

Generalized Factoriangular Numbers And Factoriangular Triangles

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Abstract: A factoriangular number is defined as the sum of corresponding factorial and triangular number. This paper aims to generalize this number as sum of any factorial and any triangular number and explore such generalization. This study is a basic research in number theory that uses mathematical exposition and exploration. The generalized factoriangular number is of the form $n! + T_k$, where $n!$ is the factorial of a natural number n and T_k is the k^{th} triangular number. When $n = k$, the sum is an ordinary factoriangular number. A consequence of the generalization is the creation of interesting Pascal-like triangles that are hereby called factoriangular triangles and formation of their corresponding integer sequences. Generalized factoriangular numbers and factoriangular triangles can be utilized as recreational mathematics for students. Further generalizations of factoriangular number and expositions on factoriangular triangles can be done next.

Keywords: factoriangular number, factoriangular triangle, generalized factoriangular number, integer sequence, recreational mathematics

1. Introduction

The sequence of natural numbers $\{2, 5, 12, 34, 135, 741, 5068, 40356, \dots\}$ has its beginnings in 2004 as a contribution of Nambi in the Online Encyclopedia of Integer Sequences (OEIS) [1]. Ten years later, some studies were devoted for the exploration of these numbers, which are called factoriangular (from the terms factorial and triangular), such numbers being sums of corresponding factorials and triangular numbers [2] – [6]. A factoriangular number, denoted by Ft_n , can be determined by the formula $Ft_n = n! + T_n$, where for natural numbers $n \geq 1$, $n!$ is the factorial of n and T_n is the n^{th} triangular number [2], [3]. Factoriangular numbers follow the recurrence relations [4]

$$Ft_{n+1} = (n+1) \left[Ft_n - (n^2 - 2) / 2 \right] \text{ for } n \geq 1 \text{ and}$$

$$Ft_n = n \left[Ft_{n-1} - (n^2 - 2n - 1) / 2 \right] \text{ for } n \geq 2,$$

which are linear non-homogenous recurrence relations with variable coefficient and of order 1. Heinz prepended $a(0) = 1$ to the original Nambi's sequence [1] and in such case, it was defined that $T_0 = 0$ and $Ft_0 = 0! + T_0 = 1$. With this, the two recurrence relations also hold true for $n = 0$ and $n = 1$, respectively. For the original sequence Ft_n with $n \geq 1$, the exponential generating function is given by the closed formula [4]

$$E(x) = \frac{2 + (2 - 5x^2 + 2x^3 + x^4)e^x}{2(1-x)^2}, \quad -1 < x < 1.$$

For the extended sequence Ft_n with $n \geq 0$, the exponential generating function is given by the closed formula [4]

$$E(x) = \frac{2 + (2x - x^2 - x^3)e^x}{2(1-x)}, \quad -1 < x < 1.$$

In another study [3], factoriangular numbers had been characterized as to its parity, compositeness, number and sum of positive divisors, abundancy or deficiency, Zeckendorf's decomposition, end digits and digital roots. The runsum representations of factoriangular numbers had been studied also [5]. This led to the addition of two new sequences in the OEIS: sequence A275928 [7], which is the number of odd divisors of A101292 [1] and the same as the number of runsums of factoriangular numbers [5]; and sequence A275929 [1], which is the sum of the first and last terms of runsums with length n of Ft_n [5] and is also the sum of sequence A038507 [9] and sequence A213169 [10] in the OEIS. A factoriangular number was also characterized as to when this can be represented as a sum of two triangular numbers. In particulars: $Ft_n = T_n + T_n$ if and only if $n = 1$ or $n = 3$; $Ft_n = T_n + T_m$ if and only if $8n! + 1$ is a square; $Ft_n = T_m + T_m$ if and only if $4Ft_n + 1$ is a square; and $Ft_n = T_k + T_m$ if and only if $8Ft_n + 2$ is a sum of two squares [6]. Furthermore, there was a conjecture given in [3] that 2, 5 and 34 are the only factoriangular numbers that are also Fibonacci numbers. This conjecture was recently confirmed in [11]. The numbers 2, 5 and 34 are now called Fibonacci factoriangular numbers. The objective of the present study is to generalize factoriangular number and to explore such generalization.

2. Methods

This study is a discipline-based scholarship of discovery or a basic research in number theory. It is expository in nature and uses mathematical experimentation and exploration. As in most researches in number theory, it involves the scientific approach of experimental mathematics, which includes computations to gain insights and recognitions of patterns and relationships.

3. Results and Discussions

In the sequence of factoriangular numbers, $\{Ft_n\}$, each entry is given by

$$Ft_n = (1 \cdot 2 \cdot 3 \cdots n) + (1 + 2 + 3 + \dots + n)$$

for natural number $n \geq 1$. This can be generalized into the sequence $\{Ft_{n,k}\}$ where each entry is given by

$$Ft_{n,k} = (1 \cdot 2 \cdot 3 \cdots n) + (1 + 2 + 3 + \dots + k)$$

for natural numbers $n, k \geq 1$. Clearly, when $n = k$, $Ft_{n,k} = Ft_n$.

To distinguish the two sets of numbers, the generalized factoriangular numbers will be called (n, k) -factoriangular numbers while the original factoriangular numbers will now be called ordinary factoriangular numbers. The (n, k) -factoriangular number is formally defined as follows: Definition. The (n, k) -factoriangular number is defined by the formula $Ft_{n,k} = n! + T_k$ where $n! = 1 \cdot 2 \cdot 3 \cdots n$ and $T_k = 1 + 2 + 3 + \dots + k = k(k+1)/2$ for natural numbers $n, k \geq 1$. With this defining formula, the sequence of (n, k) -factoriangular numbers is given by

$$\{Ft_{n,k}\} = \{2, 4, 3, 5, 7, 7, 8, 9, 12, 11, 25, 12, 27, 16, 30, 34, \dots\}$$

for $(n, k) = (1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (3,3), (1,4), (4,1), (2,4), (4,2), (3,4), (4,3), (4,4), \dots$ In this sequence, notice that the n^2 -th entry (i.e. 1st, 4th, 9th, 16th, etc.) are the ordinary factoriangular numbers 2, 5, 12, 34 and so on. This can be clearly seen in Table 1, where in the main diagonal from the top-left to the bottom-right of the table (or when $n = k$) are the ordinary factoriangular numbers.

Table 1: Table of (n, k) - factoriangular numbers

$n \setminus k$	1	2	3	4	5	...	k
1	2	4	7	11	16	...	$1+T_k$
2	3	5	8	12	17	...	$2+T_k$
3	7	9	12	16	21	...	$6+T_k$
4	25	27	30	34	39	...	$24+T_k$
5	121	123	126	130	135	...	$120+T_k$
...

n	$n!+1$	$n!+3$	$n!+6$	$n!+10$	$n!+15$...	$n!+T_k$
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The table of (n, k) - factoriangular numbers can be divided into two triangular arrays of numbers with the ordinary factoriangular numbers as common entries. As a result, Pascal-like triangles, which will be called factoriangular triangles, can be created and corresponding sequences can be formed. With this, consider the triangular array of numbers where each entry is (n, k) - factoriangular number with natural numbers $n, k \geq 1$ and $k \leq n$. This is given in Table 2.

Table 2: Triangular array of (n, k) -factoriangular numbers for $k \leq n$

$n \setminus k$	1	2	3	4	5	...	k
1	2						
2	3	5					
3	7	9	12				
4	25	27	30	34			
5	121	123	126	130	135		
...	
n	$n!+1$	$n!+3$	$n!+6$	$n!+10$	$n!+15$...	$n!+T_k$

A new sequence of numbers can now be formed as follows:

$$\{Ft_{n,k \leq n}\} = \{2, 3, 5, 7, 9, 12, 25, 27, 30, 34, \dots\}$$

for $(n, k) = (1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4), \dots$, and where $Ft_{n,k \leq n} = (1 \cdot 2 \cdot 3 \cdots n) + (1 + 2 + 3 + \dots + k)$ for natural numbers $n, k \geq 1$ and $k \leq n$. It is noticeable also that the T_k -th entries (i.e. 1st, 3rd, 6th, 10th, etc.) in the sequence are the ordinary factoriangular numbers. Consider also the other triangular array of numbers where the entries are also (n, k) - factoriangular numbers but with $k \geq n$. This is given in Table 3.

Table 3: Triangular array of (n, k) -factoriangular numbers for $k \geq n$

$k \setminus n$	1	2	3	4	5	...	n
1	2						
2	4	5					
3	7	8	12				
4	11	12	16	34			
5	16	17	21	39	135		
...	
k	T_k+1	T_k+2	T_k+6	T_k+24	T_k+120	...	$T_k+n!$

Another sequence of numbers can be formed as follows:

$$\{Ft_{n,k \geq n}\} = \{2, 4, 5, 7, 8, 12, 11, 12, 16, 34, \dots\}$$

for $(n, k) = (1,1), (1,2), (2,2), (1,3), (2,3), (3,3), (1,4), (2,4), (3,4), (4,4), \dots$, and where $Ft_{n,k \geq n} = (1 \cdot 2 \cdot 3 \cdots n) + (1 + 2 + 3 + \dots + k)$ for natural numbers $n, k \geq 1$ and $k \geq n$. In this sequence, the ordinary factoriangular numbers are also in the T_k -th places (i.e. 1st, 3rd, 6th, 10th, etc.).

4. Conclusions

An apparent generalization of the ordinary factoriangular number, $Ft_n = n! + T_n$, is the generalized (n, k) -factoriangular number, $Ft_{n,k} = n! + T_k$. From this generalization, two factoriangular triangles can be formed: one whose entries are (n, k) -factoriangular numbers with $k \leq n$ and given by the sequence

$$\{Ft_{n,k \leq n}\} = \{2, 3, 5, 7, 9, 12, 25, 27, 30, 34, \dots\}$$

and another whose entries are also (n, k) -factoriangular numbers but with $k \geq n$ and given by the sequence

$$\{Ft_{n,k \geq n}\} = \{2, 4, 5, 7, 8, 12, 11, 12, 16, 34, \dots\}.$$

On a lighter side, these generalized factoriangular numbers and factoriangular triangles can be utilized as part of recreational mathematics for students at all levels where the introduction of number theory is included in the mathematics curriculum. But it will not be surprising if in the future, this may also find some applications in more serious mathematics like combinatorics and cryptography. Further generalizations of factoriangular number and expositions on factoriangular triangles and corresponding integer sequences can be considered for future works.

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Author Profile



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