Energy in Mechanics

This set of notes contains a brief review of the laws and theorems of Newtonian mechanics, and a longer section on energy, because of its central importance in the present course. At the end are some worked-out examples.

Newton's Laws of Motion

1st Law: In an inertial frame a particle subject to no net force will move with constant velocity.

This law defines an inertial frame. Its physical content is the assertion that such frames exist.

2nd Law: If the total force on a particle is **F** then its motion in an inertial frame is subject to the relation $\mathbf{F} = m\mathbf{a}$, where m is its mass and **a** is its acceleration.

(Alternatively, in terms of the momentum $\mathbf{p} = m\mathbf{v}$ we have $\mathbf{F} = d\mathbf{p}/dt$.)

This law is in effect the definition of force. Its importance is the assertion that by experiment one can find in nature laws of force so that **F** can be known in advance, allowing the determination of **a** (or the rate of change of momentum), from which the future motion of the particle might be predicted.

3rd Law: If two particles interact in such a way that particle 1 exerts force **F** on particle 2, then particle 2 will exert force –**F** on particle 1.

This law implies that the total momentum of a system of particles is not changed by their mutual interactions. It was inferred from experiments on collisions of two objects.

Conservation Laws

Applying these laws to systems of particles, one obtains important general results.

Total momentum of a system. Because the effects of mutual interactions of particles within the system cancel in pairs, the total momentum of a system of particles is changed only by *external* forces, according to $\mathbf{F}_{ext} = d\mathbf{P}_{tot}/dt$, where $\mathbf{F}_{ext}$ is the total external force and $\mathbf{P}_{tot}$ is the total momentum of the particles. From this follows:

Conservation of momentum of a system. If the total external force on a system is zero, the total momentum of the system remains constant (is conserved).

Torque. The effect of a force in causing an object to rotate about a given point is expressed by the torque $\boldsymbol{\tau}$, defined by $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, where **F** is the force and **r** is the position vector (relative to the given point) of the place where the force is applied.

Angular momentum of a particle. About a given point, the angular momentum **L** of a particle, having momentum **p** and located relative to the given point by the vector **r**, is defined by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$

Relation between torque and angular momentum. If follows from the definitions and the 2nd law that for a single particle $\boldsymbol{\tau} = d\mathbf{L}/dt$. Analysis of systems of particles shows that this equation is generally true in an inertial frame, if **L** is the total angular momentum and $\boldsymbol{\tau}$ is the total torque due to *external* forces only. From this follows:

Conservation of angular momentum. If the total torque due to external forces about a given point is zero then the total angular momentum about that point is conserved.

The conservation laws of momentum and angular momentum involve vector quantities. They apply component by component. That is, if any one component of the total external force is zero, that component of the total momentum is conserved, and similarly for torque and angular momentum.

There are other useful results of the analysis of systems of particles, such as the role played by the center of mass of the system, but these will be of less importance in this course.

Work and Energy

The ideas of work, kinetic energy and potential energy were introduced in the 1700s to deal with situations in which the force acting on an object is known, not as a function of time, but rather in terms of the location of the object in space. Since this is the more usual situation in nature, use of energy became more important in many cases than use of force through the 2nd law. In the 19th century other forms of energy (thermal, chemical and electromagnetic) were recognized and a general law of conservation of energy was discovered, making energy the most important concept in all of science.

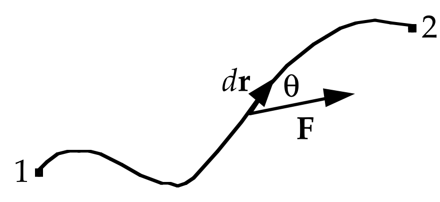
Work done by a force. If a force F acts on an object while it moves through an infinitesimal displacement $d\mathbf{r}$ then the force does *work* defined by $dW = \mathbf{F} \cdot d\mathbf{r}$.

Because it is the scalar product:

- Work is a scalar. It can be positive or negative, but has no direction.
- We can write the scalar product as $\mathbf{F} \cdot d\mathbf{r} = F\,dr\cos\theta$, where θ is the angle between the directions of the two vectors. This shows that if $\theta < \pi/2$ the work done is positive, while if $\pi > \theta > \pi/2$ the work done is negative. If $\theta = 0$ the work done is $dW = F\,dr$, which is the largest it can be. If $\theta = \pi$ the work done is $dW = -F\,dr$, which is the smallest it can be. If $\theta = \pi/2$ the work done is zero.

The work done when the object moves along a finite path from point 1 to point 2 is given by an integral:

$$W(1 \to 2) = \int_1^2 \mathbf{F} \cdot d\mathbf{r}\,.$$



Here 1 and 2 stand for the (three) coordinates of the two endpoints. This kind of integral is called a *line integral*. The drawing shows what is involved: for each infinitesimal bit of the path $d\mathbf{r}$ one calculates the scalar product to find the infinitesimal work $dW = \mathbf{F} \cdot d\mathbf{r}$; the integral is the (infinite) sum of all these infinitesimal bits. Because the path may not be simple and the force may change from point to point, this integral may be complicated.

We will carry out such calculations in detail only in cases of simple geometry or symmetry, where the integral reduces to the familiar type studied in elementary calculus courses.

Some special cases:

- If the force is the same everywhere during the object's motion (a *constant* force), the work integral simplifies:

$$W(1 \to 2) = \mathbf{F} \cdot \int_1^2 d\mathbf{r} = \mathbf{F} \cdot (\mathbf{r}_2 - \mathbf{r}_1) = \mathbf{F} \cdot \Delta\mathbf{r} .$$

  Here $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors for the initial and final points in the path.

  Gravity near the earth's surface is a case of this kind.
- If the force is always perpendicular to the path, the work done is simply zero. The radial force in circular motion is such a case. (We will see others.)
- If the force is always opposite to the motion the work done is always negative. Kinetic friction is such a case.

  Kinetic friction is sometimes mistaken for a constant force. In the simple model used in introductory courses the magnitude of the kinetic friction force is constant *if* the normal force's magnitude does not change. But for a force to be constant, its *direction* must not change. Kinetic friction changes direction when the direction of the object's motion changes.

A *closed* path is one for which the initial and final points are the same. An object making a revolution in a circular path is an example. For some forces (e.g., a constant force) the total work done for such a path is zero. But for others (e.g., kinetic friction) it is not zero.

Power. The rate at which work is done is called the *power input* by the force. This is given by

$$P = dW / dt = \mathbf{F} \cdot d\mathbf{r} / dt = \mathbf{F} \cdot \mathbf{v} .$$

Here **v** is the velocity of the object in its motion. One can write $P = F v \cos\theta = F_{||} v$, where $F_{||}$ is the component of **F** parallel to **v**. If $F_{||}$ is positive (negative), the power input is positive (negative).

Kinetic energy. In the discussion so far, **F** might be any force that acts on the object. But if it is the *total* force acting, then by Newton's 2nd Law we have $\mathbf{F} = m\mathbf{a} = m\,d\mathbf{v} / dt$. The power input then becomes

$$P = \mathbf{F} \cdot \mathbf{v} = m\mathbf{v} \cdot \mathbf{a} .$$

But it is easy to show that

$$\frac{d}{dt} v^2 = \frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = 2\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = 2\mathbf{v} \cdot \mathbf{a} ,$$

so we find

$$P = \frac{d}{dt}\left(\tfrac{1}{2} m v^2\right).$$

The quantity in ( ) is defined as the *kinetic energy* of the object: $K = \frac{1}{2} m v^2$. We have derived an important result:

The power input by the total force is equal to the rate of change of the kinetic energy.

Since the power is equal to the rate at which work is done, this can also be stated as: The work done by the total force is equal to the change in kinetic energy.

This is called the work-kinetic-energy theorem: $K(2)-K(1)=\int_1^2 \mathbf{F}_{tot}\cdot d\mathbf{r}$.

Kinetic energy is a quantity the object possesses. Work, on the other hand, represents energy being transferred (by the action of the force) to or from the object.

Conservative forces and potential energy. An important question about work done by a force: As the object moves from point 1 to point 2, does the work done depend on the path taken between those points? As we have seen, it may or may not. Work done by a constant force is independent of the path taken, but work done by kinetic friction is usually more negative if the path taken is longer.

A constant force is an example of a *conservative* force, defined as follows: The work done by a conservative force is independent of the path taken. An alternative and equivalent definition is this: The work done by a conservative force in a closed path is zero.

Proof that these definitions are equivalent: If the work is independent of path, one can choose for a closed path the simple "path" of not moving at all, for which the work is clearly zero.

There is a mathematical criterion, involving partial derivatives of the force components, by which one can test whether a force is conservative. Using it one shows that any central force — a force directed toward or away from a specific point and with magnitude dependent only on the distance from that point — is conservative. The force in Newton's law of gravity is central, as is the Coulomb force we will study, so they are conservative. Also conservative is any force of the form $\mathbf{F}=F(x)\mathbf{i}$, i.e., directed along one coordinate axis with magnitude dependent only on that coordinate. Any constant force is also conservative, as we have seen.

If $\mathbf{F}$ is conservative then the value of the integral $\int_1^2 \mathbf{F}\cdot d\mathbf{r}$ cannot depend on anything but the two endpoints 1 and 2. This makes the line integral like an ordinary integral in that we can find a function of position $f(\mathbf{r})$ such that $\int_1^2 \mathbf{F}\cdot d\mathbf{r}=f(\mathbf{r}_2)-f(\mathbf{r}_1)$. For reasons of simplicity later we let $f(\mathbf{r})=-U(\mathbf{r})$ and call U the *potential energy* associated with the conservative force $\mathbf{F}$. That is, we define the potential energy by

$$U(2)-U(1)=-\int_1^2 \mathbf{F}\cdot d\mathbf{r}\,.$$

Each conservative force has its own potential energy function. If more than one conservative force acts, the total potential energy is simply the sum of those for the individual forces.

The above equation defines only the *difference* between U at the two points. The actual value at either point is arbitrary. This arbitrariness has no physical consequences, because only changes in U have physical effects. To simplify things one usually chooses U to be zero at some point, which determines its value at all other points.

Potential energy is a property of the *whole system* including the object under consideration and all the other objects that interact with it by means of the conservative forces. It is misleading to refer to the potential energy of a single particle, for example.

General work-energy theorem and conservation of mechanical energy. Of the forces acting on a body some may be conservative and others not. Divide them into groups: $\mathbf{F}_{tot} = \mathbf{F}_{cons} + \mathbf{F}_{non-cons}$. Then use the work-kinetic-energy theorem to write

$$K(2) - K(1) = \int_1^2 \mathbf{F}_{cons} \cdot d\mathbf{r} + \int_1^2 \mathbf{F}_{non-cons} \cdot d\mathbf{r}$$
$$= U(1) - U(2) + \int_1^2 \mathbf{F}_{non-cons} \cdot d\mathbf{r}$$

(Here U stands for the total potential energy of all the conservative forces.) Now we define the *total mechanical energy* E by

$$E = K + U\,.$$

Rearranging the above equation we find

$$E(2) - E(1) = \int_1^2 \mathbf{F}_{non-cons} \cdot d\mathbf{r}\,.$$

In words:

The change in the total mechanical energy is equal to the work done by non-conservative forces.

This is an important and useful result.

It follows immediately that:

If no work is done by non-conservative forces, the total mechanical energy is conserved.

This is the law of conservation of mechanical energy, one of the cornerstones of Newtonian mechanics.

> Two major achievements of 19th century science extended this conservation law. First it was recognized that hidden in the item "work by non-conservative forces" is a large class of phenomena, from heating and cooling to chemical reactions. By including their contributions (positive or negative) into the definition of total energy, the grand law of conservation of energy was proposed, possibly the most important of all the laws of nature in terms of practical uses. The other major 19th century addition was the energy in the electromagnetic field, a central concept in this course.

Obtaining the force from the potential energy. The fundamental theorem of calculus says the integrand is the derivative of the integral. Given the definition of potential energy

$$U(2) - U(1) = -\int_1^2 \mathbf{F} \cdot d\mathbf{r}$$

it would seem that if we are given U we can find $\mathbf{F}$ by taking derivatives. That is indeed the case, but it is a bit complicated because we must deal with three dimensions.

The three components of $\mathbf{F}$ are given by *partial* derivatives of U, as follows:

$$F_x = -\partial U / \partial x,\; F_y = -\partial U / \partial y,\; F_z = -\partial U / \partial z\,.$$

(To take a partial derivative with respect to x, hold y and z constant, and so on.)

We will deal mostly with cases in which U depends on only one variable, say, x. Then we have the useful formula

$$F_x = -dU(x) / dx\,.$$

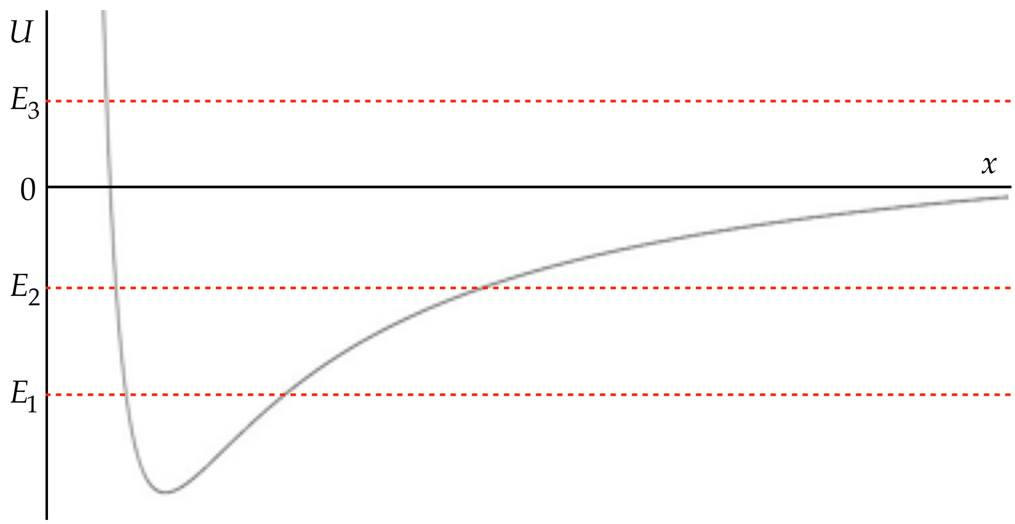
The relations discussed here contain a useful rule. Assume there are no forces other than the ones for which U is the total potential energy. Then the direction of the total force **F** is toward the region where the derivatives of U are *negative*, i.e., toward where the potential energy is *lower*. A system starting from rest will move in that direction. This gives us the rule:

Left to itself, a system starting from rest will move toward lower potential energy.

The restrictions "left to itself" and "starting from rest" are crucial. A book resting on a table does not move to lower potential energy by falling because the table exerts an upward force on it; it is not left to itself. And a ball thrown directly upward moves toward higher potential energy as it rises, but it did not start from rest; after it comes to rest at the top of its flight it does move toward lower potential energy. The rule is useful but not a panacea.

Sample problems and solutions.

1. Shown is the potential energy function for the typical interaction between two neighboring atoms in a solid, as a function of their separation distance x.



The three red dotted lines represent the total energy of the system at three different temperatures. (As the temperature rises, so does the total energy.)

a. Call the distance where U has its minimum x_0. For $x < x_0$ the force between the atoms is a repulsion, while for $x > x_0$ it is an attraction. Explain this.

b. For energy E_1 and E_2 the two atoms are bound to each other, while for energy E_3 they are unbound. How do you know?

c. The average separation of the atoms is greater for energy E_2 than for E_1. Explain how you know this and relate it to the expansion of the solid as the temperature rises.

a. The force is related to U by $F_x = -dU/dx$. So where the slope of U is positive, F_x is negative and vice versa. A negative value of F_x means an attractive force. This occurs for $x > x_0$. Similarly, a positive value of F_x means a repulsive force, which occurs for $x < x_0$. At $x = x_0$ the force is zero. If the atoms were at rest at that distance they would remain at rest. The system would be in static equilibrium. (But they are never actually at rest because of thermal energy.)

b. The kinetic energy $K = E - U$ cannot be negative (it is $\frac{1}{2}mv^2$). So the motion of the system is restricted to regions where $E > U$. Points where $E = U$ are "turning points" where K is (instantaneously) zero.

Consider first energy E_3 and imagine the atoms are initially moving toward each other (x is decreasing). Until they reach distance x_0 they are attracted and the kinetic energy increases. But then they are repelled and the kinetic energy decreases until it reaches zero at the distance where the line for E_3 crosses the curve for U. This is the turning point. Because they are still being repelled, they move apart again with increasing kinetic energy until the distance x_0, after which the are attracted and the kinetic energy decreases again. But because E_3 is positive, while U remains negative, the kinetic energy never reaches zero. There is no turning point at distances greater than x_0, so the atoms continue to move apart. They are unbound from each other.

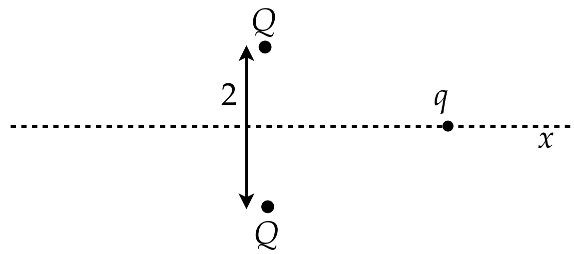
This would be the situation for two atoms in a gas; if they are in a liquid interaction with other atoms would keep them from moving far apart, but they are not bound to each other.

For the other energies the situation is different. There are two turning points. If the atoms are moving toward each other they will reach the turning point at distance less than x_0, stop instantaneously, then move apart until they reach the turning point at distance greater than x_0, where they will again reverse direction. They thus move back and forth between these two turning points. They are bound.

c. The average separation is halfway between the turning points. Because of the asymmetry of the curve for U, the average separation increases as E increases. Since increasing the temperature increases E, the average separation increases with increasing temperature. The solid expands.

2. We will find that the potential energy of the interaction between two point charges is given by the formula $U = k\frac{q_1 q_2}{r}$, where q_1 and q_1 are the charges (positive or negative), k is a constant, and r is the distance between the charges.

Consider the situation shown. the two charges Q are both positive and are fixed at the points indicated. The charges Q are each at distance 1 from the x-axis, while q is at distance x from the point ($x = 0$) between the other charges. The charge q can be either positive or negative and it will be released from rest at the position shown. We wish to determine its motion after it is released.



a. Write the potential energy of the system as a function of x. (In calculating this the potential energy of the two Qs alone can be ignored because they never move, so this contribution never changes.)

b. Find the force F_x on q in terms of x.

c. Let q be positive. It is released from rest at $x = x_0$. Show that it will move to the right along the x-axis. Find its kinetic energy when it is far from the other charges.

d. Now let q be negative. It is released from rest at the same point as before. Describe its motion. What are its turning points? For what value of x is its kinetic energy a maximum. What is that maximum value?

a. The distance q is from each of the other charges is $r = \sqrt{x^2 + 1}$, so the potential energy is $U(x) = 2kqQ / \sqrt{x^2 + 1}$.

b. We use $F_x = -dU/dx$ to find $F_x = 2kqQ\frac{x}{(x^2+1)^{3/2}}$. Note that this is zero for $x = 0$, while for $x >> 1$ it is approximately $F_x \approx 2kqQ / x^2$.

c. If q is positive, F_x is also positive, so the charge is moved to the right (toward lower potential energy). For $x >> 1$ the potential energy is essentially zero. Using conservation of energy we find $K(x \to \infty) = U(x_0) = 2kqQ / \sqrt{x_0{}^2 + 1}$. This system is unbound.

d. Now the initial force is to the left, so q moves that way. After it passes $x = 0$ the force becomes to the right, so q slows down and stops, reversing its motion. The turning point is where $U = E$, and $E = U(x_0) = 2kqQ / \sqrt{x_0^2 + 1}$. Call the new turning point x_1. We have $U(x_1) = U(x_0)$, or $x_1^2 = x_0^2$. The new turning point is $x_1 = -x_0$. The motion is oscillation between $x = \pm x_0$. The system is bound.

The kinetic energy is largest when U is smallest. Remember that U is negative (because q is), so it is smallest when $x = 0$. From conservation of energy we have $K(x=0) + U(x=0) = E = U(x_0)$, so $K(x=0) = 2kqQ\left[\frac{1}{\sqrt{x_0^2+1}} - 1\right]$. (Again, this is positive because q is negative.)