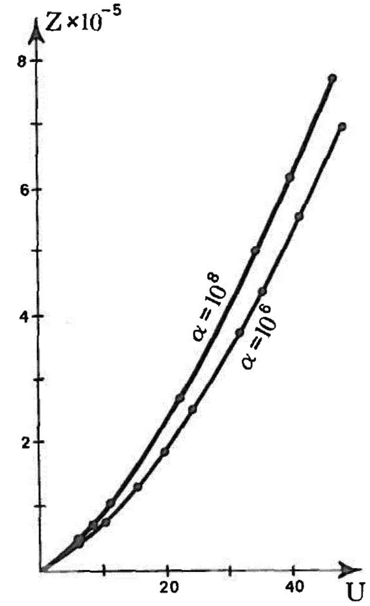

FIGURE 3

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1974-1975 MANUSCRIPT CONTEST WINNERS

The judging for the best expository papers submitted for the 1974-75 school year has now been completed. The winners are:

FIRST PRIZE ($200): Daniel Minoli and Robert Bear, Polytechnic Institute of New York, for their paper "Hyper-perfect Numbers" (this *Journal*, Vol. 6, No. 3, pp. 153-157).

SECOND PRIZE ($100): Mary Zimmerman, Western Michigan State University, for her paper "Matrix Multiplication as an Application of the Principle of Combinatorial Analysis" (this *Journal*, Vol. 6, No. 3, pp. 166-175).

THIRD PRIZE ($50): Lonnie J. Kuss, Texas Tech University, for his paper "A Conformal Group on an n-Dimensional Euclidean Space" (this *Journal*, Vol. 6, No. 3, pp. 144-152).

1976-1977 CONTEST

Papers for the 1975-76 contest are now being judged, and we are receiving papers for this year's contest, so be sure to send us your paper, or your chapter's papers (at least 5 entries must be received from the same chapter in order to qualify, with a $20 prize for the best paper in each chapter). For all manuscript contests; in order for authors to be eligible, *they* must *not have received a Master's degree* **at** *the time they submit their peeper.*

WE NEED YOUR HELP

Two of last year's contest winners were never reached by our office because of a change of address. We have the prize money, but we do not know where to send it. If anyone knows the whereabouts of Charles D. Keys and S. Brent Morris, please let us know. Our plea to all authors is to always keep us posted on any change of address.

GLEANINGS FROM CHAPTER REPORTS

ARKANSAS BETA at Hendrix College participated in the Oklahoma-Arkansas regional meeting of the Mathematical Association in March, 1976. Student members of the chapter who presented talks in the undergraduate session were *David Bonner, Janet Dillahunty, William Orton, Alma Posey,* and *Michael Tiefenback.*

CALIFORNIA ALPHA at the University of California at Los Angeles sponsored several colloquia during the year given by members of the faculty. Among others, a talk was presented by *John Garnett* on the topic "How to Tell One Cantor Set From Another."

CALIFORNIA ETA at the University of Santa Clara heard *Nicholas Knueppel* from the University of California at Davis speak on the topic 'Branching Processes" and *George Polya* from Stanford University on 'Intuitive Outline of the Solution of a Basic Combinatorial Problem." The initiation banquet guest was *John Wetzel* from the University of Illinois, who lectured on the topic "Spheres Tangent to a Tetrahedron."

COLORADO BETA at the University of Denver heard presentations by its members on the topics "Infinity of Rationals and Irrationals", "Structural Programming", and "Pascal's Triangles", and members of the faculty on such topics as "The World is Linear" and "Outlines for a Solar Energy Course."

FLORIDA EPSILON at the University of South Florida had both student members and faculty present talks during the year. The student talks were given by *David Kerr, Bruno Castellano, Robert Tubbs, Joseph Shepherd,* and *Robert Jernigan* (whose topic was "How to Win at Monopoly Using Math"). The chapter celebrated its tenth anniversary and prepared a special report on the history of the organization, on outstanding contributions of its members, and including a partial directory giving the present status of the more than 350 mathematics students granted membership during the 10 years.

GEORGIA GAMMA at Armstrong State College heard *Swarna Krishnamurti,* senior mathematics major, speak on the topic "Fourth Dimensional Fantasies", and Ben Zipperer, a sophomore, on "Fibonacci Facts."

ILLINOIS ZETA at Southern Illinois University and the Mathematics Department conducted the *Mathematics Field Day* in which 500 high school students competed for scholarships and awards in a mathematics examination. The chapter also maintained and serviced a problem board during the year.

KENTUCKY GAMMA at Murray State University heard talks by Niles E. Woods on "Inventory Control", Kathy Zettler on "The Ring of Continuous Functions on the Unit Interval", and Steve Beatty on "Mathematical Modeling of Economics."

LOUISIANA EPSILON at McNeese State College heard one of its student members, Sandra Airhart, speak on the topics "Reason Behind Russian Peasant Multiplication" and "Mathematical Reasons for the Three Shapes of Regular Tiles."

MASSACHUSETTS GAMMA at Bridgewater State College held an installation ceremony for new members at which Professor Ignatius P. Scalisi lectured on the topic "Pell's Equation--Or Is It?" concerning the solution of the equation and its ambiguous origin.

MICHIGAN ALPHA at Michigan State University held its *Annual Initiation Banquet* at the University Club and heard Professor Fritz Herzog speak on the topic "Some Examples of Unsound Deduction."

MICHIGAN DELTA at Hope College sponsored numerous student presentations during the year on a wide variety of topics. Student speakers included Roger Maitland, Laura Camp, Sherwood Quiring, Nancy Ponstein, Mahmood Masghati, James VanderMeer, Ray Lokers, Tom Westervelt, and Gary Nieuwsma.

MINNESOTA ALPHA at Carleton College conducted an active colloquium program and heard several distinguished mathematicians. Among the lectures presented were "Some Surprises in Combinatorics" by Professor David P. Roselle from Virginia Polytechnic Institute, "Equivalence Relations and Their Relationship to Groups" by Gloria Hewett from the University of Montana (a former Councillor of Pi Mu Epsilon), and "Probability Theory, or You Can't Win" by Professor Paul Halmos from the University of Indiana.

MISSISSIPPI BETA at Mississippi College heard Professor Josiah Macy from the University of Alabama at Birmingham and talks by student members, Russell Blooms on "Fibonacci Numbers", and Karen Lovell on "Applications of Mathematics in Psychology."

MISSOURI DELTA at Westminster College held its Fall Initiation ceremony during which a lecture on "Why You Can't Tell Whether a Turing Machine Ever Stops" was presented by Professor Paul Blackwell from the University of Missouri.

NEBRASKA BETA at Creighton University sponsored the *Mathematics Field Day* on February 7 in which 650 high school students participated, the competition including Marathon speed tests, a Leapfrog test for two-member teams, and the Chalk Talk which included a test on the topic of "Continuity of Functions of One Variable."

NEW JERSEY GAMMA at Rutgers University heard Professor Daniel Solomon speak on the topic "Some Elementary Results in the Theory of Topological Groups" preceding the initiation ceremony in May.

NEW JERSEY DELTA at Seton Hall University participated in the 30th annual *Eastern Colleges Science Conference* held at Rhode Island University. Student members presenting papers were Wai Man Lee., "On the Diophantine Equation $x^2 + y^2 + z^2 + c = xyz$", and Daniel Gross, Bard Rosell, and David Sabella (jointly), "Generating Functions and Partition Identities" and "kth Power Free Multiplicative Functions."

NEW JERSEY EPSILON at St. Peter's College heard Dr. David Jagerman of Bell Laboratories speak on "What Business Expects of Mathematics Majors."

NEW YORK PI at State University of New York, Fredonia, held a *Career Day* at which 3 alumni discussed their jobs and job-placement. The chapter helped to obtain a mathematics library and study area for students, to which donations of books were made by faculty members and book publishers.

NEW YORK PHI at State University of New York, Potsdam, heard members Ken Plantz and Paul Hafer speak on the results of their mathematics seminar.

NORTH CAROLINA GAMMA at North Carolina State University heard Professor J. W. Bishir speak on the topic "Gamblers, Duels, ITT, and

Mendel--Modern Applications of Probability Theory."

OHIO EPSILON at Kent State University heard Professor Kenneth Cummins speak on "'Calculus' before the Calculus--Some Moments of Ingenuity."

OKLAHOMA BETA at Oklahoma State University toured the Conoco computer center in Ponca City and met and talked to several mathematicians in industry.

PENNSYLVANIA THETA at Drexel University heard Professor Francine F. Abeles from Kean College of New Jersey present a lecture on "Lewis Carroll, Mathematician" following the initiation ceremony.

PENNSYLVANIA NU at Edinboro State College listened to lectures on "Statistical Fallacies" and "Choosing the Best" given by Professor Morris H. DeGroot from Carnegie-Mellon University.

PENNSYLVANIA XI at St. Joseph's College sponsored a series of student lectures, each a result of the student's independent study in an area of mathematics. Those lectures included "The Prisoner's Dilemma" by Terence James, "The Lattice of Finite Topologies" by Steven Kilroy, and "A Proof of Ulam's Conjecture for Unicyclic Graphs" by Edward Sweeney.

TENNESSEE BETA at the University of Tennessee at Chattanooga heard Tom McIntosh, a systems engineer at IBM, speak on the topic "Cocoanuts and Coins, or Mathematics and Computers" at the spring initiation meeting.

TEXAS ALPHA at Texas Christian University listened to both student and faculty lectures during the year, some of the meetings held in conjunction with the Parabola Club, under the leadership of Denise Heap, President, and Professor Ray Combrink, Faculty Advisor.

TEXAS DELTA at Stephen F. Austin State University heard Professor J. Dalton Tarwater from Texas Tech University lecture on the topic "American Mathematics: A Bicentennial View" at the initiation banquet.

TEXAS EPSILON at Sam Houston State University heard Mark Spearman as he demonstrated the Tektronic calculator.

TEXAS ETA at Texas A.& M. University helped conduct a mathematics contest for undergraduates in April and heard Professor B. Frank Jones lecture on "The Heat Equation" at the initiation meeting.

TEXAS IOTA at the University of Texas at Arlington heard lectures by A. Richard Mitchell on "Infinity and Beyond" and Hal Willis on "An Elementary Proof that Pi is Irrational."

TEXAS LAMBDA at the University of Texas heard student presentations by Bob Toellner on "An Algorithm for Producing Nth Powers of Integers From Partial Sums" and Kenn Askins on "A Proof of the Uniqueness of the Non-trivial Magic Hexagon."

VIRGINIA BETA at the University of Maryland listened to Brenda Cox, Vice-president of the chapter, speak on the topic "A Statistical Study of Ozone in the Stratosphere."

VIRGINIA GAMMA at Madison College heard Professor Janet Milton from Radford College (Virginia) lecture on "A Random Model for Communicable Diseases", Professor Lawrence Kurtz from Hollins College on a topic in applied mathematics, Professor Thomas Kriete from the University of Virginia on "Animal Populations and Differential Equations", and Professor Jane Sawyer from Mary Baldwin College on "Math Art."

VIRGINIA DELTA at Roanoke College heard Professor Mary Ellen Rudin from the University of Wisconsin speak on the topic "Is Set Theory Necessary?"

WEST VIRGINIA ALPHA at West Virginia University acted as the co-host for student activities at the Allegheny Moutain Section of the Mathematical Association meeting in April. The following students presented papers at this meeting: John Svedman (West Virginia University), Greg Stump (Indiana University of Pennsylvania), G. E. M. Pope (West Virginia University), Suzy Stewart (Allegheny College), Charles Wills, III (Duquesne University), Alfred Kabana (Duquesne University), Kate Boaz (Allegheny College), George Bradley (Allegheny College), and Dorothy Divers (Allegheny College).

COMMENT ON "SUMMATION OF SPECIAL CLASSES OF SERIES"

Professor Joseph M. Moser of San Diego State University has pointed out a generalization of the most general series considered by Gerard Protomastro in the article "Summation of Special Classes of Series", this *Journal*, Vol. 6, No. 4 (1976), 207-210.

Consider the series

$$S = \frac{c+d}{a(a+b)(a+2b)\dots(a+kb)} + \frac{c+kd}{(a+b)(a+2b)\dots(a+[k+1]b)} + \cdots$$

$$+ \frac{c+[(n-1)k-(n-2]d}{(a+[n-1]b)\dots(a+[k+n-1]b)} + \cdots$$

This series can be evaluated by writing

$$\left\{\frac{bc+ad+bd}{a(a+b)\dots(a+[k-1]b)} - \frac{bc+ad+(k+1)bd}{(a+b)\dots(a+kb)}\right\}$$

$$+ \left\{\frac{bc+ad+(k+1)bd}{(a+b)+\dots+(a+kb)} - \frac{bc+ad+[2k+1]bd}{(a+2b)\dots(a+[k+1]b)}\right\}$$

$$+ \cdots + \left\{\frac{bc+ad+[(n-1)k+1]bd}{(a+[n-1]b)\dots(a+[k+n-2]b} - \frac{bc+ad+(nk+1)bd}{(a+nb)\dots(a+[k+n-1]b)}\right\} + \cdots$$

with sum

$$\frac{bc+ad+bd}{a(a+b)\dots(a+[k-1]b)}.$$

Therefore

$$\frac{kb^2(c+d)}{a(a+b)\dots(a+kb)} + \cdots + \frac{kb^2\{c+[(n-1)k-(n-2)]d\}}{(a+[n-1]b)\dots(a+[k+n-1]b)} + \cdots$$

$$= \frac{bc+ad+bd}{a(a+b)\dots(a+[k-1]b)}$$

and

$$S = \frac{bc+ad+bd}{kb^2\{a(a+b)\dots(a+[k-1]b)\}}.$$

PUZZLE SECTION

Mathacrostic No. 2

submitted by R. Robinson Rowe
Sacramento, California

Identify the 30 key **words,** matching their letters in order with the opposite sequence of numbers; insert each letter of the key words in the square of the **Mathacrostic** with the same number (next two pages). Words end at the blank squares, and some words extend on to the next line.

When completed, the *Mathacrostic* will be a 217-word quotation, and the 30 initial letters of the key words will spell out the name of an author and title of his book, which is the source of the quotation. It is a commentary on a work of one of the five mathematicians named in the key words. The 30 letters ***A-Z, a-d*** correlate the squares with the key words. Thus, the *Mathacrostic* is also an anagram.

Puzzle: Missionaries and Cannibals

There is a strange story of five missionaries and five cannibals, who had to cross a river in a 3-man boat. Being acquainted with the peculiar appetites of the cannibals, the missionaries could never allow their companions to be in a majority on either side of the river. Only one of the missionaries and one of the cannibals could row the boat. How did they manage to get across, and what is the least number of crossings the boat has to make?

Solutions

The Blue **Men and** *Green* **Men** [Fall, **1974, p.** 121

The captain deduced that all the natives were Green men, because the presence of a single Blue man means the claim about what the first native said is true and hence 'We are all green men" has to be true. which is impossible with Blue men present.

Solved by **JAMES R. AMLING,** *Northern Illinois University, DeKalb, Illinois;* **PATRICK J. BROWN, Indiana** *University, Bloomington,* **Indiana; VICTOR G. FESER, St.** *Louis University,* **St.** *Louis, Missouri;* **ANDREW J.**

B 1	A 2	C 3	D 4	O 5	R 6		S 7	U 8	V 9		W 10	X 11
Z 12	c 13		d 14	A 15	E 16	J 17	Q 18	U 19		Z 20	a 21	
B 22	D 23	F 24		L 25	M 26		Z 27	a 28	b 29	D 30	E 31	M 32
R 33	W 34	a 35	c 36		d 37	A 38	G 39	H 40	I 41	J 42		O 43
B 44	E 45		F 46	H 47	B 48		O 49	B 50	P 51	R 52	S 53	B 54
	E 55	P 56		Q 57	W 58	X 59	G 60	S 61	T 62	Z 63	c 64	
A 65	B 66	G 67	T 68	U 69	V 70	W 71	d 72		C 73	D 74		B 75
H 76		L 77	M 78	S 79	W 80	Z 81	S 82	c 83	K 84		N 85	O 86
U 87	Z 88		C 89	E 90	M 91	Y 92		A 93	B 94	G 95	M 96	B 97
D 98	B 99	A 100		K 101	N 102	F 103	B 104		H 105	P 106		R 107
V 108	W 109	X 110	b 111	A 112	C 113	F 114	H 115		c 116	I 117	C 118	
D 119	G 120	I 121		O 122	S 123		Z 124	K 125		Q 126	b 127	R 128
V 129	W 130		d 131	I 132	L 133	d 134	L 135	O 136	U 137		V 138	X 139
a 140	D 141	d 142		A 143	K 144	G 145	M 146	N 147		Z 148	M 149	M 150
N 151		d 152	N 153	S 154	T 155	U 156	X 157	I 158		a 159	c 160	J 161
	X 162	C 163	K 164	T 165	b 166	L 167	W 168	A 169		R 170	T 171	
E 172	F 173	L 174		X 175	E 176	M 177	P 178	R 179		b 180	P 181	T 182
Y 183	a 184	H 185	b 186	I 187	C 188		P 189	G 190	D 191	L 192	C 193	J 194
	W 195	Q 196	B 197	N 198	Q 199	Z 200	Q 201	J 202	b 203	X 204		Y 205
C 206	J 207		Z 208	U 209	Y 210	N 211	V 212	U 213	E 214	S 215	E 216	a 217

Definitions and Key

A.	Factor of 10004 and 1000000004	15 38 65 2 112 100 169 143 93
B.	Mathematician, 1596-1650	94 44 97 66 54 50 104 197 22 99 1 48 75
C.	Sign-change transformation	73 206 89 113 363 188 193 3 118
D.	Mathematics of fluxions	191 98 23 141 4 30 119 74
E.	Mathematician, 1820-1884	45 55 31 16 216 214 172 90 176
F.	CH_3CH_2	114 46 173 103 24
G.	Base of a congruence	39 67 95 145 60 190 120
H.	Mass-less particle	40 47 76 185 105 115
I.	....	41 121 117 158 132 187
J.	Whirled in a stream	42 194 207 17 161 202
K.	Mathematician, 1815-1864	101 125 144 84 164
L.	$Ax^2 + By^2 = C$	167 135 192 25 77 133 174
M.	Math-chemist	146 149 150 78 32 26 96 177 91
N.	Detested	211 102 153 85 198 151 147
O.	Powerful	49 122 5 86 136 43
P.	Exhausted	178 56 106 181 51 189
Q.	Mathematician, 1550-1617	18 57 126 199 201 196
R.	City on Lake Winnebago	128 33 6 179 170 107 52
S.	Bridge deck	82 215 154 7 61 123 79 53
T.	Railroad tunnel	68 182 155 171 165 62
U.	One of the lilies	87 137 209 8 213 19 69 156
V.	X, if X^{18} has 20 digits	70 138 9 212 129 108
W.	F in $F(2x) = 4F(x)[1-F(x)]$	34 195 109 130 71 168 10 58 80
X.	Burlesque for PhD's.	157 162 175 139 110 59 11 204
Y.	Rochester genius	183 205 92 210
Z.	Mathematics of numbers	208 81 20 88 148 200 12 124 63 27
a.	Artichoke's wild cousin	21 28 140 217 159 35 184
b.	3, 7, 11 or 21	29 111 166 186 180 203 127
c.	Mathematician, 1789-1857	64 13 83 116 160 36
d.	Like chapparal	37 142 131 134 14 152 72

PASQUALE, *Marshall University, Huntington, West Virginia;* and HELEN SWEENEY, St. *Louis, Missouri.*

The. *Prisoner and the. Urns* [Fall, 1975, p. 1651

This puzzle was solved by R. Robinson Rowe, Sacramento, California, who determined that the best 10 options for the prisoner (yielding the lowest probability that he would be executed) occur when he places ≤ 3 white balls in each of 2 urns and the remainder in the 3rd. Placing all 24 balls in one urn yields a probability of 1/8 = 0.125, while 1 white ball in each of 2 urns and the remaining 22 balls in the 3rd yields 35/594 = 0.058922559, 3 white balls in each of 2 urns and 18 in the 3rd yields 1/18 = 0.055555556, and 2 white balls in each of 2 urns and 20 in the 3rd yields 7/135 = 0.051851852 -- the best strategy.

Mathacrostic No. 1 [Spring, 19761

Definitions and Key:

A. Geodesic	F. Ditto	K. Twentieth	P. Two-twins	U. Newton	Z. Lune
B. Heron	G. Yield	L. Hedonist	Q. Infinity	V. Stint	a. Oftenest
C. Hundredths	H. Abscissa	M. Euclid	R. Cubed	W. Airy	b. Gauss
D. Algebra	I. Method	N. Moth	S. Instant	X. Phony	c. Yellow
E. Rhumb	J. Agnesi	0. Abel	T. Addend	Y. Off-set	

First letters: G H HARDY A MATHAMATICIANS APOLOGY

Quotation: *In these days of conflict between ancient and modem studies, there must surely be something to be said for a study which did not begin with Pythagoras and will. not end with Einstein but is the oldest and youngest of all.* (From G. H. Hardy, A *Mathematician's Apology.*)

Ten Mathematicians Mentioned: Pythagoras, Einstein, Hardy, Heron, Agnesi, Euclid, Abel, Newton, Airy, and Gauss.

Solved by LINDA BALLOU, *Akron,* Ohio; JEANETTE BICKLEY, *Webster Groves High* Schooi, *Missouri;* EZRA BROWN, *Virginia Polytechnic Institute, Blacksburg, Virginia;* LOUIS H. CAIROLI, *Kansas* State *University, Manhattan, Kansas;* BRADFORD E. CARTER, *Middle Tennessee State University, Murfreesboro, Tenessee;* ALIZA DUBIN, *Fat. Rockaway, New York;* ELEANOR S. ELDER, *New Orleans, Louisiana;* JOHN T. HURT, *Bryan, Texas;* MICHAEL IACUZIO, St. *Joseph's* Cottage., *Philadelphia, Pennsylvania;* JOSEPH KONHAUSER, *Macalester College,* St. *Paul, Minnesoto;* BARBARA LEHMANN, Saint *Peters College, Jersey City,* New *Jersey;* SIDNEY PENNER, *Bronx Community College of* CUNY, *Bronx, New York;* BOB PRIELIPP, *University* of, *Wisconsin, Oshkosh, Wisconsin;* RITA PRINCI, *Bronx, New York;* RICHARD D. STRATTON, *Colorado Springs, Colorado;* LÉO SAUVÉ, *Algonquin College,* Ottawa, Canada; THOMAS F. SWEENEY, St. *Louis University, Missouri;* and CHARLES W. TRIGG, *San Diego, California.*

Editor's Note.

Some puzzle solvers did not list the names of the ten mathematicians mentioned as part of the puzzle. Three solutions were received without the name and address of the solvers.

Several favorable comments were received regarding the new Puzzle Section and a fair amount of participation was evident, so this section will be continued for the time being.

MOVING??

BE SURE TO LET THE JOURNAL KNOW!

Send your name, old address with zip code and new address with *zip code* to:

Pi Mu Epsilon Journal
601 Elm Avenue, Room 423
The University of Oklahoma
Norman, Oklahoma 73019

PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity. Occasionally we shall publish problems that should challenge the ability of the advanced undergraduate or candidate for the Master's Degree. Old problems displaying novel and elegant methods of solution are also acceptable. Proposals should be accompanied by solutions if available and by any information that will assist the editor.

Solutions should be submitted on separate sheets containing the name and address of the solver and should be mailed before the end of May 1977.

Address all communications concerning problems to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

Problems for Solution

374. *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

In a triangle ABC inscribed in a circle (O), angle bisectors AT_1, BT_2, CT_3 are drawn and extended to the circle (see Fig. 1). Perpendiculars T_1H_1, T_2H_2, T_3H_3 are drawn to sides AC, BA, CB respectively. Prove that $T_1H_1 + T_2H_2 + T_3H_3$ does not exceed $3R$, where R is the radius of the circumcircle.

375. *Proposed by Richard S. Field, Santa Monica, California.*

Approximate the value of $2^{10,000}$ without using pencil and paper (or chalk and blackboard or similar equipment).

376. *Proposed by Solomon W. Golomb, University of Southern California, Los Angeles, California.*

Let the sequence $\{a_n\}$ be defined inductively by $a_1 = 1$ and $a_{n+1} = \sin(\arctan a_n)$ for $n \geq 1$. Let the sequence $\{b_n\}$ be defined inductively by $b_1 = 1$ and $b_{n+1} = \cos(\arctan b_n)$ for $n \geq 1$. Give explicit expressions

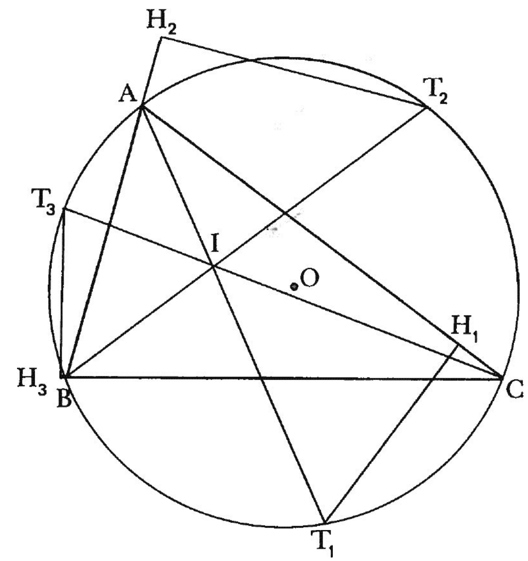

FIGURE 1

for a_n and b_n, and find $\lim a_n$ and $\lim b_n$ as n approaches ∞.

377. *Proposed by Charles W. Trigg, San Diego, California.*

From the following square array of the first 25 positive integers, choose five, no two from the same row or column, so that the maximum of the five elements is as small as possible. Justify your choice.

2	13	16	11	23
15	1	9	7	10
14	12	21	24	8
3	25	22	18	4
20	19	6	5	17

378. *Proposed by M. L. Glasser and M. S. Klamkin, University of Waterloo, Waterloo, Ontario, Canada.*

Show that

$$\left\{\frac{x^x}{(1+x)^{1+x}}\right\}^x > (1-x) + \left\{\frac{x}{1+x}\right\}^{1+x} > \frac{1}{(1+x)^{1+x}}$$

for $1 > x > 0$.

379. *Proposed by David L. Silverman, West Los Angeles, California.*

You play in a non-symmetric two-man subtractive game in which the players alternately remove counters from a single pile, the winner being the player who removes the last counter(s). At a stage when the pile contains k counters, if it is your opponent's move, he may remove 1, 2, ..., up to $[\sqrt{k}]$ counters, where $[x]$ is the largest integer $\leq x$. If it is your move, you may remove 1, 2, ..., up to $\phi(k)$ counters, where ϕ is the Euler totient function. If you play first on a pile of 1776 counters, can you assure yourself of a win against best play by your opponent?

380. *Proposed by V. F. Ivanoff, San Carlos, California.*

Form a square from a quadrangle $(ABCD)$ by bisecting segments and the angles.

381. *Proposed by Clayton W. Dodge, University of Maine, Orono, Maine.*

Solve the following wintery, slippery alphametrics (also known as cryptarithms and alphametics):

$$(ICE)^3 = ICYWHEEE.$$
$$(ICE)^3 = ICYOHOH.$$

382. *Proposed by R. Robinson Rowe, Naubinway, Michigan and Sacramento, California.*

Two cows, Lulu and Mumu, are tethered at opposite ends of a 120-foot rope threaded thru a knothole in a post of a straight fence separating two uniform pastures. How much area can they graze, presuming they eat, nap and ruminate on identical schedules and the rope length is also the extreme reach from muzzle to muzzle of Lulu and Mumu? As a sequel, if Mumu is replaced by the heifer Nunu with half the appetite, what is the area accessible to Lulu and Nunu?

383. *Proposed by Norman Schaumberger, Bronx Community College, New York.*

Find a pentagon such that the sum of the squares of its sides is equal to four times its area.

384. *Proposed by R. S. Luthar, University of Wisconsin, Janesville.*

Discuss the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{n}{p_n^2}$$

where p_n means the nth prime.

385. *Proposed by John T. Hurt, Bryan, Texas.*

Solve: $\sin\alpha = \tan(\alpha - \beta) + \cos\alpha \tan\beta.$

Comment by Editor

Problem 364 published in the Spring 1976 issue was an inadvertent duplication of Problem 325 proposed in the Spring 1974 and solved in the Spring 1975 issue.

Solutions

341. [Spring 1975; Spring 1976] *Proposed by Jack Garfunkel, Forest Hills High School, New York.*

Prove that the following construction trisects an angle of 60°. Triangle ABC is a 30°-60°-90° right triangle inscribed in a circle. Median CM is drawn to side AB and extended to M' on the circle. Using a marked straightedge, point N on AB is located such that CN extended to N' on the circle makes NN' equal to MM'. Then CN trisects the 60° angle ACM.

Comment by Charles W. Trigg, San Diego, California.

If angle $MCA = \theta$, with $45° < \theta < 90°$ then angle $MAC = \theta$, and angle $CMA = 180° - 2\theta$ (Fig. 2). Now $MN' = N'N = MC$, so triangle $N'MC$ is

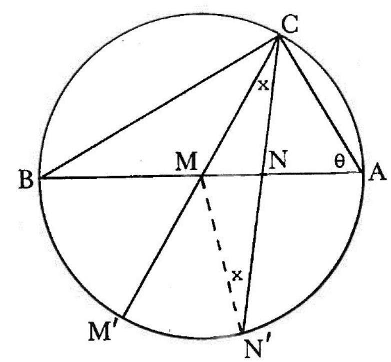

FIGURE 2

isosceles with base angles = x. Also, triangle $MN'N$ is isosceles with base angles = $(180° - x)/2$. Then angle CNA, which is exterior to triangle MCN, is equal to 180° - 26 + x. Equating the vertical angles:

$$180^\circ - 2\theta + x = (180^\circ - x)/2$$

so

$$3x = 46 - 180$$
$$x = 4\theta/3 - 60^\circ.$$

There are three cases where θ is an integral multiple of x, namely: $(x, 8) = (4^\circ, 48^\circ)$, $(20^\circ, 60^\circ)$, and (36, 72).

350. [Fall 1975] *Proposed by R. Robinson Rowe, Sacramento, California.*

In the game of ELDOS, an acronym for Each Loser Doubles Opponents' Stacks, each of n players starts with his bank (B) and at any point in the play holds his stack (S), which he bets on the next round. For each round there is just one loser; in paying the $n - 1$ winners, he doubles their stacks. Consider here a unique game when, after n rounds, each player has lost once and all players end with equal stacks.

(a) For $n = 5$, what was the minimum bank, B, for each player?

(b) How many players, n, were there if the least initial B was 11 cents?

(c) Find a general formula for B_m, the initial B of the mth player to lose, as a function of m and n.

(d) Using the formula or any other appropriate method, what was the initial bank B of the 9th of 13 players to lose?

Solution by Steve Leeland, University of, South Florida, Tampa, Florida.

The Solutions to (a), (b) and (d) can best be found after finding a general formula for B_m^n, (c), the mth player to lose out of n. If an extra player was included who started with a bank of 1 and never lost, he would have 2^n after the nth round. Hence, every player ends with 2^n, and the total amount in the game is always $n \cdot 2^n$. Furthermore, every player who has already lost has 2 at the end of the mth round at which time the mth loser must pay the $(m - 1)$ previous losers the amount they had at the end of round $(m - 1)$ and 2^{m-1}. He also must pay the winners one half of the amount they have at the end of the mth round, which is the total amount in the game, $n \cdot 2^n$, minus the amount of the losers at the end of the mth round, namely $m \cdot 2^m$. The total of m at the end of $(m - 1)$ rounds is the amount he paid at the end of the mth, plus the amount he had at the end of the mth, 2 . Since his stack doubled in each of the previous $m - 1$ rounds, his initial bank was:

$$B_m^n = \frac{(m - 1)2^{m-1} + (n \cdot 2^n - m\,2^m)/2 + 2^m}{2^{(m-1)}} = n \cdot 2^{(n-m)} + 1$$

From the foregoing general solution we obtain

(a) $B_1^5 = 81$, $B_2^5 = 41$, $B_3^5 = 2\text{-}1$, $B_4^5 = 11$, $B_5^5 = 6$.

(b) If the least initial B was 11 cents, $m = n$ and we have

$$11 = n \cdot 2^{(n-m)} + 1, \text{ whereupon } n = 10.$$

(d) $B_9^{13} = 13(2^{13-9}) + 1 = 209$

Also solved, by the Proposer, who offered algebraic, arithmetic and diophantine methods of solution for part (a). For example:

<u>*Arithmetic Method*</u>

Working backwards, anticipating repetitive division by 2, start with the assumption that end stacks were powers of 2, say 32, 32, 32, 32, 32, then the last loser had to pay 4-16 = 64 from his stack of 32 + 64 = 96. Then the 4th loser had to pay 72, so held 88 and so on, down to the initial banks as shown below.

32	32	32	32	32
16	16	16	16	96
8	8	8	88	48
4	4	84	44	24
2	82	42	22	12
81	41	21	11	6

If we had started with 64, all figures would have been doubled and 2 could have been factored out of the last line for a primitive.

351. [Fall 1975] *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, NY York.*

Angle A and angle B are acute angles of a triangle ABC. If angle $A = 30°$ and h_a, the altitude issuing from A, is equal to m_b, the median issuing from B, find angles B and C.

Solution by Zelda Katz, Beverly Hills, California.

Since the perpendicular from M upon HB = $h_a/2 = m_b/2$, it follows that angle MBC = 30° (Fig. 3). Consequently triangles MBC and ABC are similar. Let X = angle ABC = angle CUB. Then sin x = h /c = m_b/c = sin 30°/sin x. Hence $\sin^2 x = \sin 30° = 1/2$ and $x = 45°$. Since angle B = 45° it follows that angle C = 105°.

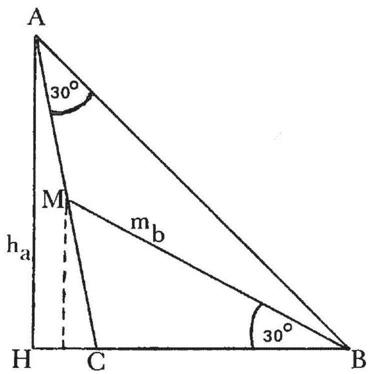

FIGURE 3

Also solved by CLAYTON W. DODGE, *University of, Maine at Orono;* JOHN TOM HURT, *Bryan, Texas;* CHARLES H. LINCOLN, *Raleigh, N. C.;* KYUNG WON PARK, *Flushing, New York;* R. ROBINSON ROWE, *Sacramento, California;* CHARLES W. TRIGG, *San Diego, California; and the Proposer.*

352. [Fall 1975] *Proposed by Charles W. Trigg, San Diego, California.*

The edges of a semi-regular polyhedron are equal. The faces consist of eight equilateral triangles and six regular octagons. In terms of the edge e, find the diameters of the following spheres: (a) the sphere touching the octagonal faces, (b) the circumsphere, and (c) the sphere touching the triangular faces. (See solution to Problem 198, on page 390 of this Journal, Vol. 4, No. 9)

Solution by the Proposer.

The polyhedron is a truncated hexahedron -- a cube that has had a trirectangular tetrahedron cut from each vertex. Hence, the edges of the tetrahedron issuing from the cube's vertex are $e/\sqrt{2}$.

a) The edge of the cube is $e(1 + 2/\sqrt{2})$ or $e(1 + \sqrt{2})$. This is the diameter of the sphere touching the octagonal faces.

b) The square of the distance from the center of the polyhedron to a vertex is $2[e(1 + \sqrt{2}/2]^2 + (e/2)^2$ or $(7+4\sqrt{2})e^2/4$. Hence the diameter of the circumsphere is $\sqrt{7+4\sqrt{2}}e \doteq 3.5576e$.

c) The volume of one of the tetrahedrons cut from the corners of the cube can be computed in two ways, so

$$(e/\sqrt{2})^3/6 = (e\sqrt{3}/2)(e/2)(h/3)$$

where h is the altitude upon the triangular face of the semiregular polyhedron. Hence, $h = e/\sqrt{6}$. Consequently the diameter of the sphere touching the triangular faces is the space diagonal of the cube less $2h$, or $e(1 + \sqrt{2})\sqrt{3} - 2e/\sqrt{6}$ or $e(\sqrt{3} + 2\sqrt{6}/3) \doteq 3.3650\,e$.

Also solved by R. ROBINSON ROWE, *Sacramento, California.*

353. [Fall 1975] *Proposed by Clayton W. Dodge, University of, Maine at Orono.*

It is easy to show that if a and b are complex numbers such that $a + b = 0$ and $|a| = |b|$, then $a^2 = b^2$. Prove that if a, b and c are complex numbers such that $a + b + c = 0$ and $|a| = |b| = |c|$, then $a^3 = b^3 = c^3$. Can this result be extended to more than three numbers?

Solution by the Proposer.

Let $|a| = |b| = |c| = k$. If $k = 0$, the result is obvious. So suppose $k \neq 0$. Since $a + b = -c$, then $|a + b| = |c|$ and

$$|a + b|^2 = a\bar{a} + a\bar{b} + \bar{a}b + \bar{b}b = c\bar{c} = k^2,$$

$$a\bar{b} + \bar{a}b + k^2 = 0,$$

$$a^2\bar{b}^2 + k^2a\bar{b} + k^4 = 0 \quad \text{(by multiplying by } a\bar{b}\text{)},$$

$$(a^2\bar{b}^2 + k^2a\bar{b} + k^4)(a\bar{b} - k^2) = 0,$$

$$a^3\bar{b}^3 - k^6 = 0,$$

$$a^3\bar{b}^3b^3 = k^6b^3,$$

$$a^3k^6 = b^3k^6,$$

and finally

$$a^3 = b^3.$$

By symmetry, $a^3 = b^3 = c^3$.

The result does not extend. Let $a = 1$, $b = -1$, and c and d be any two opposite points on the unit circle (except 1, -1, i, and $-i$). Then $a + b + c + d = 0.$ and $|a| = |b| = |c| = |d|$, but $a^4 = b^4 \neq c^4 = d^4$.

Also solved by VICTOR G. FESER, *Mary College, Bismarck, North Dakota;* JOHN TOM HUNT, *Bryan, Texas;* STEVE LEELAND, *University of South Florida, Tampa, Florida;* CHARLES H. LINCOLN, *Raleigh, North Carolina; and* AL WHITE, *St. Bonaventure University, New York.*

354. [Fall 1975] *Proposed by Arthur Bernhart and David C. Kay, University of Oklahoma, Norman, Oklahoma.*

In a triangle ABC with angles less than $2\pi/3$, the Fermat Point, defined as that point which minimizes the function $f(X) = AX + BX + CX$, may be determined as the point P of concurrence of lines AD, BE and CF, where BCD, ACE and ABF are equilateral triangles constructed externally on the sides of triangle ABC. If R, S and T are the points where PD, PE, and PF meet the sides of triangle ABC, prove that PD, PE and PF are twice the arithmetic means, and that PR, PS and *PT* are half the harmonic means of the pairs of distances (*PB*, PC), (PC, PA) and (PA, *PB*) respectively.

Solution by Charles W. Trigg, San Diego, California.

FA = *BA*, AC = AE, and angle PAC = 60° + angle BAC = angle BAE, so triangles FAC and *BAE* are congruent. Thus angle PCS = angle SAE, and since angle PSC and angle ASE are equal vertical angles, triangles PSC and ASE are similar. So angle SPC = angle SAE = 60°. Hence P lies on the circumcircle of triangle EAC and quadrilateral EAPC is concyclic. Therefore angle APE = angle ACE = 60° and triangles APE and SPC are similar.

By Ptolemy's Theorem:

$$PE \cdot AC = PA \cdot CE = PC\text{-}EA,$$

and since AC = CE = EA,

PE = PA + PC = twice the arithmetic mean of PA and PC.

In like manner, it can be shown that PD = 2[(*PC* + *PB*)/2] and *PF* = 2[(*PS* + *PA*)/2].

From the similar triangles APE and SPC,

$$PS/PA = PC/PE$$

so

$PS = PA \cdot PC/PE = PA \cdot PC/(PA + PC)$ = half the harmonic mean of PA and PC. In like manner, it can be shown that PR and *PT* are half the harmonic means of (*PC*, *PB*) and (PA, *PB*) respectively.

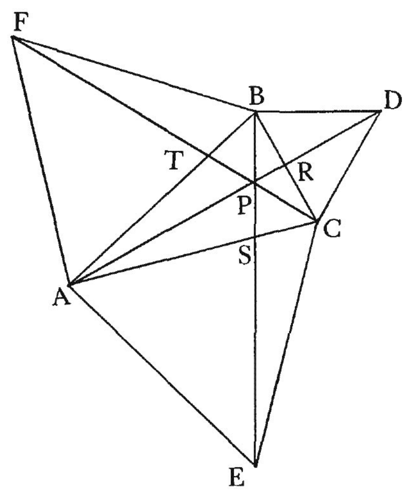

FIGURE 4

Also solved by CLAYTON W. DODGE, University of Maine., Orono, Maine;- R. ROBINSON ROWE, Sacramento, California; and the Proposers.

355. [Fall 19751 *Proposed by John M. Howell, Littlerock, California.*

On the TV game show called "Who's Who?", four panelists try to match the occupations of four contestants with signs marking their occupations.. If the first panelist matches correctly, the contestants get nothing and the game is over. If the second panelist succeeds in matching correctly, the contestants get $25. If the second panelist fails but the third succeeds, the contestants get $50. If the fourth panelist matches after the third fails, the contestants get $75. If there is no match, the contestants win $100. What is the expected value of the contestants' winnings?

Solution by Clayton W. Dodge., University of Maine, Orono, Maine and. similarly by Steve Leeland, University of Sooth Florida, Tampa, Florida.

Since there are 24 permutations of the four signs, the probability of the first panelist matching the occupations correctly is 1/24. For the second panelist the probability is (23/24)(1/23), etc. The contestants' expectations, then, is

$$\frac{1}{24}\cdot 0 + \frac{23}{24}\cdot\frac{1}{23}\cdot 25 + \frac{23}{24}\cdot\frac{22}{23}\cdot\frac{1}{22}\cdot 50 + \frac{23}{24}\cdot\frac{22}{23}\cdot\frac{21}{22}\cdot\frac{1}{21}\cdot 75$$

$$+ \frac{23}{24}\cdot\frac{22}{23}\cdot\frac{21}{22}\cdot\frac{20}{21}\cdot 100$$

$$= 0 + 1.0417 + 2.0833 + 3.1250 + 83.3333 = \$89.58.$$

Also solved by LOUIS H. CAIROLI, Kansas State. University, the. Proposer and by R. ROBINSON ROWE, Sacramento, California, who offers the. following comment:

It should be noted that the first panelist has 1/24 chance of having all four of his gusses right, no chance of having just three right, 6/24 chance of having none right. Suppose the second panelist deduced from these probabilities that the best strategy would be to assume none right and change all four. Then his chance would be 23/24 • (9/23 • 1/9 + 14/23 • 0/14) = 1/24. And, after all, strategy is ruled out by the specified 'pure guess',

356. *[Fall 1975] Proposed by Erwin Just, Bronx Community College, Bronx, New York.*

From the set of integers contained in [1, 2*n*] a subset *K* consisting of $n + 2$ integers is chosen. Prove that at least one element of K is the sum of two other distinct elements of K.

I. *Solution by Clayton W. Dodge, University of Maine, Orono, Maine.*

If the largest element of K is $2q$, then K can contain only one element from each of the pairs $(1, 2q - 1)$, $(2, 2q - 2)$, $\ldots$, $(q - 1, q + 1)$, and possibly also q, for a maximum total of $q + 1$ elements. If $2q - 1$ is the largest element of K, then K can contain only one member from each of the pairs $(1, 2q - 2)$. $(2, 2q - 3)$, $\ldots$, $(q - 1), q)$, for a total of q elements. The theorem now follows. Furthermore, if $K = \{n, n + 1, n + 2, \ldots, 2n\}$, then K contains $n + 1$ elements with no two members having a sum equal to a third member.

II. *Solution by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

We shall prove the stronger result:

<u>*Theorem.*</u> The complement K of any set of $n - 2$ integers in

$$S_{2n} = \{1, 2, \ldots, 2n\}$$

contains at least *two* elements which are the sums of two distinct elements of K.

Proof. Noting that we must have $n > 1$, we proceed by induction and observe that the result is evident for $n = 2$. Assume it is true for $n = k$ and consider S_{2k+2}. Form K' by removing $k - 1$ members from S_{2k+2}. If either $2k + 1$ or $2k + 2$ is removed then at most $k - 2$ members are removed from $S_{2k} \subset S_{2k+2}$ and we obtain the two desired sums by the induction hypothesis. Therefore we may assume that both $2k + 1$ and $2k + 2$ are in K' and hence remove $k - 1$ members from S_{2k}. Consider the sets $A = \{(1, 2k)(,(2, 2k - 1), \ldots, (k, k+1)\}$ and $B = \{(1, 2k + 1), (2, 2k), \ldots, (k, k + 2)\}$. Since each set involves $2k$ distinct integers from S_{2k+1} in k pairs, the removal of $k - 1$ members from S_{2k} will leave at least one complete pair of elements of K' in each of A and B. Hence both $2k + 1$ and $2k + 2$ will be sums of distinct elements of K' and the induction is complete.

Also solved by LOUIS H. CAIROLI, Kansas State. University; VICTOR G. FESER, Mary College, Bismarck NO; JOHN TOM HURT, Bryan, Texas; R. ROBINSON ROME, Sacramento, California; BRUCE A. YANOSHEK, University of, Cincinnati, Ohio; and the Proposer.

357. **[Fall 19751** *Proposed by David L. Silverman, West Lob Angeles, California.*

Able, Baker and Charlie, with respective speeds $a > b > c$, start at point P with Able designated *it* in a game of Tag which terminates when Able has tagged both Baker and Charlie. At time $-T$, Baker heads north and Charlie south. After a count taking time T, Able starts chasing one of the two quarries. Assuming that Baker and Charlie will maintain their speeds and directions, whom should Able chase first in order to minimize the time required to make the second and final tag?

Solution by Charles K. Lincoln, Raleigh, North Carolina.

Let x and y be the times required for Able to catch the first and the second person respectively. If Able chases Baker first, direct calculation shows that $x = bT/(a - b)$ and $y = aT(b + c)/(a - c)(a - b)$. If Able chases Charlie first, $x = cT/(a - c)$ and $y = aT(b + c)/(a - c)\cdot(a - b)$. Since both y's are equal, the x's show that the least time will elapse when Able chases Charlie first.

Also solved by LOUIS H. CAIROLI, Kansas State. University; CLAYTON W. DODGE, University of, Maine at Orono; TOM HURT, Bryan, Texas; R. ROBINSON ROME, Sacramento, California; CHARLES W. TRIGG, Sun Diego, California; and. the. Proposer. Three incorrect solutions were received.

358. **[Fall 1975]** *Proposed by Sidney Penner and H. Ian Whitlock, Bronx Community College, Bronx, New York.*

From a $2n + 1$ by $2n + 1$ checkerboard, in which the corner squares are black, two black squares and one white square are deleted. If the deleted white squares and at least one of the deleted black squares are not edge squares, then the reduced board can be tiled with 2×1 dominoes.

Solution by Clayton W. Dodge., University of, Maine at Orono.

Let the corner squares be black in any $2n + 1$ by $2n + 1$ checkerboard. Then there are $2n^2 + 2n + 1$ black and $2n^2 + 2n$ white squares in the checkerboard.

In an n by n checkerboard with $n > 2$, let us use the term contiguous square for any square that is not a boarder square but touches a border square. Thus, for $n = 3$, only the center square is a contiguous square. For $n > 3$, the contiguous squares form a hollow square of size $n - 2$ by $n - 2$.

Lemma 1. If a square of each color is removed from the border of any n by n checkerboard, the remaining border squares can be tiled with 2 by 1 dominoes.

Proof. There are an even number of squares along the border (in either direction) between two oppositely colored squares. These can be tiled in the obvious way.

Lemma 2. If a border square of color A is removed from an n by n checkerboard where $n > 3$, then an oppositely colored (color B) border square may be chosen arbitrarily and all remaining border squares tiled with 2 by 1 dominoes. Furthermore, the B square can be chosen so that a domino will tile it and a contiguous A-colored square.

Proof. Choose the color-B border square to be a non-corner square, always possible when $n > 3$, and apply Lemma 1 to the remaining border squares. Since it is not a corner square, the B square has a contiguous A square adjacent to it.

Lemma 3. If the deleted square is a corner square, then Lemma 2 also holds for $n = 3$.

Lemma 4. If, in a $2n + 1$ by $2n + 1$ checkerboard with $n > 1$, two black squares and one white square are removed from the border, then the remaining border squares along with one contiguous black square can be tiled with 2 bv 1 dominoes (as in Lemma 2).

Proof. As one travels the border between the two deleted black squares, in one direction the deleted white square intervenes. In the other direction pick a (non-corner) white square between the two deleted black squares, As in Lemma 1, now all other border squares can be tiled with dominoes. Then tile the picked white square and its contiguous black neighbor with one more domino.

Lemma 5. If, in a $2n + 1$ by $2n + 1$ checkerboard with $n > 1$, two black border squares are removed, the remaining border squares and two contiguous black squares can be tiled with 2 by 1 dominoes.

Proof. It is always possible to select two white border squares so they separate the removed black squares and so they do not abut the same black corner square (since $n > 1$). Tile the remaining border squares as in Lemma 1. Now, with two dominoes, tile the two selected white squares and their two (distinct) abutting contiguous black squares.

We now prove the theorem itself, deleting the restriction that any removed squares must not be edge squares. That is, we prove:

Theorem. From a $2n + 1$ by $2n + 1$ checkerboard in which the corner squares are black and $n > 0$, two black squares and one white square are deleted. The reduced board can be tiled with 2 by 1 dominoes.

Proof. It is trivially true that if either 1 black square or 2 black and 1 white squares are deleted from a 1 by 1 checkerboard, then the remaining squares can be tiled with dominoes. This disposes of the case $n = 0$.

Suppose it is always possible to tile the remaining squares of a $2n - 1$ by $2n - 1$ checkerboard, for some given $n > 0$, when either 1 black square or 2 black and 1 white squares are deleted. Consider a $2n + 1$ by $2n + 1$ checkerboard from which 2 black and 1 white squares have been removed. There will be 0, 1, 2, or 3 deleted squares in the border, so tile the border and 0, 1, or 2 contiguous squares by means of the preceding lemmas. Now the interior $2n - 1$ by $2n - 1$ checkerboard will have either 1 black square or 2 black and 1 white squares either removed or already tiled. The remaining squares can be tiled according to the inductive supposition.

The theorem follows by mathematical induction.

Also solved by LOUIS H. CAIROLI, Kansas State University; and the Proposers, who commented that Problem E2508 in the December 1974 issue of the American Mathematical Monthly *is a related problem.*

Comment by the Problem Editor

Louis H. Cairoli called attention to an article by David Singmaster in the March 1975 issue of *Mathematics Magazine*, (pp. 59-66), which contains a relevant theorem and proof:

Theorem 4. For any odd integers r and s, both greater than 1, an $r \times s$ chessboard, with any three squares deleted, two of the major color and one of the minor color, can be covered with dominoes.

359. [Fall 1975] *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania. (Corrected).*

Show that there are an infinitude of pairs of consecutive integers, each pair consisting of a pentagonal number $P_n^5 = n/2(3n - 1)$ and a hexagonal number $P_n^6 = m/2(4m - 2)$.

Solution by R. Robinson Rome, Sacramento, California.

Correcting an obvious typo, we are given:

$$P = P_n^5 = \frac{n}{2}(3n - 1);\ H = P_m^6 = \frac{m}{2}(4m - 2);\ H - P = \pm 1 \tag{1}$$

whence,

$$4m^2 - 2m - 3n^2 + n \pm 2 = 0 \tag{2}$$

with roots

$$m = \frac{1}{4}(1 \pm \sqrt{12n^2-4n+1\pm 8}) = \frac{1}{4}(1 \pm x),\ \text{where } x^2 = 12n^2 - 4n + 1 \pm 8 \tag{3}$$

Then

$$12n^2 - 4n + 1 \pm 8 - x^2 = 0 \tag{4}$$

with roots

$$n = \frac{1}{6}(1 \pm \sqrt{3x^2-2\pm 24}) = \frac{1}{6}(1 \pm \sqrt{3x^2+c}),\ \text{where } c = -26 \text{ or } +22 \tag{5}$$

Let $3x^2 + c = y^2$, deriving the Fermat-Pellian equation

$$y^2 - 3x^2 = c \tag{6}$$

$$2x^2 - 2xu - u^2 = -c, \quad \text{from } y = x + u \text{ in } (7) \tag{8}$$

$$u^2 - 2uv - 2v^2 = c, \quad \text{from } x = u + v \text{ in } (8) \tag{9}$$

$$w^2 - 3v^2 = c, \quad \text{from } u = v + w \text{ in } (9), \tag{10}$$

which is in the form of (7). So let $w = y_0$ and $v = x_0$; then

$$u = x_0 + y_0, \quad x_1 = 2x_0 + y_0, \quad y_1 = 3x_0 + 2y_0. \tag{11}$$

That is, from any solution (x_0, y_0) equation (11) derives a larger one (x_1, y_1). From (3) and (5), $m_0 = \frac{1}{4}(x_0 + 1)$ and $n_0 = \frac{1}{6}(y_0 + 1)$, so

$$x_0 = 4m_0 - 1 \quad\text{and}\quad y_0 = 6n_0 - 1. \tag{12}$$

Then

$$m_1 = \frac{1}{4}(x_1 + 1) = \frac{1}{4}(2x_0 + y_0 + 1) = \frac{1}{2}(4m_0 + 3n_0 - 1) \tag{13}$$

and

$$n_1 = (y_1 + 1)/6 = (3x_0 + 2y_0 + 1)/6 = 2(3m_0 + 3n_0 - 1)/3 \tag{14}$$

Noting that (14) would give a fractional value for n_1, (13) and (14) can be used recursively to find:

$$m_2 = \frac{1}{2}(14m_0 + 12n_0 - 5) \qquad n_2 = 8m_0 + 7n_0 - 3 \tag{15}$$

$$m_3 = \frac{1}{2}(52m_0 + 45n_0 - 20) \qquad n_3 = (90m_0 + 78n_0 - 35)/3 \tag{16}$$

$$m_4 = 97m_0 + 84n_0 - 38 \qquad n_4 = 112m_0 + 97n_0 - 44 \tag{17}$$

So m_2 and n_3 would always be fractional, but m_4 and n_4 will always be integers if derived from an integral set (m_0, n_0). Thus beginning with any such primitive set, (17) can be used recursively to generate an infinitude of sets (m, n), whence an infinitude of figurate sets derived from (1) satisfying $H - P = \pm 1$.

There are 4 such primitive sets (m_0, n_0), viz: (0, 1), (1, 0), (2, 2), (7, 8), designated A, B, C and D in the following tabulation to the limit of my computer.

Set	m	n	n - m	Hexagonal	Pentagonal	$H - P$
A	0	1	1	0	1	-1
B	1	0	-1	1	0	1
C	2	2	0	6	5	1
D	7	8	1	91	92	-1
A	46	53	7	4 186	4 187	-1
B	59	68	9	6 903	6 902	1
C	324	374	50	209 628	209 627	1
D	1 313	1 516	203	3 446 625	3 446 626	-1
A	8 876	10 249	1 373	157 557 876	157 557 877	-1
B	11 397	13 160	1 763	259 771 821	259 771 820	1
C	62 806	72 522	9 716	7 889 124 466	7 889 124 465	1
D	254 667	294 064	39 397	129 710 307 111	129 710 307 112	-1

Comment

Equation (17) will recursively generate an infinitude for any primitive value of $H - P$, e.g. for H = P, m = n = 1 generates 1; 40755 and 1; 533 776 805. Also negative values of m and/or n derive positive values for H and P in similar sequences, but they are not figurate numbers.

Also solved by JEFFREY BERGEN, Brooklyn College, NY York; LOUIS H. CAIROLI, Kansas State University; CLAYTON DODGE, University of Maine at Orono; and the Proposer.

360. *[Fall 19751 Proposed by Paul Erdos and Ernst Straus, University of California at Los Angeles.*

Denote by A_n the least common multiple of the integers $\le n$ and denote by $d(n)$ the number of divisors of n.

(a) Prove that $\sum_{n=1}^{\infty} \frac{1}{A_n}$ is irrational.

(b) Prove that $\sum_{n=1}^{\infty} \frac{d(n)}{A_n}$ is irrational.

(c) Prove that $\sum_{n=1}^{\infty} \frac{f(n)}{A_n}$ is irrational, where $f(x)$ is a polynomial with integer coefficients.

Solution by the Proposers.

(a) Put $\frac{a}{b} = \sum_{n=1}^{\infty} \frac{1}{A_n}$. Let $n = p_k$, where p_k is the kth prime and

$p_k > b$, and multiply both sides by A_{n-1}. Since $b|A_{n-1}$ we get

$$\text{integer} = \frac{a}{b}A_{n-1} = A_{n-1}\sum_{n=1}^{\infty}\frac{1}{A_n}$$

$$= \text{integer} + A_{n-1}\sum_{i=0}^{\infty}\frac{1}{A_{n+i}}.$$

Hence $A_{n-1}\sum_{i=0}^{\infty}\frac{1}{A_{n+i}}$ is a positive integer and therefore

(1) $A_{n-1}\sum_{i=0}^{\infty}\frac{1}{A_{n+i}} \geq 1.$

On the other hand we have

$$(2)\quad A_{n-1}\sum_{i=0}^{\infty}\frac{1}{A_{n+i}} = A_{n-1}\left(\sum_{m=p_k}^{p_{k+1}-1}\frac{1}{A_m} + \sum_{m=p_{k+1}}^{p_{k+2}-1}\frac{1}{A_m} + \cdots\right)$$

$$\leq \frac{p_{k+1}-p_k}{p_k} + \frac{p_{k+2}-p_{k+1}}{p_kp_{k+1}} + \frac{p_{k+3}-p_{k+2}}{p_kp_{k+1}p_{k+2}} + \cdots.$$

We now use the fact that for all large p_i (in fact for $p_i \geq 11$) we have $p_{i+1} - p_i < 1/2\, p_i$. Substituting this inequality in (2) we get (for $p_k \geq 11$)

$$A_{n-1}\sum_{i=0}^{\infty}\frac{1}{A_{n+i}} < \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\cdots = 1$$

in contradiction to (1). Hence the sum cannot be rational.

(b) Start as in part (a) to get

$$(3)\quad A_{n-1}\sum_{i=0}^{\infty}\frac{d(n+i)}{A_{n+i}} = A_{n-1}\left(\sum_{m=p_k}^{p_{k+1}-1}\frac{d(m)}{A_m} + \sum_{m=p_k}^{p_{k+2}-1}\frac{d(m)}{A_m} + \cdots\right)$$

= integer for $n = p_k > b$.

Now set $D(m) = d(1) + d(2) + \cdots + d(m)$ and write

$$\sum_{m=p_i}^{p_{i+1}-1}\frac{d(m)}{A_m} \leq \frac{D(p_{i+1}) - D(p_i)}{A_{p_i}}$$

so (3) becomes

$$(4)\quad \frac{D(p_{k+1}) - D(p_k)}{p_k} + \frac{D(p_{k+2}) - D(p_{k+1})}{p_kp_{k+1}} + \cdots > 1.$$

It therefore suffices to show that (4) is false for infinitely many values of k.

We use $d(n) < 2\sqrt{n}$ so that

(5) $D(p_{i+1}) - D(p_i) < 2\sqrt{p_{i+1}}\,(p_{i+1} - p_i).$

Using Bertrand's Postulate, $p_{i+1} < 2p_i$, we get

(6) $D(p_{i+1}) - D(p_i) < 2\sqrt{2p_i}\,(2p_i - p_i) < 2\sqrt{4p_{i-1}p_i} < 4p_ip_{i-1}.$

By the prime number theorem we have $\pi(m) > \frac{m}{2\log m}$ for all large m. Thus the difference $p_{i+1} - p_i$ between consecutive primes is on the average less than a multiple of $\log p_i$. There are therefore arbitrarily large values of k with

$$p_{k+1} < p_k + \sqrt[4]{p_k}$$

and

(7) $D(p_{k+1}) - D(p_k) < 2\sqrt{p_{k+1}}\,\sqrt[4]{p_k} < 4p_k^{3/4}.$

We now substitute the estimates (6) and (7) in (4) to get

$$\frac{4p_k^{3/4}}{p_k} + \frac{4}{p_k} + \frac{4}{p_kp_{k+1}} + \frac{4}{p_kp_{k+1}p_{k+2}} + \cdots > 1$$

which is clearly false for sufficiently large values of k.

(c) The solution for the problem involving polynomials with integer coefficients will appear in the Spring 1977 issue of this Journal.

361. *[Fall 1975] Proposed by Carl A. Argila, De La Salle College, Manila, Philippines.*

Consider any triangle ABC such that the midpoint P of side BC is joined to the midpoint Q of side AC by the line segment PQ. Suppose R and S are the projections of P and Q respectively on AB, extended if necessary. What relationship must hold between the sides of the triangle if the figure $PQRS$ is a square? (The projections R, S, should have been transposed so that S is the projection of P and R the projection of Q -- Problem Editor.)

Solution by John Tom Hunt, Bryan, Texas.

The construction gives $PQ = AB/2$ and $QR = h_c/2$. So if $PQRS$ is a square the altitude on the base AB is equal to AB.

From the Cosine Law we obtain

$$\cos B = (c^2 + a^2 - b^2)/2ac = (a^2 - b^2)^{1/2}/a,$$

which yields the quartic

$$a^4 + b^4 + 5c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 = 0.$$

Solving for c^2, we get

$$c^2 = \frac{(a^2 + b^2) \pm 2\sqrt{3a^2b^2 - a^4 - b^4}}{5}$$

Since c is real we must have $3a^2b^2 - a^2 - b^2 \geq 0$, and this gives

$$\frac{\sqrt{5} - 1}{2} \leq a/b \leq \frac{\sqrt{5} + 1}{2}$$

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* VICTOR G. FESER, *Mary College, Bismarck, North Dakota;* R. ROBINSON ROWE, *Sacramento, California;* CHARLES W. TRIGG, *San Diego, California; and the Proposer.*

Comment by Problem Editor

The solution $c = h$ is necessary for the construction of the square but is not sufficient. If the ratio a/b (or b/a) lies outside the bounds of the Golden Ratio and its reciprocal, P and Q together with their projections S and R form a rectangle instead of a square. In an acute triangle, the bounds are further restricted so that we have

$$\sqrt{2}/2 \leq a/b \leq \sqrt{2}.$$

Problem Editor's Note. Credit for a solution to Problem 338 should have been given to C. B. A. Peck, State College, Pennsylvania.

LOCAL CHAPTER AWARDS WINNERS

CALIFORNIA ETA (University of Santa Clara). The George W. *Evans, II, Memorial* Prizes awarded annually to the male and female Santa Clara students who score highest in the William Lowell Putnam Mathematics Competition was won by

Brian Conrey,
Rita Robbins.

COLORADO DELTA (University of Northern Colorado). The Outstanding *Freshman* Award was presented to

Van Endres

and the Outstanding *Senior* Award went to

Chris Ivey.

FLORIDA EPSILON (University of South Florida). The Outstanding *Scholar* award for work in mathematics of the highest quality went to

Edward V. Baker, III,

in 1975 and to

Bruno Michael Castellano,
Robert Earl Tubbb

in 1976.

GEORGIA BETA (Georgia Institute of Technology). Recipients of a book award for attaining a grade point average of at least 3.8 (4.0, perfect) in all mathematics courses taken were

John W. Endsley,
Michael E. Hoffman,
Richard S. John,
William V. Luedtke,
Michael 3. Schramm,
Martin K. Smith,
Lee S. Tadelman.

GEORGIA GAMMA (Armstrong State college). Named the Outstanding Senior for 1976 was

Michael Brennan.

College sponsored memberships in the American Mathematical Society were awarded to

Marshall Hinds,

Charles Leska,
Larry Parker,
Frank Yellin,

and memberships in the Mathematical Association of America were awarded to

Terry Anderson,
Michael Brennan,
Swarna Krishnamurti,
Mary Ann Reiser,
Ben Zipperer.

IOWA ALPHA (Iowa State University). Scholarship awards of $50 each were awarded to

David C. Challener,
Gregory D. Dougal

who scored highest on a competitive examination which was written and scored by members of the mathematics faculty.

MISSISSIPPI BETA (Mississippi College). The *Strange Memorial Scholarship Award* for 1975-76 was presented to

Edward C. Nichols

and the *Outstanding Senior Mathematics Major* was

John A. Lever.

NEW JERSEY EPSILON (St. Peter's College). The *James B. Collins Award* was established in honor of Professor Emeritus Collins, to be presented annually to the sophomore who has achieved the highest academic average in the 2-year calculus sequence, one year physics course, 2 algebra courses, and a data processing course. The first recipient of the award was

Maria Rodriguez.

NEW YORK BETA (Hunter College). The *Pi Mu Epsilon Scholarship Award* in the amount of $100 went to

Joseph M. Criscione.

NEW YORK PHI (State University of New York, Potsdam). The winner of the *Senior Award* was

Paul Hafer.

OHIO DELTA (Miami University). Students who placed highest on a mathematics examination were

Jeffrey Millard

who won *First Prize* and $30, and

Wayne. Heym

who won *Second Prize* and $20.

OHIO EPSILON (Kent State University). The winner of the 1976 *Pi Mu Epsilon Mathematics Award* was

Susan Zaborowski.

OKLAHOMA BETA (Oklahoma State University). The *Mathematical Sciences Faculty Award* for seniors was presented to

Kathy Stewart,
Kay Schwendimann,

and the *Mathematical Sciences Alumni Award* for sophomores went to

Karen Sonder,
Mary Stone..

The *William R. Pogue Award* for juniors was given to

Christy Gelmers,
Kathy Sullivan,

and the *AMOCO Foundation Scholarship* was received by

Brian Bickerstaff,
Martha Shahp.

William R. Pogue Certificates of Merit were awarded to

Marsha Currie,	Mary Launer,
Robert G. Hayes,	Clayton Morgan,
Deborah Huffman,	Emily Wonderly,

and a *Mathematical Sciences Faculty Achievement* award went to

Michael Coats,	Gahy Reeder,
Kim Cowherd,	Michael Thornberry,
Richard Embree,	Gloria Wilson.

PENNSYLVANIA BETA (Bucknell University). The *John Steiner Gold Mathematical Competition* involved 106 individuals from 36 high schools and resulted in the winners listed below.

INDIVIDUAL WINNERS:

Tom Blackador (First Place), STATE COLLEGE AREA HIGH SCHOOL,
Amy Kaufman (Second Place), LOCK HAVEN SENIOR HIGH SCHOOL,
Neil Bechdel (Third Place), LOCK HAVEN SENIOR HIGH SCHOOL,

John C. Hampton (Fourth Place), *MILTON AREA HIGH SCHOOL*,
David Bennett (Fifth Place), *STATE COLLEGE AREA HIGH SCHOOL*.

TEAM WINNERS:
(Division A)

STATE COLLEGE AREA HIGH SCHOOL (First Place)
(*David Bennett, Tom Blackador, Doug King*)

LOCK HAVEN SENIOR HIGH SCHOOL (Second Place)
(*Neil Bechdel, Amg Kaufman, Mark R. Schaitkin*)

(Division B)

LINE MOUNTAIN HIGH SCHOOL (First Place)
(*Thomas Arner, Ruth Dreibelbis, Randy Snyder*)

LEWISBURG BIGS SCHOOL (Second Place)
(*Jonathan Chenoweth, Kurt Weist, Theresa Yuschok*)

RHODE ISLAND BETA (Rhode Island College). The *Mitchell Award*, named in honor of a former faculty member, was presented to

Sharon Remington

for being the best senior mathematics major according to gradepoint average.

TEXAS DELTA (Stephen F. Austin State University). The *Outstanding Senior Mathematics Student* was

Anna Jones.

TEXAS ETA (Texas A. 6 M. University). Winners of a mathematics contest for undergraduates were as follows:

SOPHOMORE CONTEST

Robbie W. W. Lou (First Place),
Jon Juneau (Second Place),
Yuk-Lin Chu (Third Place),

FRESHMAN CONTEST

David C. Taylor (First Place),
John. D. Bremsteller (Second Place),
Curtis F. Feeny (Third Place).

TEXAS LAMBDA (University of Texas). The *Exxon Award*, presented to outstanding mathematics students, was won by

Kenn Askins.

VIRGINIA GAMMA (Madison College). The winner of the *Outstanding Senior Mathematics Student* award of *$50* was

Hope. Harbeck.

SUMMER MEETING IN TORONTO

Pi Mu Epsilon Fraternity held its annual summer meeting in conjunction with the American Mathematical Association in Toronto, Canada August 24-28, 1976. On Wednesday, the Governing Council held its annual luncheon and business meeting at Wilson Hall Cafeteria anddiscussed initiation procedures for Councilors who are asked to induct new Chapters, considered methods of improving the activities of the Fraternity, were advised of the continuing high cost of publishing the *Journal* and possible ways to finance it, were informed of the excellent financial status of the Fraternity, and voted to increase the amount of travel money available for delegates and speakers attending next year's meeting in Seattle due to the greater distances involved.

Wednesday evening the Fraternity heard the second J. Sutherland Frame Lecture at Sidney Smith Hall. The lecture was a most interesting and vigorous slide presentation on the topic *The. Pappus Configuration and Its Groups* by H. S. M. Coxeter, from the University of Toronto. Thursday morning, the annual Dutch Treat Breakfast was held at Wilson Hall Cafeteria.

The very excellent student papers presented during the Wednesday and Thursday afternoon sessions were as follows:

1. *A Chainable Continuum Not Homeomorphic to an Inverse Limit on [0, 1] with Only One. Bonding Map*, Dorothy Marsh, Texas Theta.
2. *An Informal Math Lab*, Kevin Bucol, Nebraska Beta.
3. *[0, 1] Is Not Compact: A Discussion of the Hyper-reals*, Thomas Sweeney, Missouri Gamma.
4. *On the. Problem of the. Lion and the Man*, Mark Showers, Illinois Zeta.
5. *Ridge Regression*, Dale Borowiak, Ohio Nu.
6. *Fugue in $Z^{\#}$ Major*, Willian Stone, Utah Alpha.
7. *Fixed Point Theorems in Metric Spaces*, Carol Collins, North Carolina Delta.
8. *Additions and Corrections to "Elementary Number Theory in Certain Subsets of the. Integers I and II"*, William Lenhart (speaker) and Karen McConloque, Pennsylvania Xi.

9. *Topology on Geometries with Betweenness*, Anees Rozzouk, Michigan Gamma.

10. *High School Mathematical Models*, Ellen Hearn, New Jersey Epsilon.

11. *The Whitney Theory for Maps Between 2-Manifolds*, Jane Hawkins, Massachusetts Beta.

12. *On Difunctional and Circular Relations*, Alma E. Posey, Arkansas Beta.

13. *Operations Research--An Approach to the Solution of Problems in the Urban System*, Elaine Flowers, Alabama Zeta.

POSTERS AVAILABLE FOR LOCAL ANNOUNCEMENTS

At the suggestion of the Pi Mu Epsilon Council we have had a supply of 10 x 14-inch Fraternity crests printed. One in each color will be sent free to each local chapter on request. Additional posters may be ordered at the following rates:

(1) Purple on goldenrod stock - - - - - - $1.50/dozen,

(2) Purple and lavendar on goldenrod- - - $2.00/dozen.

REGIONAL MEETINGS OF MAA

Many regional meetings of the Mathematical Association regularly have sessions for undergraduate papers. If two or more colleges and at least one local chapter help sponsor or participate in such undergraduate sessions, financial help is available up to $50 for one local chapter to defray postage and other expenses. Send request to:

Dr. Richard A. Good
Secretary-Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742

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