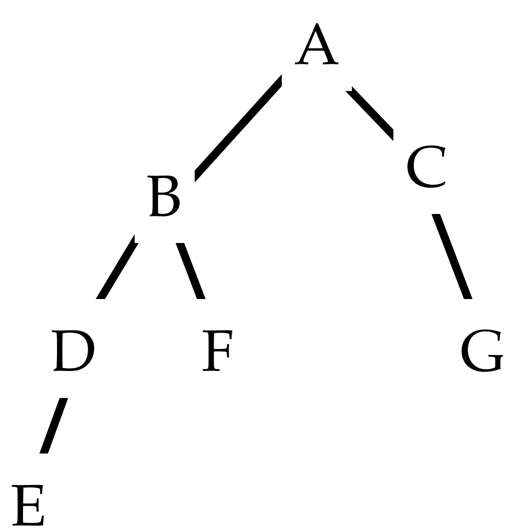
Trees

CMSC 420: Lecture 5

Hierarchies

Many ways to represent tree-like information:



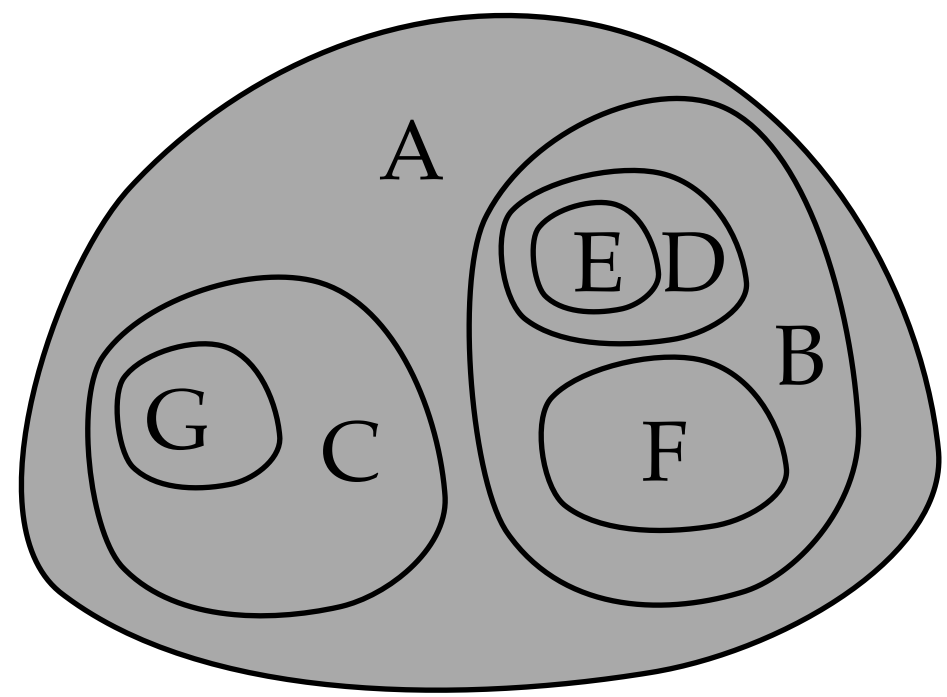
linked hierarchy

I. A
  1. B
    a. D
      i. E
    b. F
  2. C
    a. G

outlines, indentations

(((E):D), F):B, (G):C):A

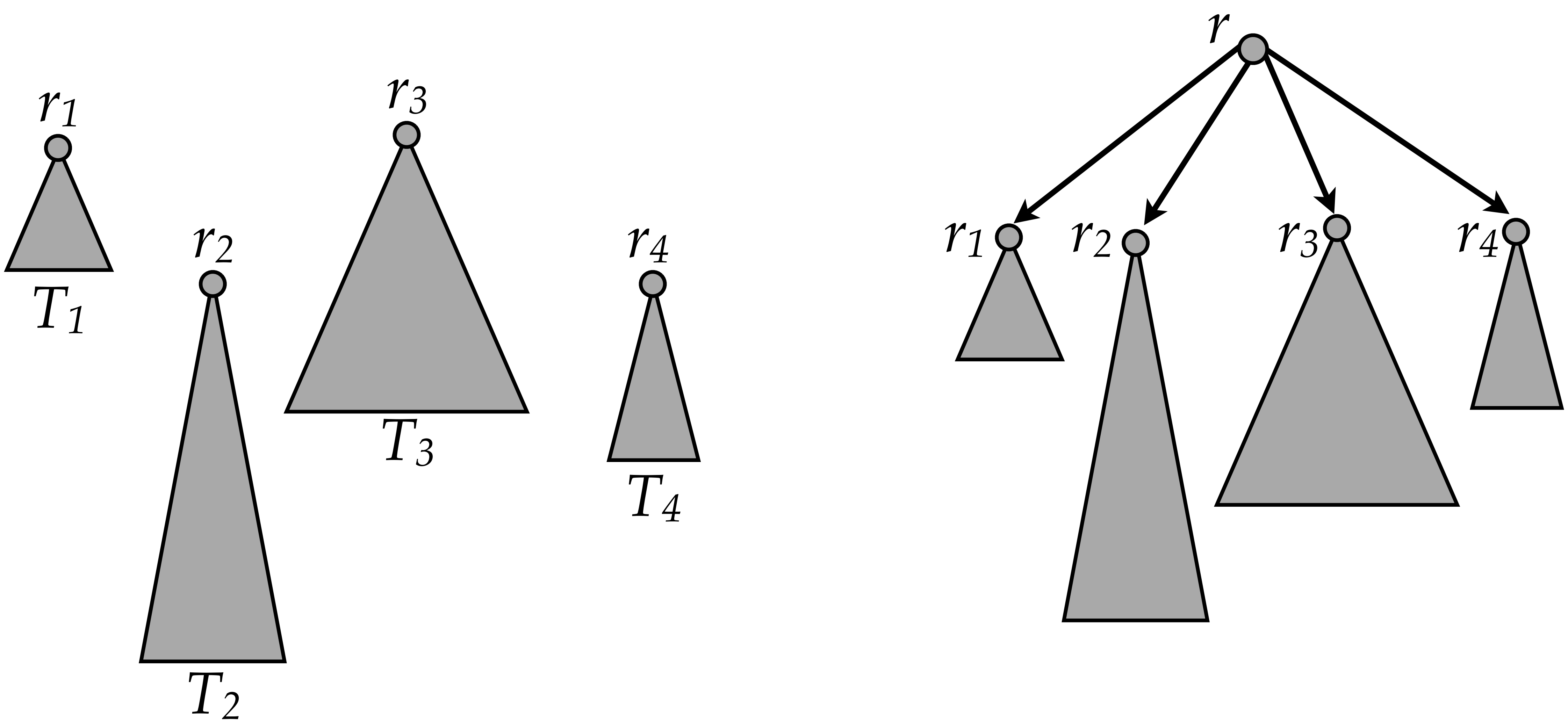
nested, labeled parenthesis



nested sets

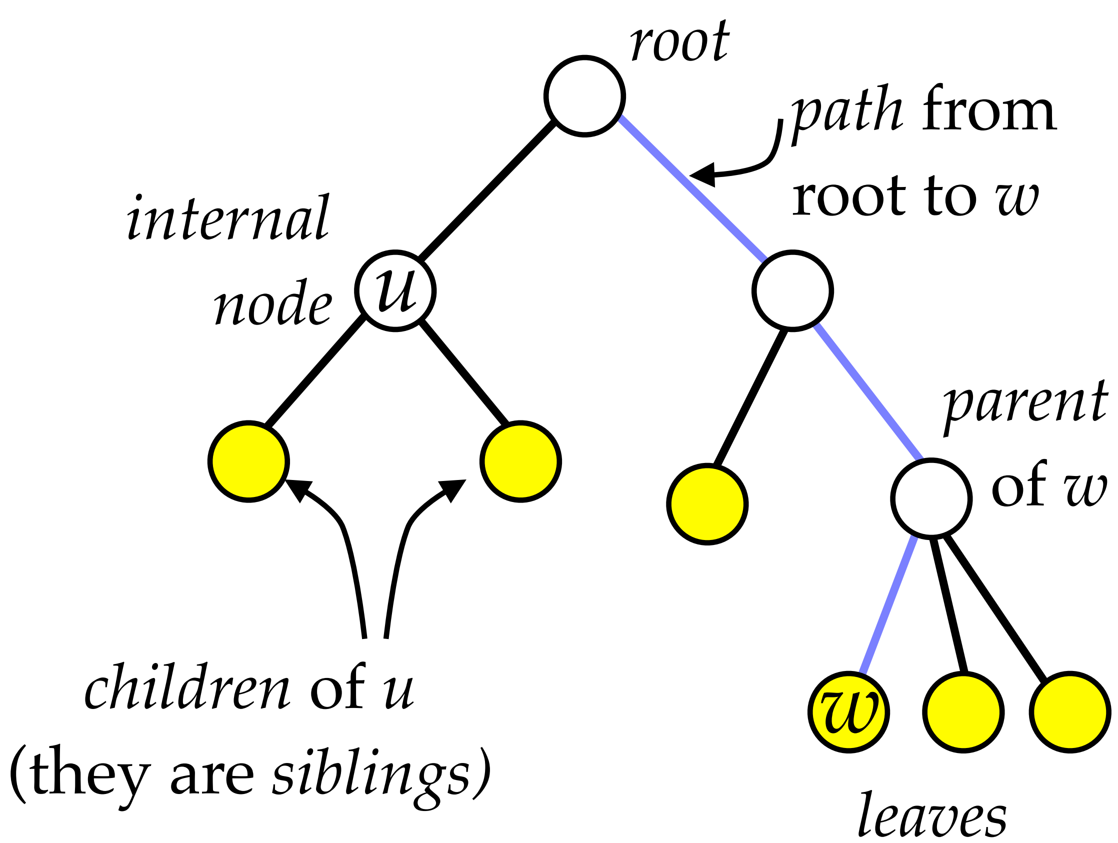
Definition – Rooted Tree

- Λ is a tree
- If $T_1, T_2, \ldots, T_k$ are trees with roots $r_1, r_2, \ldots, r_k$ and r is a node $\notin$ any T_i, then the structure that consists of the T_i, node r, and edges (r, r_i) is also a tree.



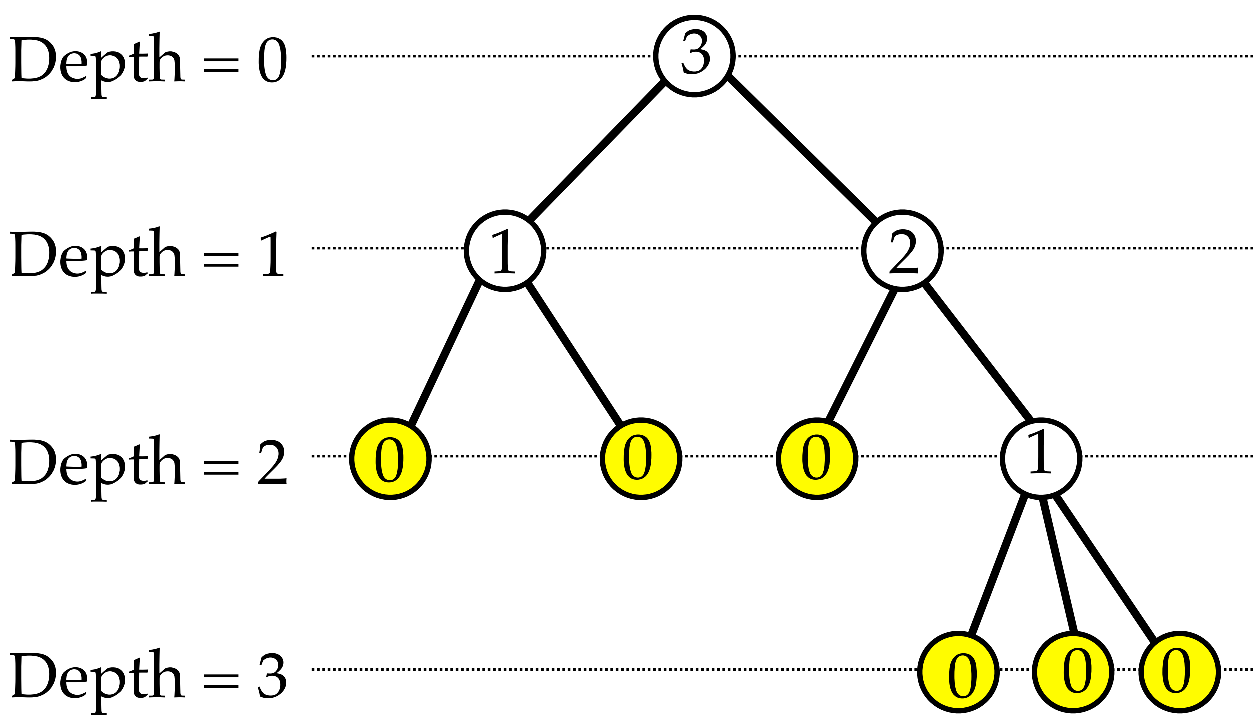
Terminology

- r is the <u>*parent*</u> of its <u>*children*</u> $r_1, r_2, \ldots, r_k$.
- $r_1, r_2, \ldots, r_k$ are <u>*siblings*</u>.
- <u>*root*</u> = distinguished node, usually drawn at top. Has no parent.
- If all children of a node are Λ, the node is a <u>*leaf*</u>. Otherwise, the node is a <u>*internal node*</u>.
- A <u>*path*</u> in the tree is a sequence of nodes $u_1, u_2, \ldots, u_m$ such that each of the edges (u, u_{i+1}) exists.
- A node u is an <u>*ancestor*</u> of v if there is a path from u to v.
- A node u is a <u>*descendant*</u> of v if there is a path from v to u.



Height & Depth

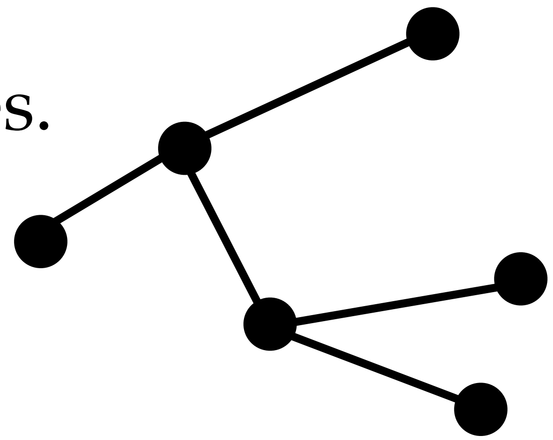
- The *height* of node u is the length of the longest path from u to a leaf.
- The *depth* of node u is the length of the path from the root to u.
- Height of the tree = maximum depth of its nodes.
- A *level* is the set of all nodes at the same depth.



Numbers in nodes give heights

Subtrees, forests, and graphs

- A *subtree* rooted at u is the tree formed from u and all its descendants.
- A *forest* is a (possibly empty) set of trees.
  The set of subtrees rooted at the children of r form a forest.
- As we've defined them, trees are **not** a special case of graphs:
  - Our trees are *oriented* (there is a root which implicitly defines directions on the edges).
  - A *free tree* is a connected graph with no cycles.



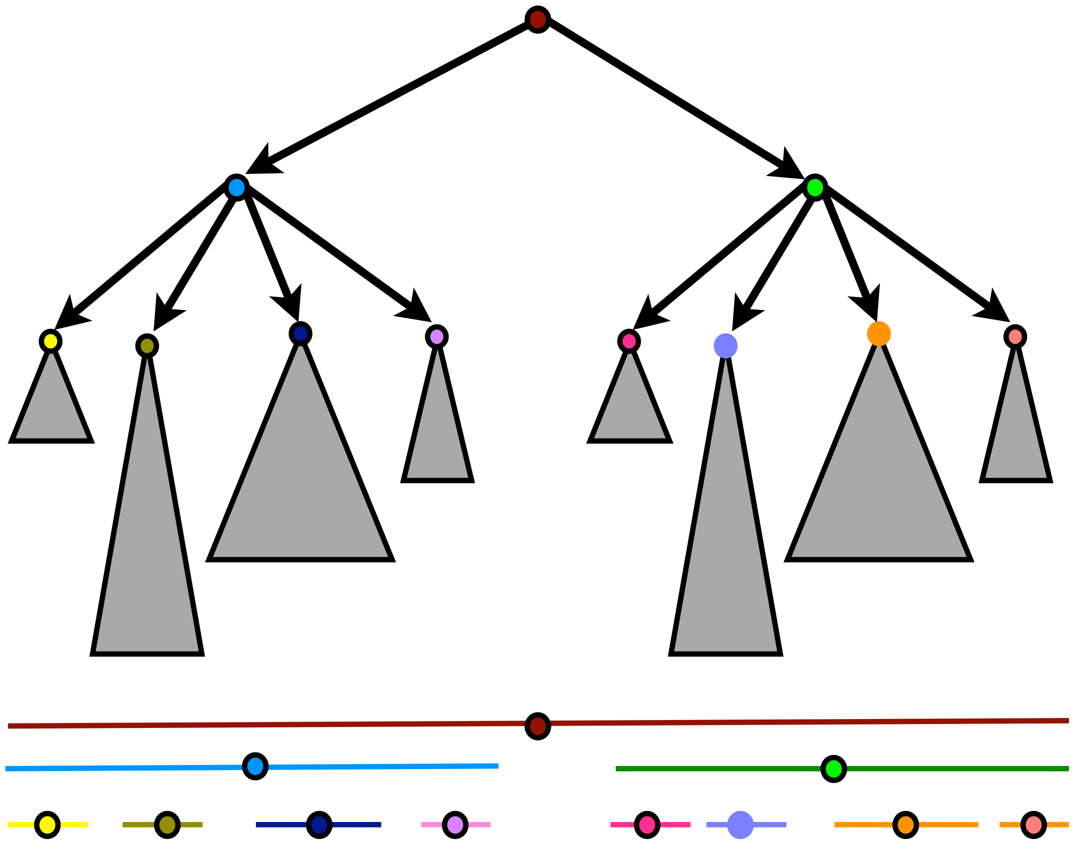
Alternative Definition – Rooted Tree

- A *tree* is a finite set T such that:
  - one element $r \in T$ is designated the *root*.
  - the remaining nodes are partitioned into $k \geq 0$ disjoint sets $T_1, T_2, \ldots, T_k$, each of which is a tree.

This definition emphasizes the ***partitioning*** aspect of trees:

As we move down the we're dividing the set of elements into more and more parts.

Each part has a distinguished element (that can represent it).



Basic Properties

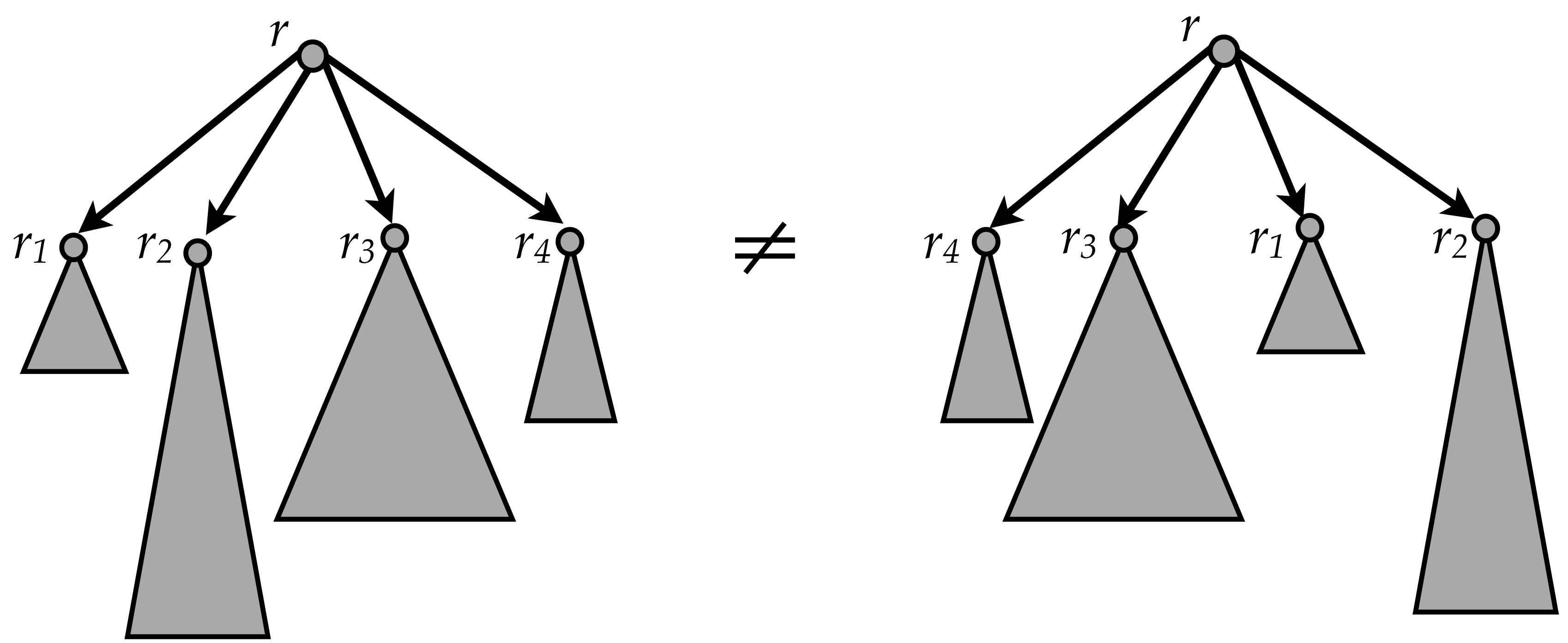
- Every node except the root has exactly one parent.

- A tree with *n* nodes has *n*-1 edges
(every node except the root has an edge to its parent).

- There is exactly one path from the root to each node.
(Suppose there were 2 paths, then some node along the 2 paths would have 2 parents.)

Binary Trees – Definition

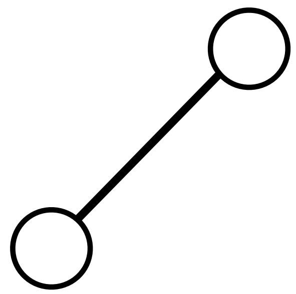
- An *ordered* tree is a tree for which the order of the children of each node is considered important.



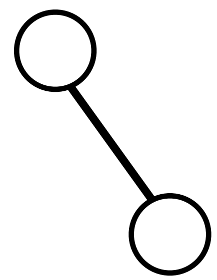
- A *binary tree* is an ordered tree such that each node has ≤ 2 children.
- Call these two children the *left* and *right* children.

Example Binary Trees

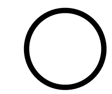
The edge cases:



Only left child



Only right child

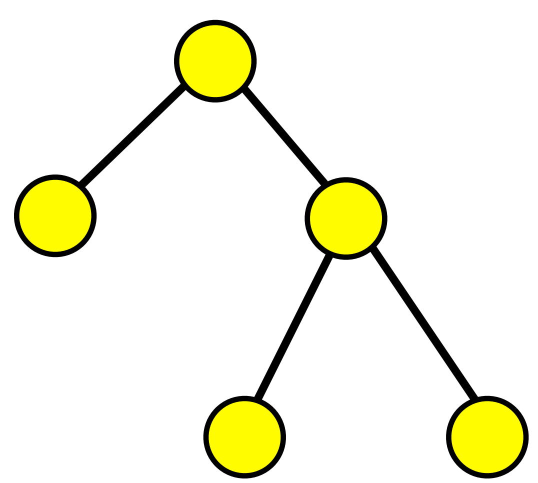


Single node

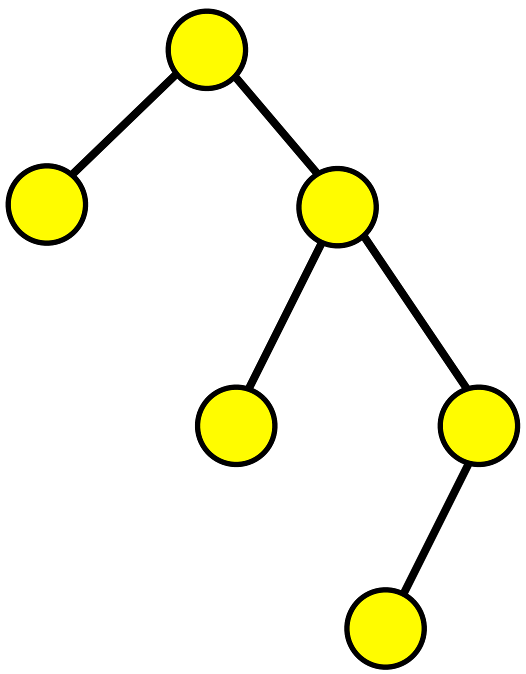
Λ

Empty Binary Tree

Small binary tree:

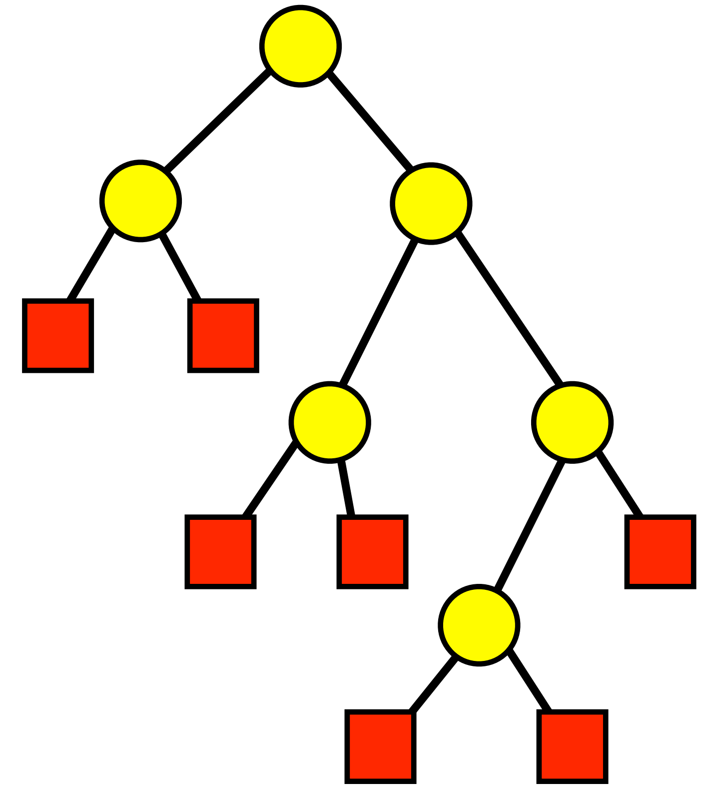


Extended Binary Trees



Replace each missing child with *external node*

Do you need a special flag to tell which nodes are external?



Binary tree

Extended binary tree

Every internal node has exactly 2 children.

Every leaf (external node) has exactly 0 children.

Each external node corresponds to one Λ in the original tree – let's us distinguish different instances of Λ.

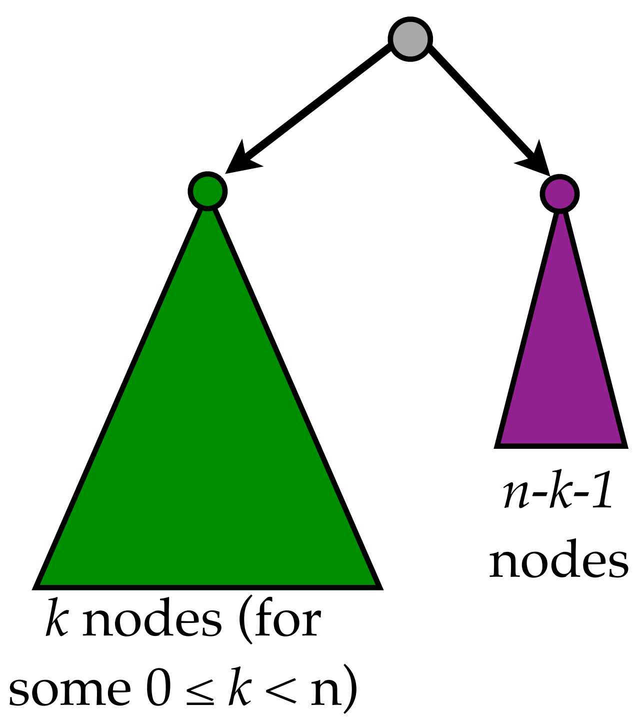
of External Nodes in Extended Binary Trees

Thm. *An extended binary tree with* n *internal nodes has* n+1 *external nodes.*

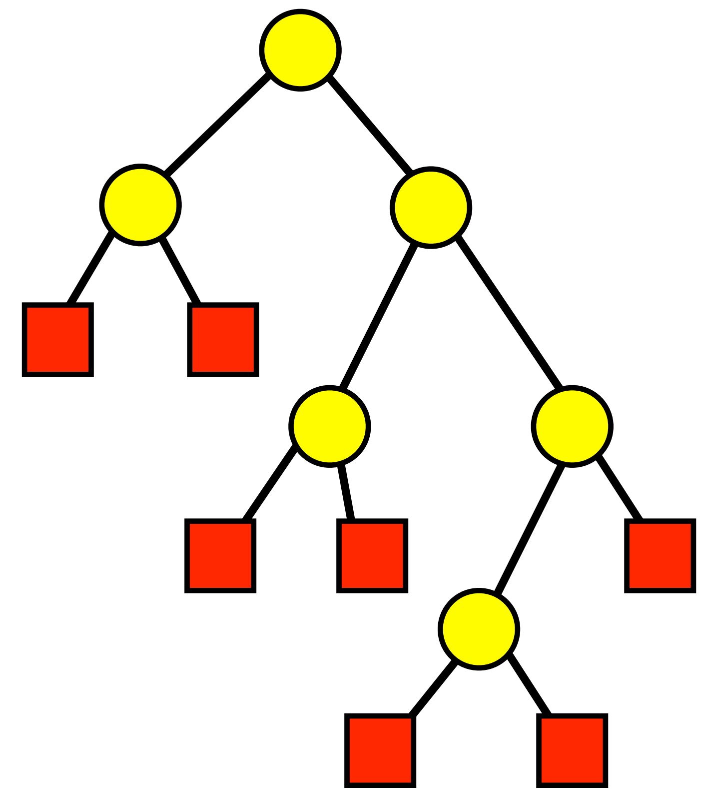
Proof. By induction on n.
X(n) := number of external nodes in binary tree with n internal nodes.

<u>Base case:</u> $X(0) = 1 = n + 1$.

<u>Induction step:</u> Suppose theorem is true for all $i < n$.
Because $n \geq 1$, we have:



$$\begin{aligned} X(n) &= X(k) + X(n\text{-}k\text{-}1) \\ &= k\text{+}1 + n\text{-}k\text{-}1 + 1 \\ &= n + 1 \quad \square \end{aligned}$$



Extended binary tree

Related to Thm 5.2
in your book.

Alternative Proof

Thm. *An extended binary tree with* n *internal nodes has* n+1 *external nodes.*

Proof. Every node has 2 children pointers, for a total of $2n$ pointers.

Every node except the root has a parent, for a total of n - 1 nodes with parents.

These n - 1 parented nodes are all children, and each takes up 1 child pointer.

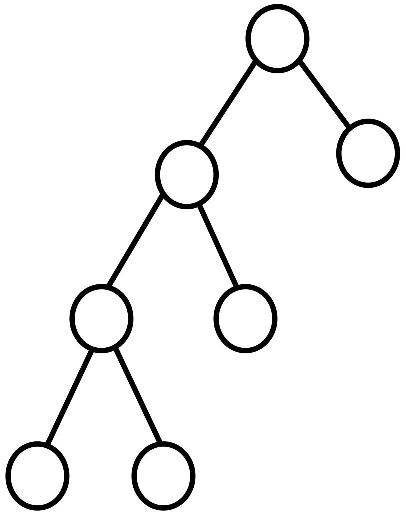
$$\text{(pointers)} - \text{(used child pointers)} = \text{(unused child pointers)}$$
$$2n - (n-1) = n + 1$$

Thus, there are $n + 1$ null pointers.

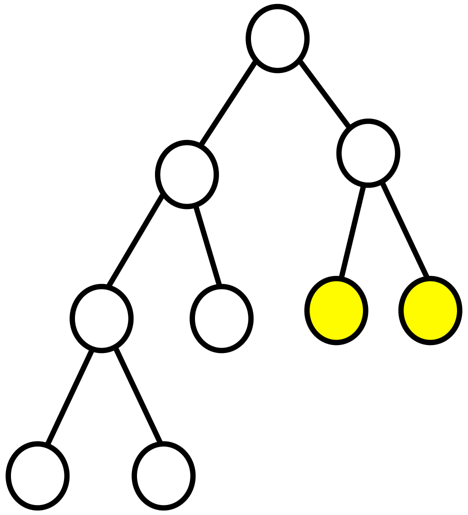
Every null pointer corresponds to one external node by construction. □

Full and Complete Binary Trees

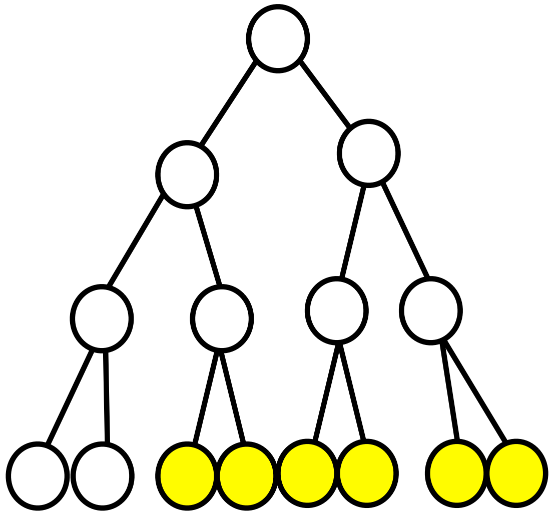
- If every node has either 0 or 2 children, a binary tree is called *full*.
- If the lowest *d-1* levels of a binary tree of height *d* are filled and level *d* is partially filled from left to right, the tree is called *complete*.
- If all *d* levels of a height-*d* binary tree are filled, the tree is called *perfect*.



full



complete



perfect

Nodes in a Perfect Tree of Height *h*

Thm. *A perfect tree of height h has* $2^{h+1} - 1$ *nodes.*

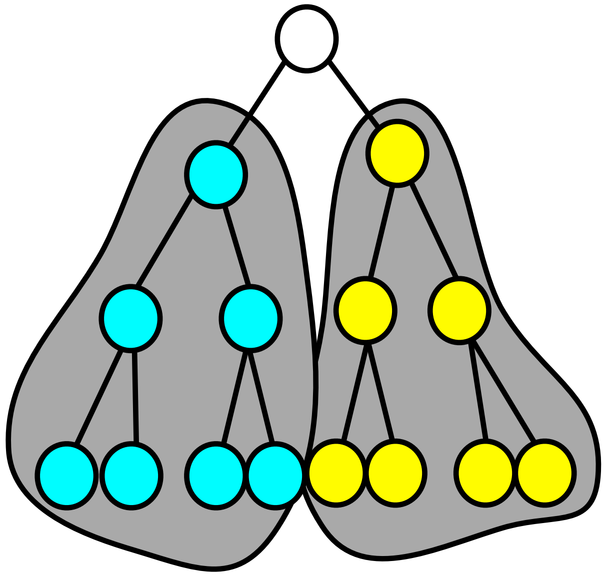
Proof. By induction on *h*.

Let N(h) be number of nodes in a perfect tree of height h.

<u>Base case:</u> when h = 0, tree is a single node. $N(0) = 1 = 2^{0+1} - 1$.

<u>Induction step:</u> Assume $N(i) = 2^{i+1} - 1$ for $0 \le i < h$.

A perfect binary tree of height h consists of 2 perfect binary trees of height *h*-1 plus the root:



$$
\begin{aligned}
N(h) &= 2 \times N(h-1) + 1 \\
&= 2 \times (2^{h-1+1} - 1) + 1 \\
&= 2 \times 2^{h} - 2 + 1 \\
&= 2^{h+1} - 1 \quad \square
\end{aligned}
$$

2^h are leaves

$2^h - 1$ are internal nodes

Full Binary Tree Theorem

Thm. *In a non-empty, full binary tree, the number of internal nodes is always 1 less than the number of leaves.*

Proof. By induction on n.
L(n) := number of leaves in a non-empty, full tree of n internal nodes.

Base case: $L(0) = 1 = n + 1$.

Induction step: Assume $L(i) = i + 1$ for $i < \mathrm{n}$.

Given T with n internal nodes, remove two sibling leaves.

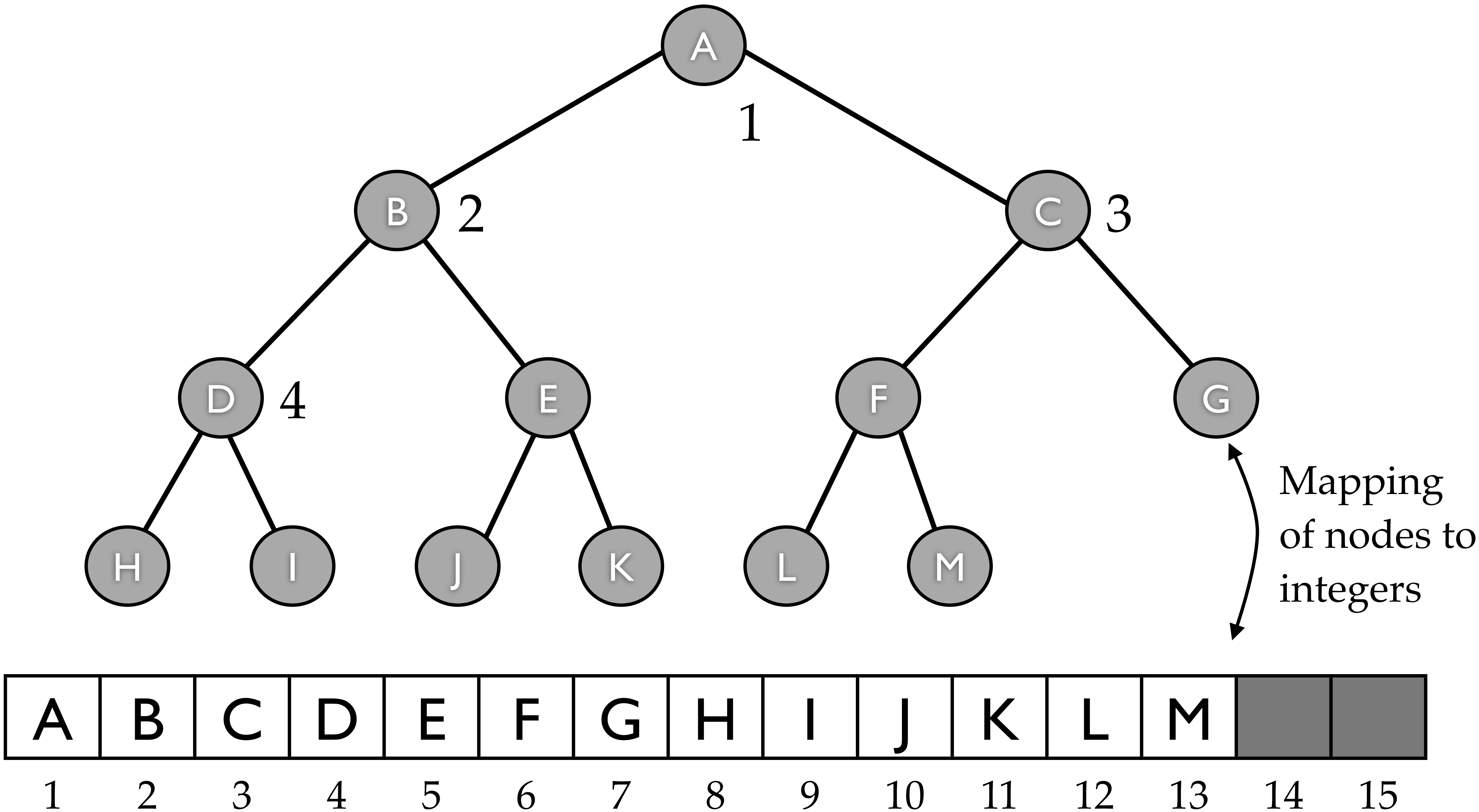
T′ has n-1 internal nodes, and by induction hypothesis, $L(n\text{-}1) = n$ leaves.

Replace removed leaves to return to tree T.
Turns a leaf into an internal node, adds two new leaves.

Thus: $L(n) = n + 2 - 1 = n + 1$.

Thm 5.1 in your book.

Array Implementation for Complete Binary Trees



left(i): $2i$ **if** $2i \le n$ **otherwise** 0

right(i): $(2i + 1)$ **if** $2i + 1 \le n$ **otherwise** 0

parent(i): $\lfloor i/2 \rfloor$ **if** i ≥ 2 **otherwise** 0

Binary Tree ADT

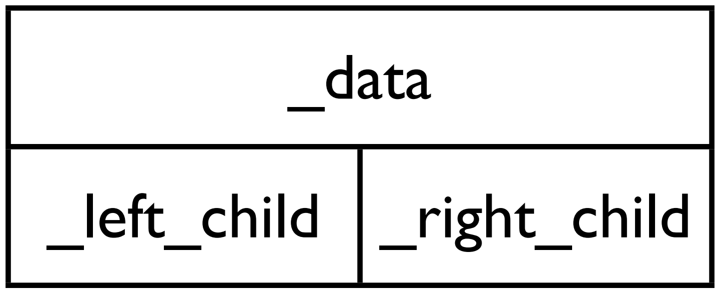
A tree can be represented as a linked collection of its nodes:

```
template <class ValType>
class BinaryTree {
 public:

   virtual ValType & value() = 0;
   virtual void set_value(const ValType &) = 0;

   virtual BinaryTree * left() const = 0;
   virtual void set_left(BinNode *) = 0;

   virtual BinaryTree * right() const = 0;
   virtual void set_right(BinNode *) = 0;
   virtual bool is_leaf() = 0;
};
```

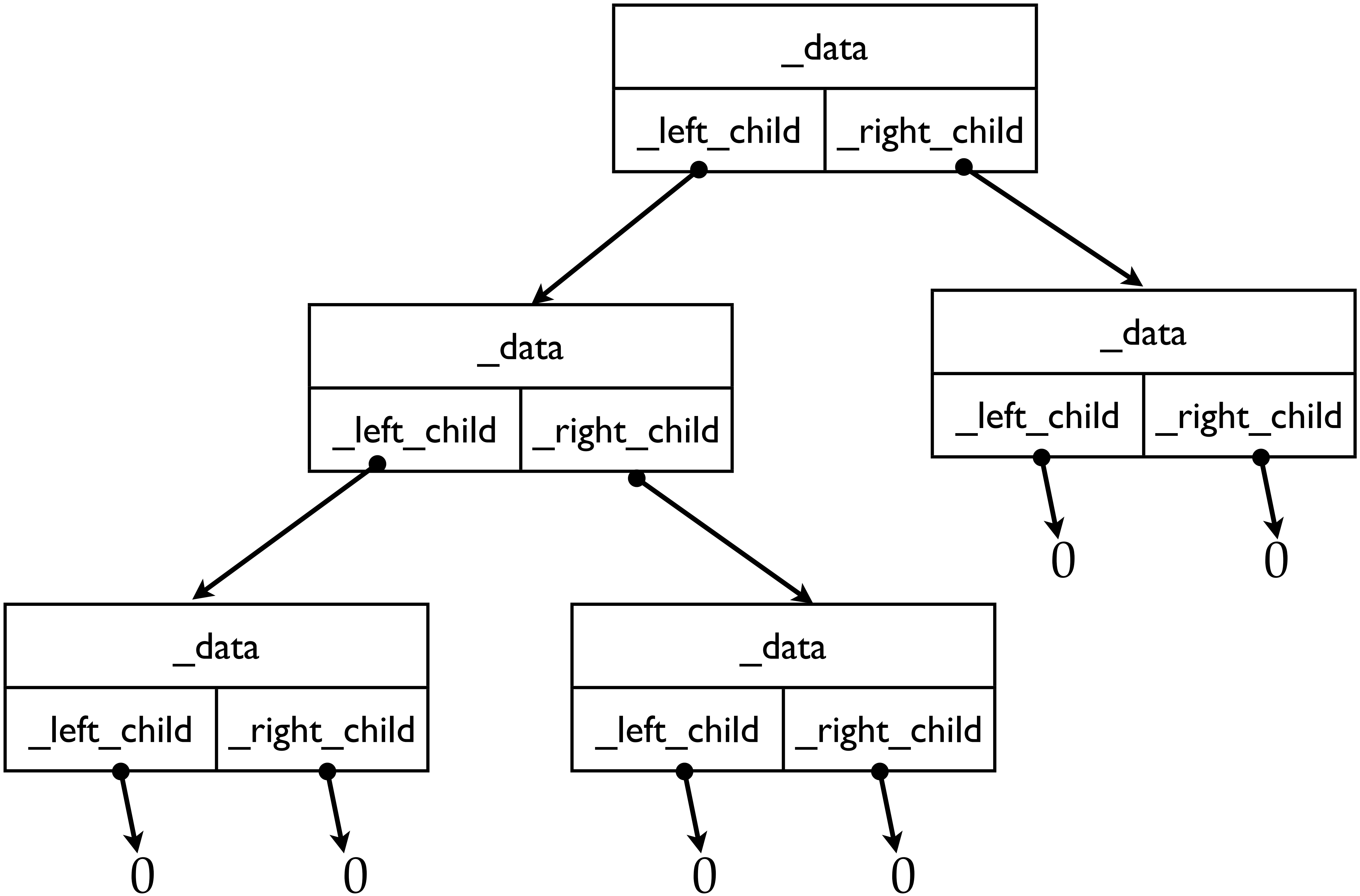
virtual ⇒ this function can be overridden by subclassing.
"= 0" ⇒ a *pure* function with no implementation. Must subclass to get implementation.

Linked Binary Tree Implementation

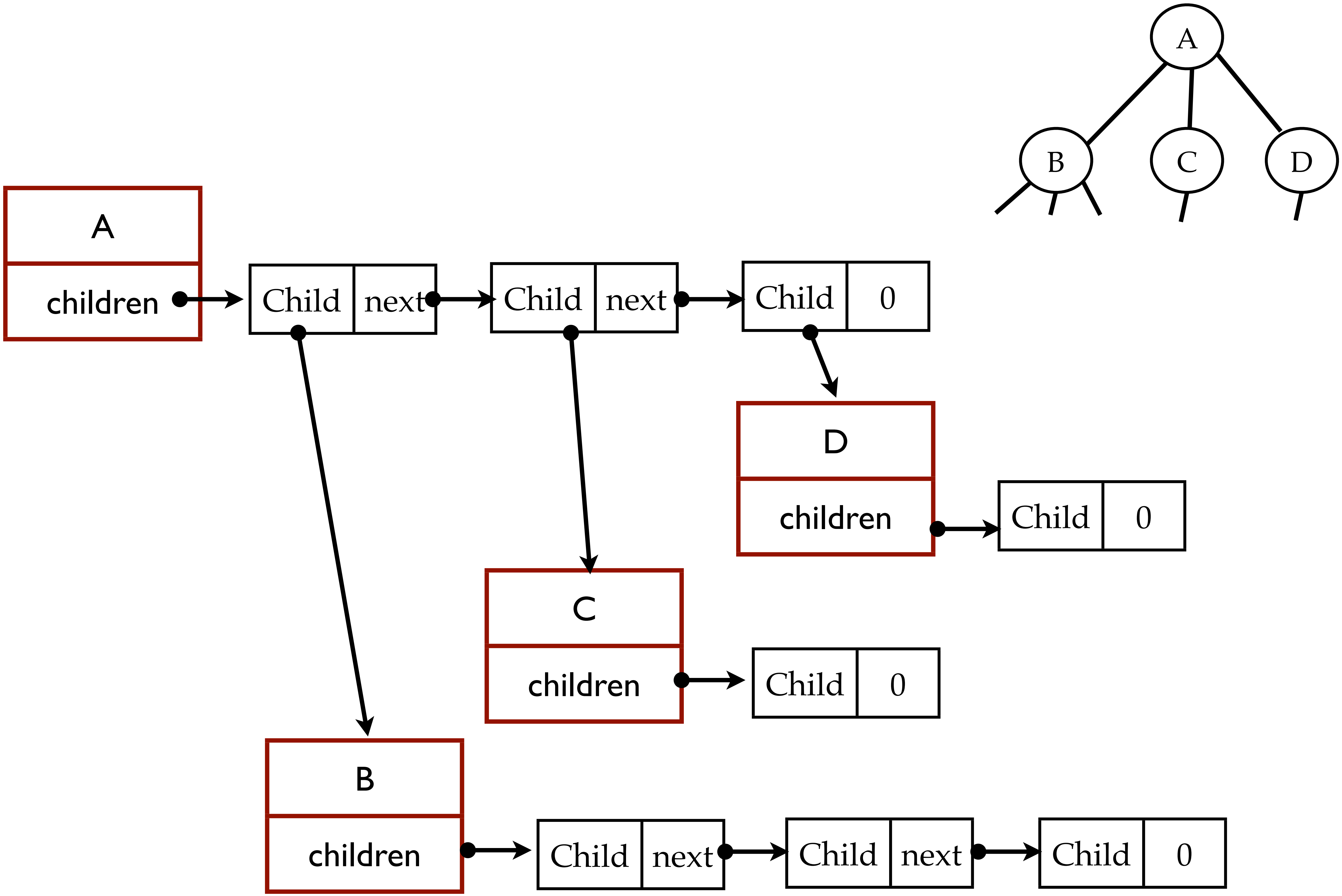
```
template <class ValType>
class BinNode : public BinaryTree<ValType>
{
 public:
   BinNode(ValType * v);
   ~BinNode();

   ValType & value();
   void set_value(const ValType&);

   BinNode * left() const;
   void set_left(BinNode *);

   BinNode * right() const;
   void set_right(BinNode *);
   bool is_leaf();

 private:
   ValType * _data;
   BinNode<ValType> * _left_child;
   BinNode<ValType> * _right_child;
};
```



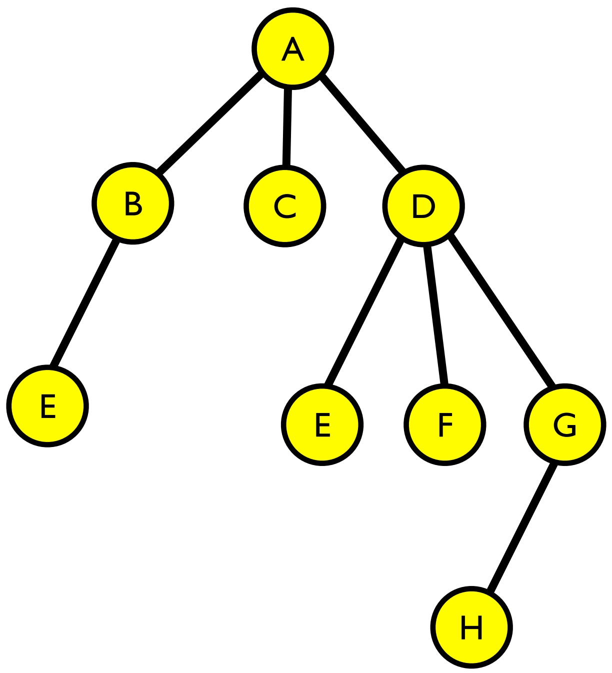
Binary Tree Representation



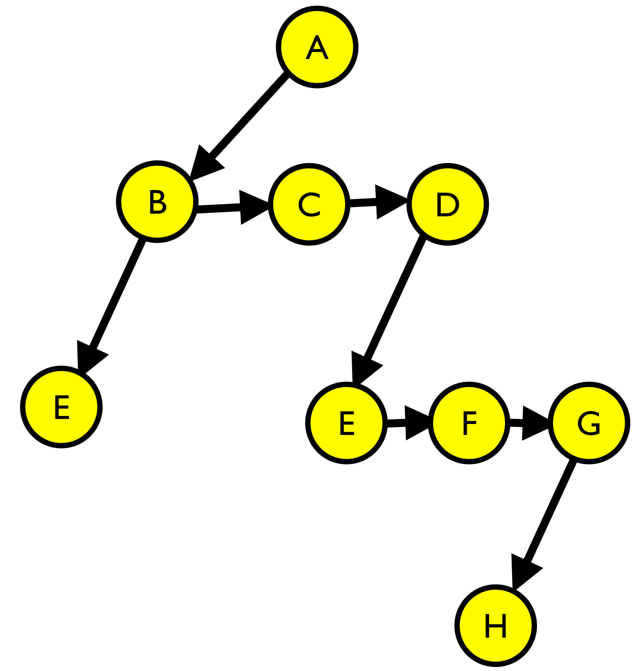
List Representation of General Trees



Representing General Trees with Binary Trees

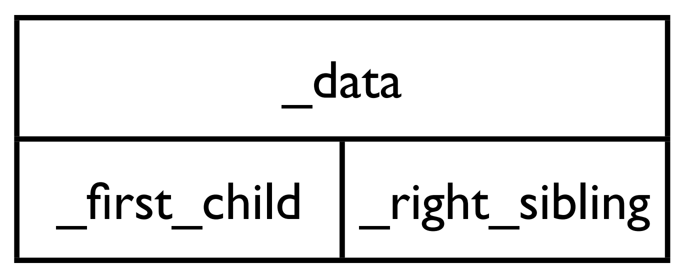


General K-ary Tree



Representation as Binary Tree

Each node represented by:



How would you implement an *ordered* general tree using a binary tree?