First Order Differential Equations

1 Finding Solutions

1.1 Linear Equations

$$\frac{dy}{dt} + p(t)y = g(t), \quad y(0) = y_0.$$

First Step: Compute the Integrating Factor.
We want the integrating factor to satisfy

$$\mu'(t) = p(t)\mu(t). \tag{1.1}$$

The formula which works is

$$\mu(t) := \exp \int p(t)dt.$$

Second Step: Multiply the ODE by the Integrating Factor.

$$\begin{aligned}
\mu(t)g(t) &= \mu(t)\left[y'(t) + p(t)y(t)\right] \\
&= \mu(t)y'(t) + \mu'(t)y(t) && \text{by equation (1.1)} \\
&= \frac{d}{dt}\left[\mu(t)y(t)\right] && \text{by the product rule.}
\end{aligned}$$

Third Step: Integrate or Take Antiderivatives.
By the Fundamental Theorem of Calculus

$$\mu(t)y(t) - \mu(0)y(0) = \int_0^t \mu(s)g(s)ds,$$

so by solving for $y(t)$ we get

$$y(t) = \frac{\mu(0)y_0 + \int_0^t \mu(s)g(s)ds}{\mu(t)}.$$

or

$\mu(t)y(t) = \int \mu(t)g(t)dt$, and $y(0) = y_0$ is used to compute the undetermined constant of the antiderivative.

1.2 Separable Equations

$$M(x)dx = N(y)dy, \quad y(x_0) = y_0.$$

With separable equations, the technique is totally trivial: Integrate both sides of the equation. (There is a tiny bit of work involving the substitution formula to justify that the resulting formula for y really **is** a solution of $M(x)/N(y(x)) = y'(x)$.) Unfortunately, the precalculus aspects of the problem are often very difficult.

$\int M(x)dx = \int N(y)dy$ leads to $m(x) + c = n(y)$. c can be found very easily by using $y(x_0) = y_0$, but then the function $n(y)$ needs to be inverted. It may be impossible to find the inverse or $n(y)$ may "have" several inverses, in which case the initial data $y(x_0) = y_0$ needs to be used again to determine the right inverse to use.

1.3 Homogeneous Equations

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right), \quad y(x_0) = y_0.$$

The Substitution.
Let $v = \frac{y}{x}$, so $y = xv$. Then $F\left(\frac{y}{x}\right) = F(v)$, and

$$\frac{dy}{dx} = \frac{d}{dx}(xv) = v + x\frac{dv}{dx}\ .$$

So

$$x\frac{dv}{dx} = F(v) - v = G(v),$$

and thus

$$\frac{dv}{G(v)} = \frac{dx}{x}\ ,$$

which is separable. After we solve this equation we will have $v = f(x)$, so $y = xv = xf(x)$.

The Symmetries.
Note that if $f(x)$ is continuous, then $y(0) = 0$. Hence it is unreasonable to prescribe $y(x)$ at 0. Also note that on lines through the origin (which automatically have the form $y = mx$), $\frac{y}{x}$ is constant, and therefore F is constant, and therefore $\frac{dy}{dx}$ is constant. So, the direction field segments which lie on the same line through the origin have the same slope, and in particular, the integral curves are symmetric with respect to the origin.

1.4 Exact Equations

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0, \quad y(x_0) = y_0$$

with

$$M(x,y) = \psi_x(x,y), \quad N(x,y) = \psi_y(x,y).$$

With the conditions above it turns out that if we give y implicitly as a function of x by the equation $\psi(x,y) \equiv c$, then we will get a solution of our ODE. To see this fact, take $\frac{d}{dx}$ of both sides of the proposed solution:

$$\begin{aligned} 0 = \frac{d}{dx}c = \frac{d}{dx}\left[\psi(x, y(x))\right] \\ &= \psi_x(x, y(x)) + \psi_y(x, y(x))\frac{dy}{dx} \\ &= M(x,y) + N(x,y)\frac{dy}{dx} \ . \end{aligned}$$

We can view the solution ($\psi \equiv c$) as equivalent to finding the existence of some conserved quantity.

Since exact equations don't come with little signs saying, "This Equation Is Exact!" we need a fast way of determining whether an equation of the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is exact. We take advantage of the fact that if ψ is smooth, then $\psi_{xy} = \psi_{yx}$. This theorem is sort of a converse to that fact.

1.1 Theorem. *If M and N are smooth enough, then $M + N\frac{dy}{dx} = 0$ is exact if and only if $M_y = N_x$.*

Thus, when we are given an equation of the form $M + N\frac{dy}{dx} = 0$ we simply compare M_y to N_x to see if it is exact. If it is exact, then we try to find ψ by taking antiderivatives:

$$\psi(x,y) = \int \psi_x(x,y)dx + h(y) = \int M(x,y)dx + h(y).$$

($h(y)$ is an undetermined function here analogous to the undetermined constants which appear with one variable antiderivatives: $\frac{\partial}{\partial x}[\psi(x,y) - h(y)] = \psi_x(x,y) + 0$.) Next, to find $h(y)$, do the following computation:

$$N(x,y) = \psi_y(x,y) = \frac{\partial}{\partial y}\left[\int M(x,y)dx + h(y)\right].$$

The last constant is found using the initial conditions: $y(x_0) = y_0$.

2 Qualitative Properties of Solutions

2.1 Existence and Uniqueness

2.1 Theorem. ***Linear ODE Existence and Uniqueness.*** *If the functions p and q are continuous on an open interval $I := \{\alpha < t < \beta\}$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation*

$$y' + p(t)y = q(t) \ \textit{for each} \ t \in I, \tag{2.1}$$

and that also satisfies the initial condition

$$y(t_0) = y_0 \tag{2.2}$$

where y_0 is a prescribed initial value.

2.2 Theorem. ***Nonlinear ODE Existence and Uniqueness.*** *If the functions f AND $\frac{\partial f}{\partial y}$ are continuous in some rectangle: $\{\alpha < t < \beta, \gamma < y < \delta\}$ which contains the point (t_0, y_0), then in some interval $\{t_0 - h < t < t_0 + h\}$ contained in $\{\alpha < t < \beta\}$, there exists a unique function $y = \phi(t)$ satisfying the initial value problem:*

$$y' = f(t, y), \quad y(t_0) = y_0. \tag{2.3}$$

Interval of Existence. Since the solution of (2.1) and (2.2) is given explicitly by

$$y(t) = \frac{\mu(0)y_0 + \int_0^t \mu(s)g(s)ds}{\mu(t)} , \tag{2.4}$$

where $\mu(t) := \exp \int p(t)dt$, it is clear that the domain of this solution will be independent of y_0. On the other hand, the "blow-up time" (if there is one) of solutions of (2.3) will typically depend on y_0.

Implicit Solutions. (2.4) gives the solution to our linear ODE explicitly. Often the best that we can do for (2.3) is find a relation $F(t, y) \equiv$ constant that our solution satisfies.

2.2 Definitions and Comments

Integrating Factor: An Integrating Factor for a given ODE is a function which, when multiplied by the ODE, transforms the ODE into a form which is immediately integrable. Integrating factors are essential for solving first order linear ODEs, and sometimes can be used to make a first order ODE into a form which is exact.

Critical Initial Value: A Critical Initial Value or Critical Initial Data for a given ODE is a value which divides the data which yield solutions with one kind of long term behavior from data that yield solutions with another kind of long term behavior.

Autonomous Differential Equations: An Autonomous Differential Equation is an ODE of the form $\frac{dy}{dt} = f(y)$ (as opposed to $\frac{dy}{dt} = f(y, t)$). Note that Autonomous ODEs are always separable: $\frac{dy}{f(y)} = dt$. Often geometric methods can be used to obtain important qualitative information directly from the ODE without solving it. In particular for Autonomous ODEs we can define the concepts of stability, instability, and equilibrium solutions.

Equilibrium Solution: In the ODE $\frac{dy}{dt} = f(y)$ an Equilibrium Solution is a constant solution: $y(t) \equiv c$. Note that $y(t) \equiv c$ is a solution of the ODE if and only if $f(c) = 0$. The zeros of $f(y)$ are called critical points of the ODE.

Stable Solution: An equilibrium solution $y(t) \equiv y_*$ is a Stable Solution or Asymptotically Stable Solution if solutions to the same ODE with initial data close enough to y_* approach y_* as $t \to +\infty$. (Note: In more advanced treatments of differential equations, there is a difference between *stability* and *asymptotic stability.*)

Unstable Solution: An equilibrium solution $y(t) \equiv y_*$ is an Unstable Solution if the only way to guarantee that a solution "remains near" to y_* is to make sure that its initial data is *exactly* equal to y_*.

Note the Following: Suppose $\frac{dy}{dt} = f(y),\ f(\widehat{y_0}) = 0,$ and $f(y)$ is continuously differentiable so that the existence-uniqueness theorem applies. Then

1. $y(t) \equiv \widehat{y_0}$ is a solution.

2. As a consequence, $\widehat{y_0}$ forms a "barrier" for all the solutions by the uniqueness theorem. In other words, if $\bar{y}(t)$ is a solution with $\bar{y}(0) < \widehat{y_0},$ then $\bar{y}(t) < \widehat{y_0}$ for all time, and if $\bar{y}(0) > \widehat{y_0},$ then $\bar{y}(t) > \widehat{y_0}$ for all time.

3. The stability of $y(t) \equiv \widehat{y_0}$ is determined by the sign of $f(y)$ near $\widehat{y_0}$. Since $\frac{dy}{dt} = f(y),$ if $f(y(\tilde{t})) > 0,$ then $y(t)$ is increasing at $t = \tilde{t},$ and if $f(y(\tilde{t})) < 0,$ then $y(t)$ is decreasing at $t = \tilde{t}$. Thus, if $f(y) > 0$ for $y < \widehat{y_0}$ and $f(y) < 0$ for $y > \widehat{y_0},$ then $y(t) \equiv \widehat{y_0}$ is a stable solution. If $f(y) < 0$ for $y < \widehat{y_0}$ and $f(y) > 0$ for $y > \widehat{y_0},$ then $y(t) \equiv \widehat{y_0}$ is an unstable solution.

2.3 Phase Portraits and Stability

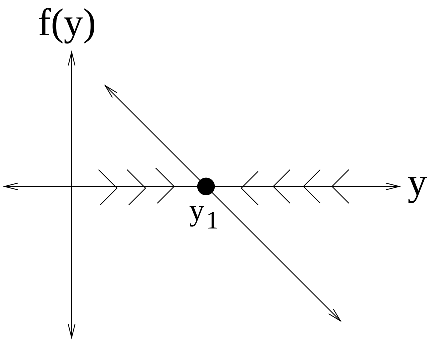
We study the ODE

$$\frac{dy}{dt} = f(y)$$

and assume

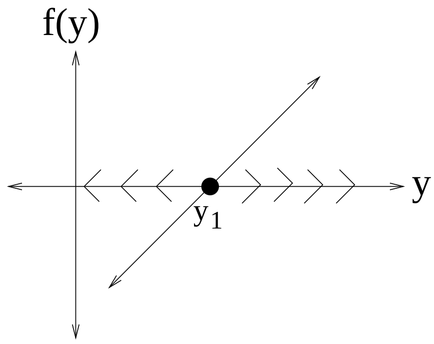
$$f(y_1) = 0 \ .$$

Case I: y_1 is a simple zero or a zero with odd multiplicity. Then the phase portrait of our ODE will look like:



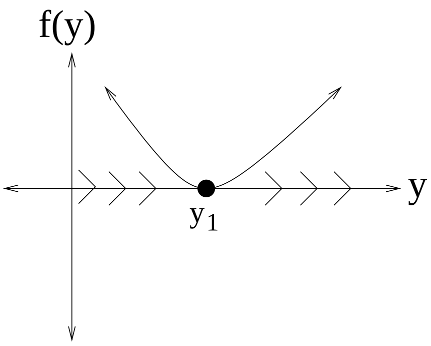
y_1 is stable

OR



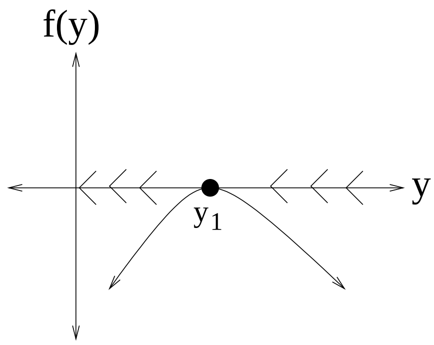
y_1 is unstable

Case II: y_1 is a zero with even multiplicity. Then the phase portrait of our ODE will look like:



y_1 is semistable
y is increasing

OR



y_1 is semistable
y is decreasing

3 Examples and their Direction Fields

3.1 Descriptions of the Direction Fields

1 Direction Field.

$$\frac{dy}{dt} = y, \quad y(0) = y_0.$$

The solution is:

$$y(t) = y_0 e^t.$$

0 is critical initial data: If $y(0)$ is a tiny bit larger than 0, then $\lim_{t\to\infty} y(t) = +\infty$. On the other hand, if $y(0)$ is a tiny bit less than 0, then $\lim_{t\to\infty} y(t) = -\infty$.

2 Direction Field.

$$\frac{dy}{dt} = y^3, \quad y(0) = y_0.$$

The solution is:

$$y(t) = \frac{y_0}{\sqrt{1 - 2ty_0^2}}\ .$$

The solution "blows up" at time $\frac{1}{2y_0^2} < +\infty$. This fact should not be too surprising: The solution to $\frac{dy}{dt} = y$ grows exponentially (i.e. VERY fast) and $|y^3|$ is MUCH larger than $|y|$ once $|y|$ is large enough, so the solutions to $\frac{dy}{dt} = y^3$ grow much faster than exponentially. y_0 is still critical.

3 Direction Field.

$$\frac{dy}{dt} = -y, \quad y(0) = y_0.$$

The solution is:

$$y(t) = y_0 e^{-t}.$$

So $\lim_{t\to\infty} y(t) = 0$ no matter what y_0 is. Hence 0 is not critical initial data in this case.

4 Direction Field.

$$\frac{dy}{dt} = -y^3, \quad y(0) = y_0.$$

The solution is:

$$y(t) = \frac{y_0}{\sqrt{1 + 2ty_0^2}}\ .$$

Again $\lim_{t\to\infty} y(t) = 0$, so there is no critical initial data. Note that $|y^3|$ is MUCH smaller than $|y|$ once $|y|$ is small enough, so the solutions of $\frac{dy}{dt} = -y^3$ approach 0 much more slowly than the solutions of $\frac{dy}{dt} = -y$.

5 Direction Field.

$$\frac{dy}{dt} = .2y^2.$$

In the first two examples, we had an unstable equilibrium at 0. In the next two examples, we had a stable equilibrium at 0. In this example 0 is semistable. In all of our examples so far, our ODE has been autonomous. (i.e. of the form $\frac{dy}{dt} = f(y)$ with f independent of t.) In each of these examples we have also had $f(0) = 0$. The sign of f switched at 0 in the first four examples:

- from $-$ to $+$ in the first two examples (unstable), and
- from $+$ to $-$ in the next two examples (stable)

In this example, because the root (the zero of f) is repeated an even number of times, the sign of f does not switch. This fact explains the semi-stability of 0.

6 Direction Field.

$$\frac{dy}{dt} = .2(y-4)^2.$$

Again semistable. Again the zero of f (4 in this case) is repeated twice. Note the vertical shift.

7 Direction Field.

$$\frac{dy}{dt} = r\left(1 - \frac{y}{k}\right)y = (r - ay)y = f(y).$$

$$r = \text{Intrinsic Growth Rate, and}$$

$$k = \text{Saturation Level or Environmental Carrying Capacity.}$$

To find inflection points take $\frac{d}{dy}$ of our ODE to get:

$$\frac{d^2y}{dt^2} = r\left(1 - \frac{2y}{k}\right).$$

So $y(t)$ inflects when it equals $\frac{k}{2}$. (The inflection occurs when y = 3 in our specific example. Look at the integral curves.)

8 Direction Field.

$$\frac{dy}{dt} = -ry\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{k}\right) = f(y), \quad \text{where} \quad T = \text{the Threshold.}$$

3.2 The Figures

Figure 1

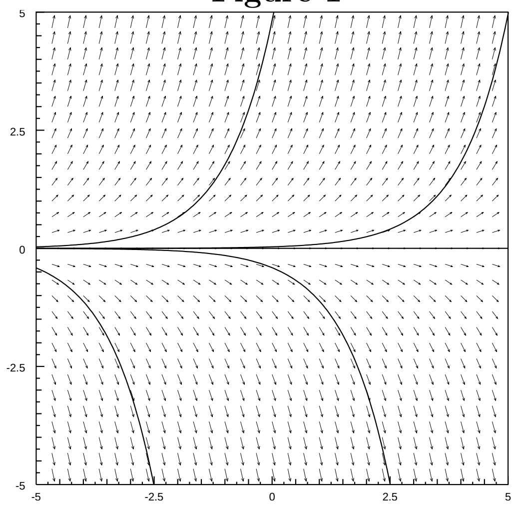


Figure 2

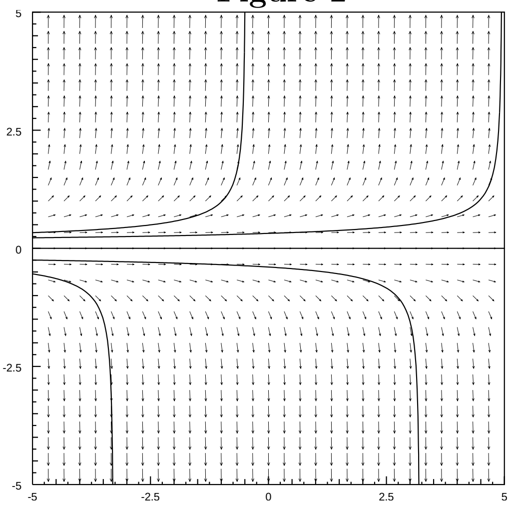


Figure 3

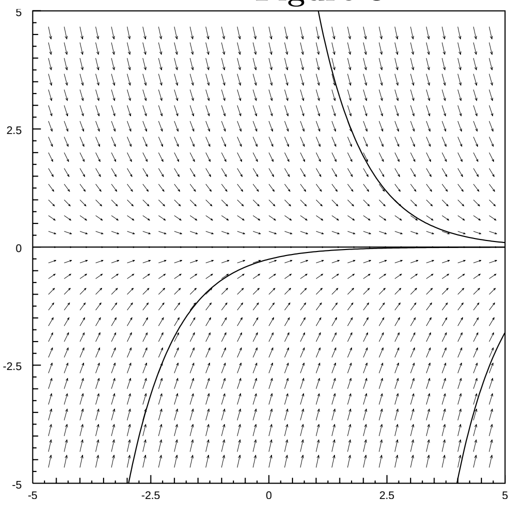


Figure 4

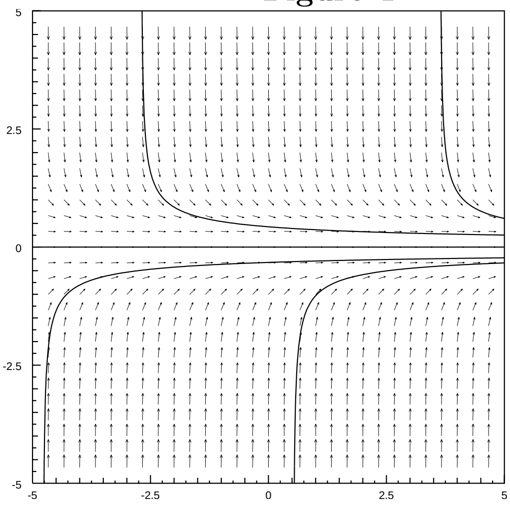


Figure 5

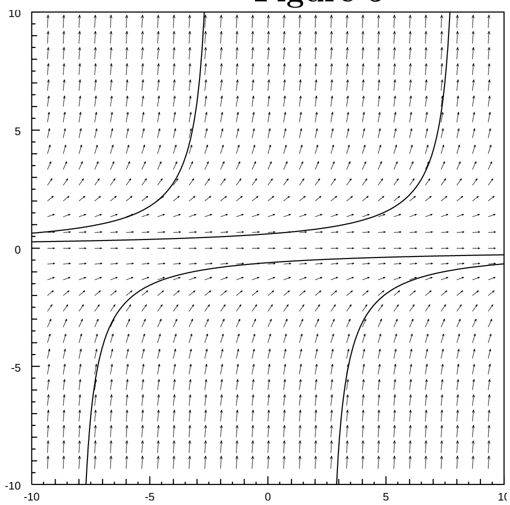


Figure 6

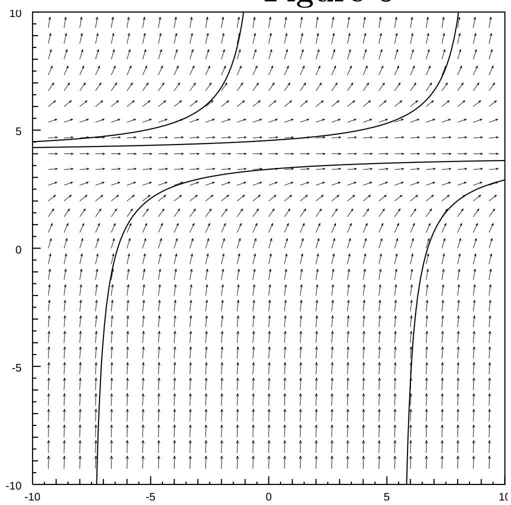


Figure 7

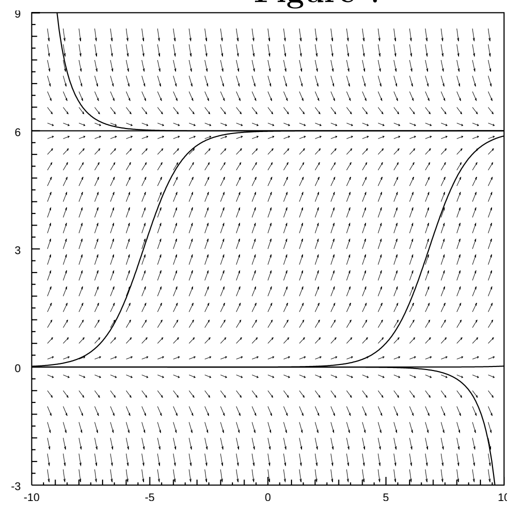


Figure 8

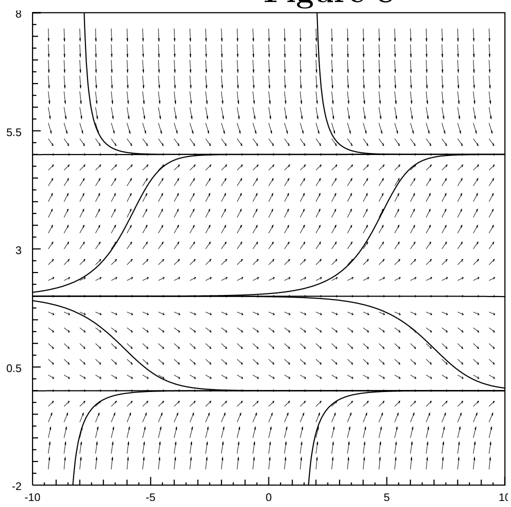
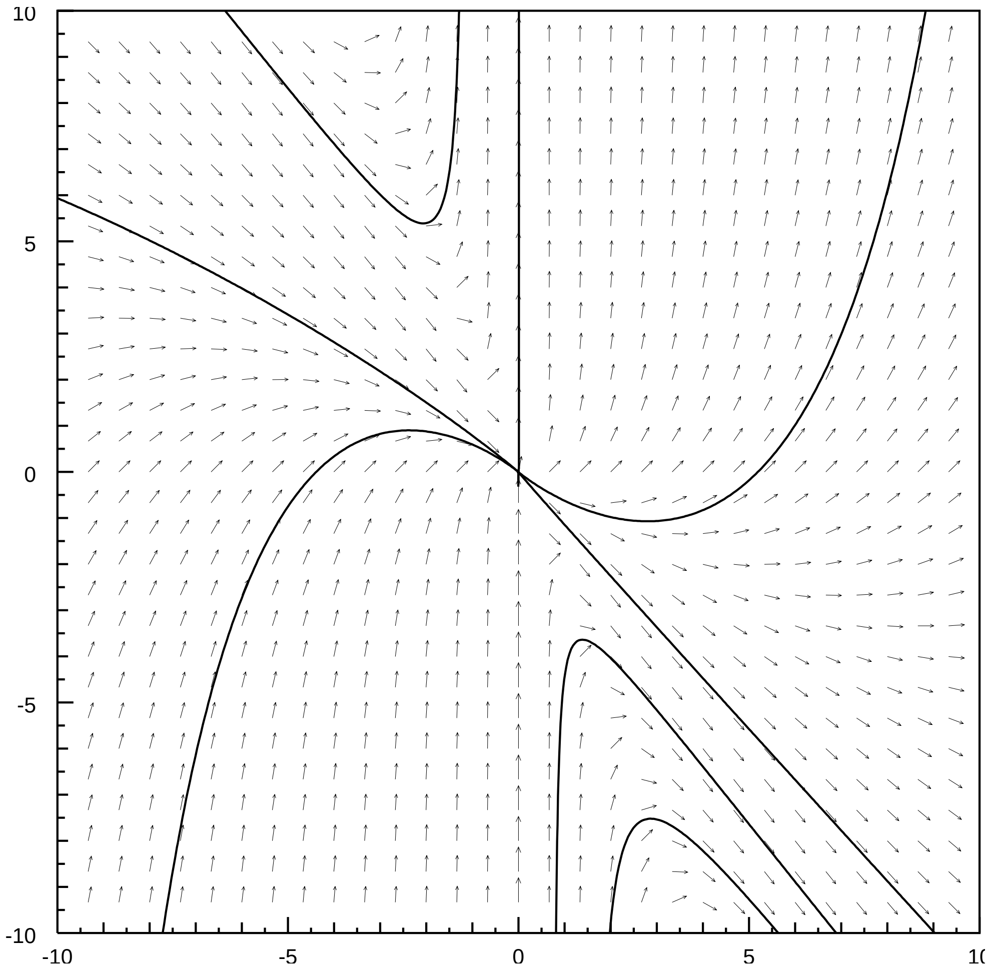


Figure 9

Here we have the Homogeneous Equation:

$$\frac{dy}{dt} = \frac{t^2 + 3yt + y^2}{t^2} = 1 + 3\left(\frac{y}{t}\right) + \left(\frac{y}{t}\right)^2.$$



Note that any integral curve can be rotated 180^o about the origin to form a new integral curve. Note the "node" at zero.