Basics of Logic Design: Boolean Algebra, Logic Gates

Computer Science 104

Today's Lecture

- Projects (groups of 2 or 3)

Outline

- Building the building blocks…
- Logic Design
  - Truth tables, Boolean functions, Gates and Circuits

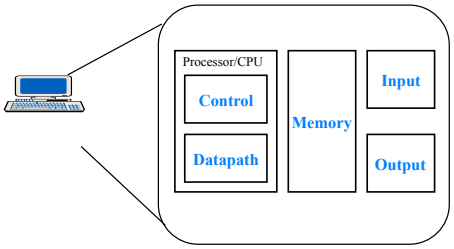
Reading

Appendix C (link off course web page/documents)

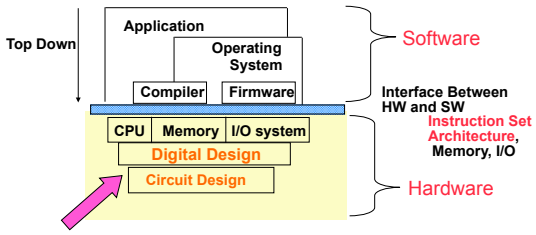
http://arch.cs.duke.edu/local/COD4ED/resources/Appendix/Appendix-C-P374493.pdf

The Big Picture

- The Five Classic Components of a Computer



What We've Done, Where We're Going



Bottom UP to CPU

Digital Design

- Logic Design, Switching Circuits, Digital Logic

Recall: Everything is built from transistors

- A transistor is a switch
- It is either on or off
- On or off can represent True or False

Given a bunch of bits (0 or 1)…

- Is this instruction a lw or a beq?
- What register do I read?
- How do I add two numbers?
- Need a method to reason about complex expressions

Boolean Algebra

- Boolean functions have arguments that take two values ({T,F} or {1,0}) and they return a single or a set of ({T,F} or {1,0}) value(s).
- Boolean functions can always be represented by a table called a "Truth Table"
- Example: F: $\{0,1\}^3$ -> $\{0,1\}^2$

a	b	c	f_1	f_2
0	0	0	0	1
0	0	1	1	1
0	1	0	1	0
0	1	1	0	0
1	0	0	1	0
1	1	0	0	1
1	1	1	1	1

Boolean Functions

- Example Boolean Functions: NOT, AND, OR, XOR, . . .

a	NOT(a)
0	1
1	0

a	b	AND(a,b)
0	0	0
0	1	0
1	0	0
1	1	1

a	b	OR(a,b)
0	0	0
0	1	1
1	0	1
1	1	1

a	b	XOR(a,b)
0	0	0
0	1	1
1	0	1
1	1	0

a	b	XNOR(a,b)
0	0	1
0	1	0
1	0	0
1	1	1

a	b	NOR(a,b)
0	0	1
0	1	0
1	0	0
1	1	0

Boolean Functions and Expressions

- Boolean algebra notation: Use * for AND, + for OR, ~ for NOT.
  - NOT is also written as A' and $\overline{A}$
- Using the above notation we can write Boolean expressions for functions

  F(A, B, C) = (A * B) + (~A * C)

- We can evaluate the Boolean expression with all possible argument values to construct a truth table.
- What is truth table for F?

Boolean Function Simplification

- Boolean expressions can be simplified by using the following rules (bitwise logical):
  - A*A = A
  - A* 0 = 0
  - A*1 = A
  - A*~A = 0
  - A+A = A
  - A+0 = A
  - A+1 = 1
  - A+~A = 1
  - A*B = B*A
  - A*(B+C) = (B+C)*A = A*B + A*C

Boolean Function Simplification

a b c	$f_1 f_2$
0 0 0	0 1
0 0 1	1 1
0 1 0	0 0
0 1 1	1 0
1 0 0	0 0
1 0 1	1 0
1 1 0	0 1
1 1 1	1 1

f_1 = ~a*~b*c + ~a*b*c + a*~b*c + a*b*c

f_2 = ~a*~b*~c + ~a*~b*c + a*b*~c + a*b*c

Simplify these functions...

Boolean Functions and Expressions

- The Fundamental Theorem of Boolean Algebra: Every Boolean function can be written in disjunctive normal form as an OR of ANDs (Sum-of products) of it's arguments or their complements.

"Proof:" Write the truth table, construct sum-of -product from the table.

a	b	XNOR(a,b)
0	0	1
0	1	0
1	0	0
1	1	1

XNOR = (~a * ~b) + (a * b)

Boolean Functions and Expressions

- Example-2:

a b c	$f_1 f_2$
0 0 0	0 1
0 0 1	1 1
0 1 0	1 0
0 1 1	0 0
1 0 0	1 0
1 1 0	0 1
1 1 1	1 1

f_1 = ~a*~b*c + ~a*b*~c + a*~b*~c + a*b*c

f_2 = ~a*~b*~c + ~a*~b*c + a*b*~c + a*b*c

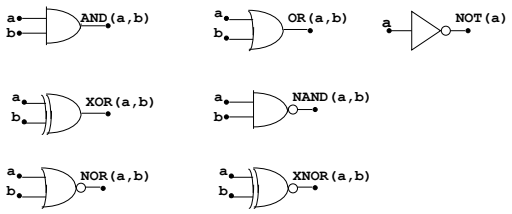
Applying the Theory

- Lots of good theory
- Can reason about complex boolean expressions
- Now we have to make it real…

Boolean Gates

- Gates are electronic devices that implement simple Boolean functions

Examples



Reality Check

- Basic 1 or 2 Input Boolean Gate 1- 4 Transistors

Pentium III
- Processor Core 9.5 Million Transistors
- Total: 28 Million Transistors

Pentium 4
- Total: 42 Million Transistors

Core2 Duo (two processors)
- Total: 290 Million Transistors

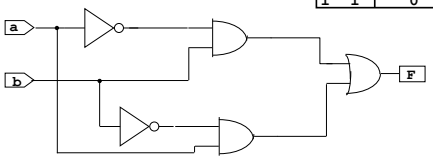
Core2 Duo Extreme (4 processors, 8MB cache)
- Total: 590 Million Transistors

Boolean Functions, Gates and Circuits

- Circuits are made from a network of gates. (function compositions).

F = ~a*b + ~b*a

a	b	XOR(a,b)
0	0	0
0	1	1
1	0	1
1	1	0



Digital Design Examples

Input: 2 bits representing an unsigned number (n)
Output: n^2 as 4-bit unsigned binary number

Input: 2 bits representing an unsigned number (n)
Output: 3-n as unsigned binary number

Design Example

- Consider machine with 4 registers
- Given 2-bit input (register specifier, I_1, I_0)
- Want one of 4 output bits (O_3-O_0) to be 1
  - E.g., allows a single register to be accessed
- What is the circuit for this?

More Design Examples

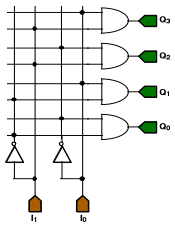
- X is a 3-bit quantity

1. Write a logic function that is true if and only if X contains at least two 1s.
2. Implement the logic function from problem 1. using only AND, OR and NOT gates. (Note there are no constraints on the number of gate inputs.) By implement, I mean draw the circuit diagram.
3. Write a logic function that is true if and only if X, when interpreted as an unsigned binary number, is greater than the number 4.
4. Implement the logic function from problem 3. using only AND, OR and NOT gates. (Note there are no constraints on the number of gate inputs.)

Parity Example

- The parity code of a binary word counts the number of ones in a word. If there are an even number of ones the parity code is 0, if there are an odd number of ones the parity code is 1. For example, the parity of 0101 is 0, and the parity of 1101 is 1.
- Construct the truth table for a function that computes the parity of a four-bit word. Implement this function using AND, OR and NOT gates. (Note there are no constraints on the number of gate inputs.)

Circuit Example: Decoder

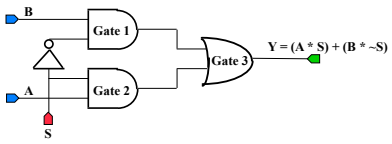


I_1	I_0	Q_0	Q_1	Q_2	Q_3
0	0	1	0	0	0
0	1	0	1	0	0
1	0	0	0	1	0
1	1	0	0	0	1

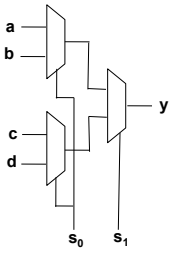
Circuit Example: 2x1 MUX

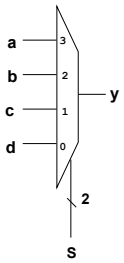
Multiplexor (MUX) selects from one of many inputs

MUX(A, B, S) = (A * S) + (B * ~S)



Example 4x1 MUX





Arithmetic and Logical Operations in ISA

- What operations are there?
- How do we implement them?
  - Consider a 1-bit Adder

Summary

- **Boolean Algebra & functions**
- **Logic gates (AND, OR, NOT, etc)**
- **Multiplexors**

Reading

- **Appendix C**